Forecasting Economic Time Series

Economic forecasts rely on time series data – observations which are recorded sequentially over time.
Time Series Data

- A time series is written as $y_t$
- The index $t$ denotes the time period.
- A time period may be a year, quarter, month, week, day, transaction, or any other time unit.
- We call this the data frequency.
Lags and Leads

• We will often talk about lags and leads
  • The first lag of $y_t$ is written $y_{t-1}$
    – It is the observation from the previous period
    – For example, the lag of November is October
  • The second lag is $y_{t-2}$, the k’th lag is $y_{t-k}$
• The first lead of $y_t$ is $y_{t+1}$
• The k’th lead is $y_{t+k}$
Time Series Samples

• A historical sample is a set of observations in contiguous time, written as
  \[ \{y_1, y_2, \ldots, y_T\} \]
  
• \( T \) is the number of observations in-sample.

• The number of observations does not equal the number of years, unless the frequency is annual.
Examples

- Number of Unemployed
- Unemployment Rate
- GDP
- Real GDP
- Price Level
- Inflation Rate
- New Housing Starts
Wisconsin Unemployment Rate
Forecast Period

• In-sample observations: \( \{y_1, y_2, \ldots, y_T\} \)
• Out-of-sample period: \( \{y_{T+1}, y_{T+2}, \ldots, y_{T+h}\} \)
• \( h \) is called the forecast horizon
Forecast Notation

• We denoted $\hat{y}$ as the point forecast for $y$.
• This suggests $\hat{y}_{T+h}$ as the point forecast for $y_{T+h}$.
• But is not enough. While it is the forecast for the time series at time period $T+h$, it is not clear when the forecast is made.
  – At time period $T$
  – At time period $T+1$
  – At time period $T+h-1$
Notation

• We will use the notation

\[ \hat{y}_{t+h|t} \]

to refer to the forecast of \( y_{t+h} \) made at time \( t \).

• Thus \( \hat{y}_{T+h|T} \), \( \hat{y}_{T+h|T+1} \), \( \hat{y}_{T+h|T+2} \), etc. are the sequence of forecasts of \( y_{T+h} \) made in time periods \( T, T+1, T+2 \), etc.
Notation

• Similarly, the forecast distribution and density for $y_{t+h}$ made at time $t$ will be written as

$$F_{t+h|t}(y)$$

and

$$f_{t+h|t}(y)$$
Extrapolative Forecasts and Fan Charts

• At time $T$, make a sequence of forecasts for time periods $T+1$, $T+2$, $T+3$, ..., $T+h$

• Point forecasts: $\hat{y}_{T+1|T}$, $\hat{y}_{T+2|T}$, ..., $\hat{y}_{T+h|T}$

• Add interval forecasts

• Plot over forecast horizon

• This is called a fan chart.

• The intervals tend to fan out with the forecast horizon.
Extrapolative Forecasts

<table>
<thead>
<tr>
<th>Year</th>
<th>Point Forecast</th>
<th>50% Interval Forecast</th>
<th>80% Interval Forecast</th>
</tr>
</thead>
<tbody>
<tr>
<td>2013:12</td>
<td>6.2%</td>
<td>(6.2%, 6.3%)</td>
<td>(6.1%, 6.3%)</td>
</tr>
<tr>
<td>2014:1</td>
<td>6.2%</td>
<td>(6.1%, 6.3%)</td>
<td>(6.0%, 6.3%)</td>
</tr>
<tr>
<td>2014:2</td>
<td>6.1%</td>
<td>(5.9%, 6.2%)</td>
<td>(5.8%, 6.3%)</td>
</tr>
<tr>
<td>2014:3</td>
<td>6.0%</td>
<td>(5.8%, 6.1%)</td>
<td>(5.6%, 6.3%)</td>
</tr>
<tr>
<td>2014:4</td>
<td>5.9%</td>
<td>(5.7%, 6.1%)</td>
<td>(5.4%, 6.4%)</td>
</tr>
<tr>
<td>2014:5</td>
<td>5.9%</td>
<td>(5.6%, 6.1%)</td>
<td>(5.3%, 6.4%)</td>
</tr>
<tr>
<td>2014:6</td>
<td>5.8%</td>
<td>(5.6%, 6.1%)</td>
<td>(5.3%, 6.3%)</td>
</tr>
<tr>
<td>2014:7</td>
<td>5.8%</td>
<td>(5.5%, 6.1%)</td>
<td>(5.2%, 6.4%)</td>
</tr>
<tr>
<td>2014:8</td>
<td>5.8%</td>
<td>(5.5%, 6.1%)</td>
<td>(5.2%, 6.5%)</td>
</tr>
<tr>
<td>2014:9</td>
<td>5.8%</td>
<td>(5.5%, 6.1%)</td>
<td>(5.2%, 6.5%)</td>
</tr>
<tr>
<td>2014:10</td>
<td>5.8%</td>
<td>(5.4%, 6.1%)</td>
<td>(5.1%, 6.6%)</td>
</tr>
<tr>
<td>2014:11</td>
<td>5.8%</td>
<td>(5.4%, 6.2%)</td>
<td>(5.1%, 6.7%)</td>
</tr>
</tbody>
</table>
Information Set

- To forecast $y_{t+h}$ at time $t$ we use relevant information.
- Most information is values of other economic variables.
- Those observed up to time $t$.
- This includes previous values of the variable $y_t$.
- It can also include other relevant variables.
- All this information is the Information Set, written as $\Omega_t$.
- For example $\Omega_t = \{y_1, y_2, y_3, ..., y_t\}$ is the set of previous values.
- $\Omega_t = \{y_1, x_1, y_2, x_2, y_3, x_3, ..., y_t, x_t\}$ includes another variable $x_t$. 
Conditional Mean Forecasts

- Under squared error loss, the optimal *unconditional* point forecast is the mean $E_y$:
  $$E(y) = \int y f(y) dy$$

- Under squared error loss, the optimal *conditional* point forecast is the *conditional mean*:
  $$E(y \mid \Omega_t) = \int y f(y \mid \Omega_t) dy$$

where $f(y \mid \Omega_t)$ is the conditional density of $y$ given $\Omega_t$. 
Conditional Mean

• The unconditional mean $E(y)$ is the average value of $y$ in the entire population.

• The conditional mean $E(y | x)$ given a set of variables $x$ is the average value of $y$ for the subpopulation with the variables $x$.

• The conditional mean $E(y | \Omega_t)$ given an information set is the average value of $y$ in the subhistory with the previous history $\Omega_t$. 
Conditional Mean

• What is $E(y|\ \Omega_t)$?
• When $\Omega_t$ is discrete, then $E(y|\ \Omega_t)$ is simply the mean in the subpopulation.
• For example, the average wage among white male college graduates.
• When $\Omega_t$ large and/or is continuously distributed, this definition does not work.
Conditional Mean

• Let $f(y, \Omega_t)$ denote the joint density of $(y, \Omega_t)$. Then the conditional density of $y$ given $\Omega_t$ is

$$f(y \mid \Omega_t) = \frac{f(y, \Omega_t)}{f(\Omega_t)}$$

• It is the distribution of $y$ holding $\Omega_t$ fixed.

• It is a slice of the joint density

• Then the conditional mean is

$$E(y \mid \Omega_t) = \int y f(y \mid \Omega_t) dy$$
Forecast Intervals

• Forecast intervals are constructed from the conditional distribution $F(y \mid \Omega_t)$.
• The endpoints are the conditional quantiles.
• Definition: The $\alpha$’th conditional quantile of $y$ given $\Omega_t$ is the number $q_\alpha(\Omega_t)$ which satisfies

$$\alpha = F(q_\alpha(\Omega_t) \mid \Omega_t)$$

• It looks more complicated, but it is identical with the case with no $\Omega_t$. 
When does Conditioning Help?

• It helps if $y$ and $\Omega_t$ are dependent or correlated.

• If $y$ and $\Omega_t$ are independent, then $f(y \mid \Omega_t) = f(y)$ and $E(y \mid \Omega_t) = E(y)$.
  – There is no gain from conditioning.

• To optimally forecast unknown $y$ we want to use observable variables $\Omega_t$ which are highly correlated with $y$. 
Summary – Conditional Forecasts

• Information improves forecasts
• Conditioning on relevant variables reduces the risk (expected loss) of forecasts
• Point and interval forecasts are functions of the conditioning variables
Forecast Density and Distribution
Actual Forecasting

• Even if the variables in the information set $\Omega_t$ are known, the conditional mean function $E(y_{t+h} \mid \Omega_t)$ is unknown
  – The functional form is unknown
  – The parameters of the function are unknown

• Thus to make an actual forecast, we need to:
  – Create an approximate model for $E(y_{t+h} \mid \Omega_t)$
  – Estimate the model parameters from data.
Time-Series Components

• Recall that the optimal point forecast of a series $y_{t+h}$ is its conditional mean.

• It is useful to decompose this mean into components

$$E(y_{t+h} | \Omega_t) = T_t + S_t + C_t$$

- $T_t$ = Trend
- $S_t$ = Seasonal
- $C_t$ = Cycle
Components

- **Trend**
  - Very long term (decades)
  - Smooth

- **Seasonal**
  - Patterns which repeat annually
  - May be constant or variable

- **Cycle**
  - Business cycle
  - Correlation over 2-7 years

- It is useful to consider the components separately
Mean Forecasting

• The simplest forecasting model has no trend, seasonal or cycle, only a constant mean
  \[ E(y_{t+h} \mid \Omega_t) = \beta_0 \]

• This might seem overly simple, but can be appropriate for a random stationary time-series
  – A series not growing or changing over time
  – Many series reported as percentage changes

• In this model, the optimal point forecast for \( y_{t+h} \)
  is \( E(y_{t+h} \mid \Omega_t) = \beta_0 \).

• An actual forecast is an estimate of \( \beta_0 \).
U.S. Real Personal Consumption (Quarterly) 
Percentage Change from Previous Period
Estimation

• If \( E(y_{t+h} \mid \Omega_t) = \beta_0 \) then the optimal forecast is the mean \( \beta_0 = E(y_{t+h}) \)

• The estimate of \( \beta_0 \) is the sample mean

\[
b_0 = \frac{1}{T} \sum_{t=1}^{T} y_{t+h}
\]

• This is the estimate of the optimal point forecast when \( E(y_{t+h} \mid \Omega_t) = \beta_0 \)

• \( b_0 \) is also the least-squares estimate in an intercept-only model
Estimation

• In STATA, use the **regress** command
• See *STATA Handout* on website
• Sample mean is estimated “constant”

```
. use gdp2013

. regress pce

                        Source |        SS      df    MS
-------------+------------------+---------+---------+
        Model |         0         0   .
Residual   |     3088.35013   265  11.6541514
-------------+------------------+---------+---------+
        Total |     3088.35013   265  11.6541514

Number of obs = 266
F(  0,   265) =  0.00
Prob > F      = .
R-squared     =  0.0000
Adj R-squared =  0.0000
Root MSE      =  3.4138

                      pce      Coef.  Std. Err.     t    P>|t|    [95% Conf. Interval]
-------------+----------------+---------+---------+------------------+
        _cons  |    3.408647    .2093146   16.28     0.000   2.996515   3.820778
```
Fitted Values

• Fitted values are the sample mean
  \[ \hat{y}_t = b_0 \]

• In STATA use the \textbf{predict} command

  . predict yp
  (option xb assumed; fitted values)

• This creates a variable “yp” of fitted values
Plot actual against fitted

```
tsline pce yp
```
Out-of-Sample

• Point forecasts are the sample mean

\[ \hat{y}_{T+h} = b_0 \]

• In STATA, use `tsappend` to expand sample, and `predict` to generate point forecasts.

\[
. \text{tsappend, add(12)}
\]

\[
. \text{predict p if time>tq(2013q3)}
\]
(option xb assumed; fitted values)
(266 missing values generated)

\[
. \text{label variable p "point forecast"}
\]

\[
. \text{tsline pce yp p if time>tq(2000q4)}
\]
Out-of-Sample

![Graph showing personal consumption expenditure, fitted values, and point forecast over time from 2000q1 to 2015q1.]

- **Personal Consumption Expenditure**
- **Fitted Values**
- **Point Forecast**
Forecast Errors

• The forecast error $e_t$ is the difference between the realized value and the conditional mean.

$$e_t = y_{t+h} - \mathbb{E}(y_{t+h} | \Omega_t)$$

or equivalently

$$y_{t+h} = \mathbb{E}(y_{t+h} | \Omega_t) + e_t$$

• We call $e_t$ the forecast error.
Residuals

- The residuals are the in-sample fitted errors.
- The difference between the realized value and the in-sample forecast.
  \[
  \hat{e}_t = y_{t+h} - \hat{y}_{t+h} = y_{t+h} - b_0
  \]
- In general, it is useful to plot the residuals against time, to see if any time series pattern remains.
Calculate and Plot Residuals

```
predict e, residuals
(12 missing values generated)

tslines e
```