Volatility

• Many economic series, and most financial series, display conditional volatility
  – The conditional variance changes over time
  – There are periods of high volatility
    • When large changes frequently occur
  – And periods of low volatility
    • When large changes are less frequent
Weekly Stock Prices
Levels and Returns

Graph 1: Weekly Stock Prices (sp) over time (t) from 1950w1 to 2010w1.

Graph 2: Weekly Returns (r) over time (t) from 1950w1 to 2010w1.
Conditional Mean

• The conditional mean of $y$ is

$$E(y_t \mid \Omega_{t-1})$$

• The regression error is mean zero and unforecastable

$$E(e_t \mid \Omega_{t-1}) = 0$$
Conditional Variance

• The conditional variance of $y$ is

$$\text{var}(y_t \mid \Omega_{t-1}) = E\left( (y_t - E(y_t \mid \Omega_{t-1}))^2 \mid \Omega_{t-1} \right)$$

$$= E\left( e_t^2 \mid \Omega_{t-1} \right)$$

• The squared regression error can be forecastable
Forecastable Conditional Variance

• If the squared error is forecastable, then the conditional variance is time-varying and correlated.
  – The magnitude of changes is predictable
  – The sign is not predictable
Stock returns are unpredictable

```
. reg r L(1/4).r r

Linear regression

Number of obs = 3120
F( 4, 3115) = 1.22
Prob > F = 0.3009
R-squared = 0.0033
Root MSE = 2.0793

Coef. Std. Err. t P>|t| [95% Conf. Interval]
_cons  .1337402 .0410395 3.26 0.001 .053273 .2142074
L1.  -.01737 .032557 -0.53 0.594 -.0812053 .0464653
L2.  .0466548 .0273551 1.71 0.088 -.0069811 .1002907
L3.  -.0044898 .0282283 -0.16 0.874 -.0598378 .0508581
L4.  -.0298138 .0264335 -1.13 0.259 -.0816427 .0220151

. testparm L(1/4).r

( 1)  L. r = 0
( 2)  L2. r = 0
( 3)  L3. r = 0
( 4)  L4. r = 0

F( 4, 3115) = 1.22
Prob > F = 0.3009
```
Squared Returns are predictable

```
. gen y=(r-.1334364)^2
(1 missing value generated)

. reg y L(1/4).y, r

Linear regression

|          | Coef.  | Std. Err. | t     | P>|t|     | [95% Conf. Interval]         |
|----------|--------|-----------|-------|---------|-----------------------------|
| y        |        |           |       |         |                             |
| L1.      | .2332184 | .1248813  | 1.87  | 0.062   | -.0116395                  | .4780763                          |
| L2.      | .0627729 | .0308486  | 2.03  | 0.042   | .0022873                   | .1232586                          |
| L3.      | .125441  | .0364307  | 3.44  | 0.001   | .0540103                   | .1968717                          |
| L4.      | .0517234 | .0506949  | 1.02  | 0.308   | -.0476755                 | .1511222                          |
| _cons    | 2.282243 | .3480455  | 6.56  | 0.000   | 1.599821                   | 2.964665                          |

. testparm L(1/4).y

( 1)  L. y = 0
( 2)  L2. y = 0
( 3)  L3. y = 0
( 4)  L4. y = 0

F(  4,  3115) =  9.72
Prob > F =  0.0000
```

Number of obs = 3120
F(  4,  3115) =  9.72
Prob > F =  0.0000
R-squared = 0.1092
Root MSE = 11.618
Squared Returns

Bartlett's formula for MA(q) 95% confidence bands
ARCH

• Robert Engle (1982) proposed a model for the conditional variance
  – AutoRegressive Conditional Heteroskedasticity
  – “ARCH” now describes volatility models

• Nobel Prize 2003
ARCH(1) Model

\[ y_t = \mu + e_t \]

\[ \sigma_t^2 = \text{var}(e_t | \Omega_{t-1}) = \omega + \alpha e_{t-1}^2 \]

\[ \omega > 0 \]

\[ \alpha \geq 0 \]

- \(\alpha > 0\) means that the conditional variance is high when the lagged squared error is high.
- Large errors (either sign) today mean high expected errors (in magnitude) tomorrow.
- Small magnitude errors forecast next period small magnitude errors.
Unconditional variance

• A property of expectations is that expected (average) conditional expectations are unconditional expectations.

• So the average conditional variance is the average variance – the variance of the regression error.

\[ \sigma^2 = E\left(\sigma_t^2\right) = \omega + \alpha E\left(e_{t-1}^2\right) = \omega + \alpha \sigma^2 \]

• Solving for the variance:

\[ \sigma^2 = \frac{\omega}{1 - \alpha} \]
• Rewriting, this implies
\[ \omega = \sigma^2 (1 - \alpha) \]
• Substituting into ARCH(1) equation
\[ \sigma_t^2 = (1 - \alpha) \sigma^2 + \alpha e_{t-1}^2 \]
or
\[ \sigma_t^2 = \sigma^2 + \alpha (e_{t-1}^2 - \sigma^2) \]
• This shows that the conditional variance is a combination of the unconditional variance, and the deviation of the squared error from its average value.
ARCH(1) as AR(1) in squares

• The model

\[ \text{var}(e_t \mid \Omega_{t-1}) = E(e_t^2 \mid \Omega_{t-1}) = \omega + \alpha e_{t-1}^2 \]

implies the regression

\[ e_t^2 = \omega + \alpha e_{t-1}^2 + u_t \]

where \( u \) is white noise

• Thus e-squared is an AR(1)
Estimation

- `.arch r, arch(1)`

ARCH family regression

Sample: 1950w2 - 2010w5
Distribution: Gaussian
Log likelihood = -6525.268

|        | Coef.    | Std. Err. | z    | P>| z|   | [ 95% Conf. Interval ] |
|--------|----------|-----------|------|-------|----------------------------|
| r      |          |           |      |       |                            |
| _cons  | .1996426 | .0314495  | 6.35 | 0.000 | .1380027 - .2612825       |
| ARCH   |          |           |      |       |                            |
| arch   | .3006982 | .0216209  | 13.91| 0.000 | .2583219 - .3430745       |
| L1.    |          |           |      |       |                            |
| _cons  | 2.926873 | .0686021  | 42.66| 0.000 | 2.792416 - 3.061331       |
Variance Forecast

• Given the parameter estimates, the estimated conditional variance for period $t$ is

\[ \hat{\sigma}_t^2 = \hat{\omega} + \hat{\alpha}\hat{e}_{t-1}^2 = \hat{\omega} + \hat{\alpha}(y_{t-1} - \hat{\mu})^2 \]

• The forecasted out-of-sample variance is

\[ \hat{\sigma}_{n+1}^2 = \hat{\omega} + \hat{\alpha}(y_n - \hat{\mu})^2 \]
Forecast Interval for the mean

- You can use the estimated conditional standard deviation to obtain forecast intervals for the mean
  \[ \hat{y}_{n+1|n} \pm Z_{\alpha/2} \hat{\sigma}_{n+1} \]

- These forecast intervals will vary in width depending on the estimated conditional variance.
  - Wider in periods of high volatility
  - More narrow in periods of low volatility
ARCH(p) model

• Allow $p$ lags of squared errors

\[
y_t = \mu + e_t
\]

\[
\sigma_t^2 = \omega + \alpha_1 e_{t-1}^2 + \alpha_2 e_{t-2}^2 + \cdots + \alpha_p e_{t-p}^2
\]

• Similar to AR(p) in squares

• Estimation: ARCH(8)
  – .arch r, arch(1/8)
  – ARCH model with lags 1 through 8
ARCH(8) Estimates

- `.arch r, arch(1/8)`

ARCH family regression

Sample: 1950w2 - 2010w5
Distribution: Gaussian
Log likelihood = -6368.552

|       | Coef.    | Std. Err. | z     | P>|z|   | [ 95% Conf. Interval ] |
|-------|----------|-----------|-------|-------|-----------------------|
| r     |          |           |       |       |                       |
| _cons | .2027503 | .0290226  | 6.99  | 0.000 | .1458671 .2596335     |
| ARCH  |          |           |       |       |                       |
| arch  |          |           |       |       |                       |
| L1.   | .1867283 | .0163376  | 11.43 | 0.000 | .1547071 .2187495     |
| L2.   | .1099957 | .0203355  | 5.41  | 0.000 | .0701388 .1498526     |
| L3.   | .1541191 | .0225999  | 6.82  | 0.000 | .1098241 .1984142     |
| L4.   | .0912413 | .0192753  | 4.73  | 0.000 | .0534625 .1290202     |
| L5.   | .0284588 | .0171987  | 1.65  | 0.098 | -.0052501 .0621677    |
| L6.   | .0811242 | .0192012  | 4.22  | 0.000 | .0434906 .1187578     |
| L7.   | .041083  | .0147083  | 2.79  | 0.005 | .0122553 .0699107     |
| L8.   | .0706622 | .0171741  | 4.11  | 0.000 | .0370015 .1043229     |
| _cons | 1.144063 | 1.1001062 | 11.43 | 0.000 | .9478582 1.340267      |
ARCH needs many lags

• Notice that we included 8 lags, and all appeared significant.

• This is commonly observed in estimated ARCH models
  – The conditional variance appears to be a function of many lagged past squares
GARCH Model

- Tim Bollerslev (1986)
  - A student of Engle
  - Current faculty at Duke

proposed the GARCH model to simplify this problem

\[
\sigma_t^2 = \omega + \beta \sigma_{t-1}^2 + \alpha e_{t-1}^2
\]

- \( \beta > 0 \)
- \( \omega > 0 \)
- \( \alpha \geq 0 \)
GARCH(1,1)

• This makes the variance a function of all past lags:
  \[ \sigma_t^2 = \omega + \beta \sigma_{t-1}^2 + \alpha e_{t-1}^2 \]
  \[ = \sum_{j=0}^{\infty} \beta^j \left( \omega + \alpha e_{t-1-j}^2 \right) \]

• It is also smoother than an ARCH model with a small number of lags
GARCH($p,q$)

- $p$ lags of squared error
- $q$ lags of conditional variance

\[ \sigma_t^2 = \omega + \beta_1 \sigma_{t-1}^2 + \cdots + \beta_q \sigma_{t-q}^2 + \alpha_1 e_{t-1}^2 + \cdots + \alpha_p e_{t-p}^2 \]

- GARCH(1,1):
  - `.arch r, arch(1) garch(1)`
- GARCH(3,2):
  - `.arch r, arch(1/3) garch(1/2)`
GARCH(1,1)

ARCH family regression

Sample: 1950w2 - 2010w5
Distribution: Gaussian
Log likelihood = -6359.118

Number of obs = 3124

| r       | Coef.  | Std. Err. | z     | P>|z|  | [95% Conf. Interval] |
|---------|--------|-----------|-------|------|----------------------|
| _cons  | 0.1935287 | 0.0296365 | 6.53  | 0.000 | 0.1354421 - 0.2516152 |
| r      | 0.1317621 | 0.0094385 | 13.96 | 0.000 | 0.113263 - 0.1502613 |
| arch   | 0.8444868 | 0.0117076 | 72.13 | 0.000 | 0.8215404 - 0.8674333 |
| L1.    | 0.1207574 | 0.0221764 | 5.45  | 0.000 | 0.0772924 - 0.1642223 |

Common GARCH features

- Lagged variance has large coefficient
- Sum of two coefficients very close to (but less than) one
### GARCH(2,2) for Stock Returns

**ARCH family regression**

|       | Coef.   | Std. Err. | z      | P>|z|   | [95% Conf. Interval] |
|-------|---------|-----------|--------|-------|---------------------|
| r     | _cons   | 0.1913999 | 0.0295859 | 6.47  | 0.000              | 0.1334126 to 0.2493872 |

**ARCH**

|       | Coef.     | Std. Err. | z      | P>|z|   | [95% Conf. Interval] |
|-------|-----------|-----------|--------|-------|---------------------|
| arch  | L1.      | 0.1658834 | 0.0149416 | 11.10 | 0.000          | 0.1365984 to 0.1951684 |
|       | L2.      | -0.0277704 | 0.042739 | -0.65 | 0.516             | -0.1115373 to 0.0559964 |
| garch | L1.      | 0.5991681 | 0.2941188 | 2.04  | 0.042             | 0.227059 to 1.17563     |
|       | L2.      | 0.2373325 | 0.2478461 | 0.96  | 0.338             | -0.248437 to 0.7231021  |
|       | _cons    | 0.1233949 | 0.0428309 | 2.88  | 0.004             | 0.0394478 to 0.207342   |

**Distribution:** Gaussian

Log likelihood = -6356.166

Number of obs  = 3124

Wald chi2(7)  = .

Prob > chi2   = .

Sample: 1950w2 - 2010w5

Number of obs  = 3124
GARCH(1,1)

- The GARCH(1,1) often fits well, and is a useful benchmark.
  - Daily, weekly, or monthly asset returns, exchange rates, or interest rates
Extensions

• There are many extensions of the basic GARCH model, developed to handle a variety of situations
  – Asymmetric Response
  – Garch-in-mean
  – Explanatory variables in variance
  – Non-normal errors
Asymmetric GARCH

• Threshold GARCH

$$\sigma_t^2 = \omega + \beta \sigma_{t-1}^2 + \alpha e_{t-1}^2 + \gamma e_{t-1}^2 1(e_{t-1} > 0)$$

• The last term is dummy variable for positive lagged errors

• This model specifies that the ARCH effect depends on whether the error was positive or negative
  – If the error is negative, the effect is $\alpha$
  – If the error is positive, the full effect is $\alpha + \gamma$
TARCH estimation

- `.arch r, arch(1) tarch(1) garch(1)`
- Negative errors have coefficient of 0.19
- Positive errors have coefficient of 0.05
- Negative returns increase volatility much more than positive returns

**ARCH family regression**

|          | Coef.  | Std. Err. | z      | P>|z|  | [95% Conf. Interval] |
|----------|--------|-----------|--------|------|---------------------|
| r        |        |           |        |      |                     |
| _cons    | 0.1474826 | 0.0313132 | 4.71   | 0.000 | [0.0861099, 0.2088552] |
| ARCH     |        |           |        |      |                     |
| arch L1. | 0.1879679 | 0.0154078 | 12.20  | 0.000 | [0.1577692, 0.2181665] |
| tarch L1.| -0.1408097 | 0.0160892 | -8.75  | 0.000 | [-0.1723439, -0.1092754] |
| garch L1.| 0.8437111 | 0.0132294 | 63.78  | 0.000 | [0.817782, 0.8696403] |
| _cons    | 0.1540714 | 0.0219836 | 7.01   | 0.000 | [0.1109843, 0.1971585] |
Leverage Effect

• This model describes what is called the “leverage effect”
  – A negative shock to equity increases the ratio debt/equity of investors
  – This increases the leverage of their portfolios
  – This increases risk, and the conditional variance
  – Negative shocks have stronger effect on variance than positive shocks
GARCH-in-mean

• If investors are risk averse, risky assets will earn higher returns (a risk premium) in market equilibrium

• If assets have varying volatility (risk), their expected return will vary with this volatility
  – Expected return should be positively correlated with volatility
GARCH-M model

\[ y = \beta_1 + \beta_1 \sigma_{t-1}^2 + e_t \]

\[ \sigma_t^2 = \omega + \beta \sigma_{t-1}^2 + \alpha e_{t-1}^2 \]

• .arch arch(1) garch(1) archm
GARCH-M for Stock Returns

- Marginally positive effect

**ARCH family regression**

Sample: 1950w2 - 2010w5  
Distribution: Gaussian  
Log likelihood = -6357.259  
Number of obs = 3124  
Wald chi2(1) = 2.96  
Prob > chi2 = 0.0853

|       | Coef.    | Std. Err. | z     | P>|z| | [95% Conf. Interval] |
|-------|----------|-----------|-------|------|----------------------|
| r     | .1211625 | .0512489  | 2.36  | 0.018| .0207165 -.2216085  |
| _cons |          |           |       |      |                      |
| ARCHM |          |           |       |      |                      |
| sigma2| .024739  | .0143783  | 1.72  | 0.085| -.003442 .0529199   |
| ARCH  |          |           |       |      |                      |
| arch  | .1315334 | .0096454  | 13.64 | 0.000| .1126287 .1504381   |
| L1.   |          |           |       |      |                      |
| garch | .8450762 | .0118319  | 71.42 | 0.000| .8218862 .8682662   |
| L1.   |          |           |       |      |                      |
| _cons | .1193442 | .022376   | 5.33  | 0.000| .075488 .1632004    |
**TARCH and GARCH-M**

- `.arch arch(1) tarch(1) garch(1) archm`
- archm term appears insignificant

---

**ARCH family regression**

|     | Coef.    | Std. Err. | z    | P>|z| | [95% Conf. Interval] |
|-----|----------|-----------|------|------|----------------------|
| _r_ | 0.1309694| 0.0512904 | 2.55 | 0.011| 0.030442 - 0.2314967 |
| _cons|          |           |      |      |                      |
| ARCH |          |           |      |      |                      |
| _arch_ | 0.1872044| 0.0155283 | 12.06| 0.000| 0.1567694 - 0.2176393 |
| L1.  |          |           |      |      |                      |
| _tarch_ | -.1391533| 0.0161185 | -8.63| 0.000| -.1707449 - -.1075617 |
| L1.  |          |           |      |      |                      |
| _garch_ | .8425617 | 0.0137719 | 61.18| 0.000| .8155693 - .8695542  |
| L1.  |          |           |      |      |                      |
| _cons| .1565728 | 0.0228305 | 6.86 | 0.000| 0.1118259 - 0.2013197 |

**OPG**

Log likelihood = -6332.324                         Prob > chi2 = 0.6776
Distribution: Gaussian                             Wald chi2(1) = 0.17
Sample: 1950w2 - 2010w5                            Number of obs = 3124
Log likelihood = -6332.324                         Prob > chi2 = 0.6776
Number of obs = 3124

**Distribution:** Gaussian

**Sample:** 1950w2 - 2010w5
Estimated standard deviation

• Estimated TARCH model
• `.predict v, variance`
• `.gen s=sqrt(v)`
• Unconditional variance is 2.1
S&P, returns, and standard deviation
2006-2010