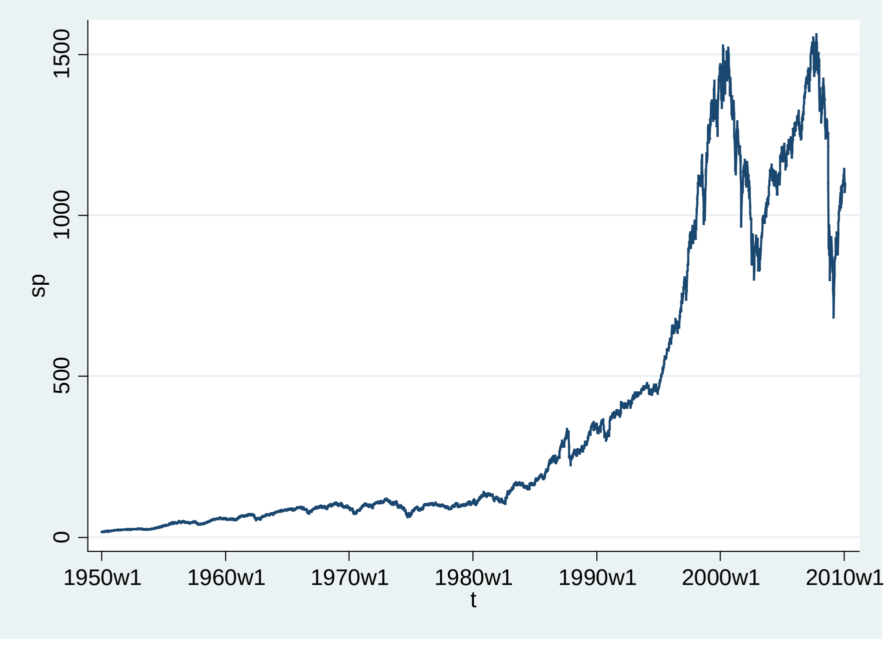
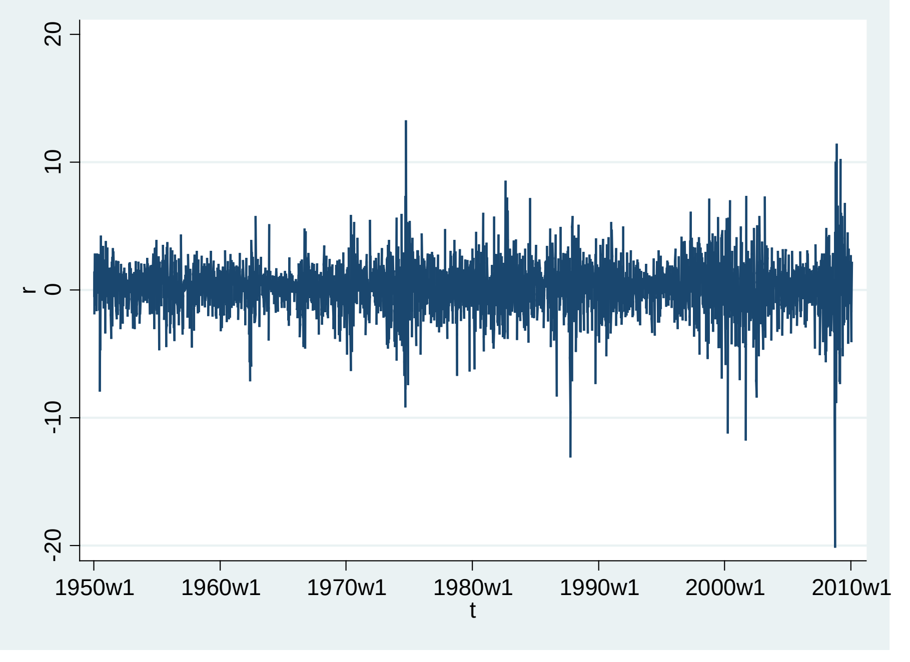
# Volatility

- Many economic series, and most financial series, display conditional volatility
  - The conditional variance changes over time
  - There are periods of high volatility
    - When large changes frequently occur
  - And periods of low volatility
    - When large changes are less frequent

# Weekly Stock Prices
# Levels and Returns



# Conditional Mean

- The conditional mean of y is

$$E\left(y_t \mid \Omega_{t-1}\right)$$

- The regression error is mean zero and unforecastable

$$E\left(e_t \mid \Omega_{t-1}\right) = 0$$

# Conditional Variance

- The conditional variance of y is

$$\operatorname{var}(y_t \mid \Omega_{t-1}) = E\left(\left(y_t - E(y_t \mid \Omega_{t-1})\right)^2 \mid \Omega_{t-1}\right)$$
$$= E\left(e_t^2 \mid \Omega_{t-1}\right)$$

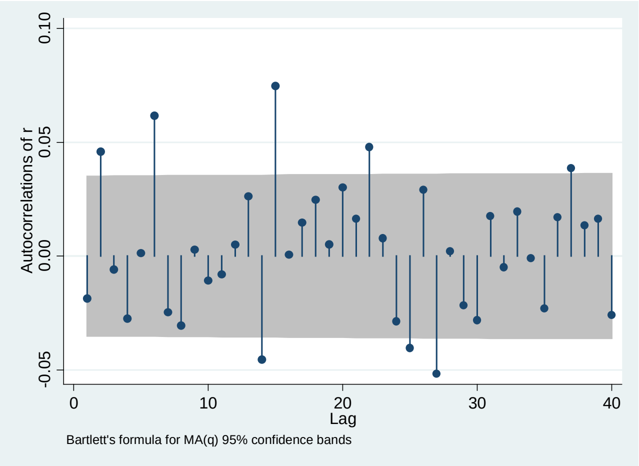
- The squared regression error can be forecastable

# Forecastable Conditional Variance

- If the squared error is forecastable, then the conditional variance is time-varying and correlated.
  - The magnitude of changes is predictable
  - The sign is not predictable

# Stock returns are unpredictable

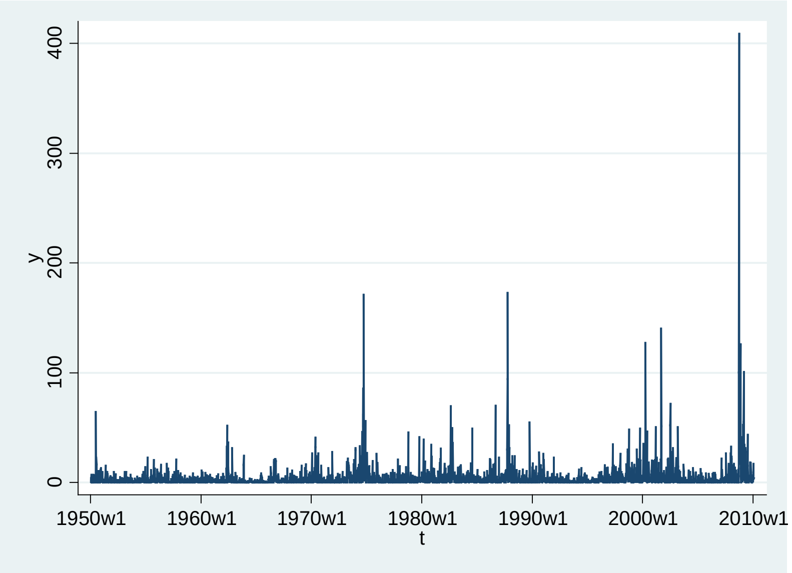
```
. reg r L(1/4).r,r

Linear regression                               Number of obs =      3120
                                                F(  4,  3115) =      1.22
                                                Prob > F      =    0.3009
                                                R-squared     =    0.0033
                                                Root MSE      =    2.0793

------------------------------------------------------------------------------
             |               Robust
           r |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           r |
         L1. |    -.01737    .032557    -0.53   0.594    -.0812053    .0464653
         L2. |   .0466548   .0273551     1.71   0.088    -.0069811    .1002907
         L3. |  -.0044898   .0282283    -0.16   0.874    -.0598378    .0508581
         L4. |  -.0298138   .0264335    -1.13   0.259    -.0816427    .0220151
             |
       _cons |   .1337402   .0410395     3.26   0.001      .053273    .2142074
------------------------------------------------------------------------------

. testparm L(1/4).r

 ( 1)  L.r = 0
 ( 2)  L2.r = 0
 ( 3)  L3.r = 0
 ( 4)  L4.r = 0

       F(  4,  3115) =    1.22
            Prob > F =    0.3009
```

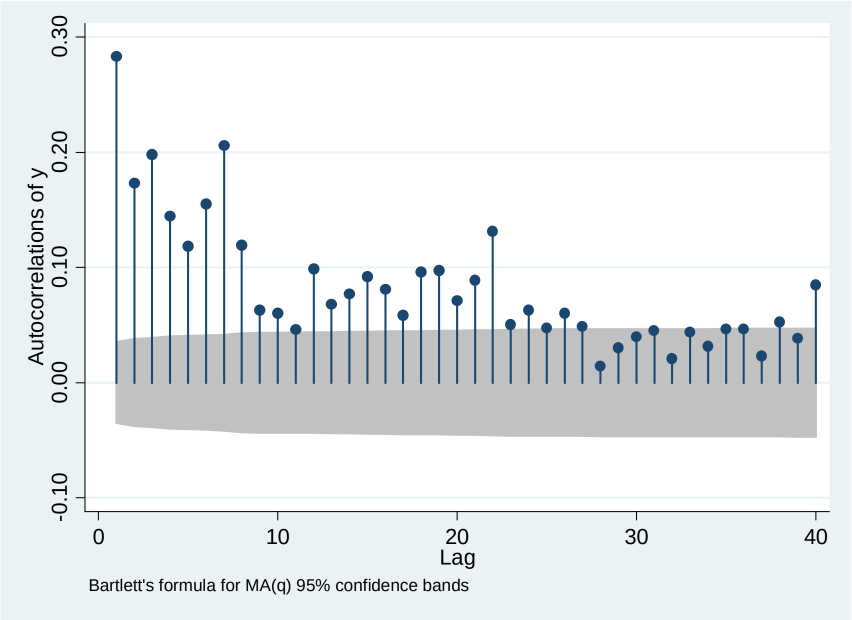


# Squared Returns are predictable

```
. gen y=(r-.1334364)^2
(1 missing value generated)

. reg y L(1/4).y,r

Linear regression                                      Number of obs =    3120
                                                       F(  4,  3115) =    9.72
                                                       Prob > F      =  0.0000
                                                       R-squared     =  0.1092
                                                       Root MSE      =  11.618

------------------------------------------------------------------------------
             |               Robust
           y |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           y |
         L1. |   .2332184   .1248813     1.87   0.062    -.0116395    .4780763
         L2. |   .0627729   .0308486     2.03   0.042     .0022873    .1232586
         L3. |    .125441   .0364307     3.44   0.001     .0540103    .1968717
         L4. |   .0517234   .0506949     1.02   0.308    -.0476755    .1511222
             |
       _cons |   2.282243   .3480455     6.56   0.000     1.599821    2.964665
------------------------------------------------------------------------------

. testparm L(1/4).y

 ( 1)  L.y = 0
 ( 2)  L2.y = 0
 ( 3)  L3.y = 0
 ( 4)  L4.y = 0

       F(  4,  3115) =    9.72
            Prob > F =    0.0000
```

# Squared Returns



# ARCH

- Robert Engle (1982) proposed a model for the conditional variance
  – AutoRegressive Conditional Heteroskedasticity
  – “ARCH” now describes volatility models
- Nobel Prize 2003

# ARCH(1) Model

$$y_t = \mu + e_t$$

$$\sigma_t^2 = \mathrm{var}\left(e_t \mid \Omega_{t-1}\right) = \omega + \alpha e_{t-1}^2$$

$$\omega > 0$$

$$\alpha \geq 0$$

- $\alpha > 0$ means that the conditional variance is high when the lagged squared error is high
- Large errors (either sign) today mean high expected errors (in magnitude) tomorrow.
- Small magnitude errors forecast next period small magnitude errors.

# Unconditional variance

- A property of expectations is that expected (average) conditional expectations are unconditional expectations.
- So the average conditional variance is the average variance – the variance of the regression error.

$$\sigma^2 = E\left(\sigma_t^2\right) = \omega + \alpha E\left(e_{t-1}^2\right) = \omega + \alpha\sigma^2$$

- Solving for the variance: $\sigma^2 = \dfrac{\omega}{1-\alpha}$

- Rewriting, this implies

$$\omega = \sigma^2 (1 - \alpha)$$

- Substituting into ARCH(1) equation

$$\sigma_t^2 = (1 - \alpha)\sigma^2 + \alpha e_{t-1}^2$$

or

$$\sigma_t^2 = \sigma^2 + \alpha\left(e_{t-1}^2 - \sigma^2\right)$$

- This shows that the conditional variance is a combination of the unconditional variance, and the deviation of the squared error from its average value.

# ARCH(1) as AR(1) in squares

- The model

$$\mathrm{var}\left(e_t \mid \Omega_{t-1}\right) = E\left(e_t^2 \mid \Omega_{t-1}\right) = \omega + \alpha e_{t-1}^2$$

implies the regression

$$e_t^2 = \omega + \alpha e_{t-1}^2 + u_t$$

where u is white noise

- Thus e-squared is an AR(1)

# Estimation

- **.arch r, arch(1)**

```
ARCH family regression

Sample: 1950w2 - 2010w5                         Number of obs   =       3124
Distribution: Gaussian                          Wald chi2(.)    =          .
Log likelihood = -6525.268                      Prob > chi2     =          .
```

| r | Coef. | OPG Std. Err. | z | P>\|z\| | [95% Conf. | Interval] |
|---|---|---|---|---|---|---|
| r | | | | | | |
| _cons | .1996426 | .0314495 | 6.35 | 0.000 | .1380027 | .2612825 |
| ARCH | | | | | | |
| arch L1. | .3006982 | .0216209 | 13.91 | 0.000 | .2583219 | .3430745 |
| _cons | 2.926873 | .0686021 | 42.66 | 0.000 | 2.792416 | 3.061331 |

# Variance Forecast

- Given the parameter estimates, the estimated conditional variance for period t is

$$\hat{\sigma}_t^2 = \hat{\omega} + \hat{\alpha}\hat{e}_{t-1}^2 = \hat{\omega} + \hat{\alpha}\left(y_{t-1} - \hat{\mu}\right)^2$$

- The forecasted out-of-sample variance is

$$\hat{\sigma}_{n+1}^2 = \hat{\omega} + \hat{\alpha}\left(y_n - \hat{\mu}\right)^2$$

# Forecast Interval for the mean

- You can use the estimated conditional standard deviation to obtain forecast intervals for the mean

$$\hat{y}_{n+1|n} \pm Z_{\alpha/2}\hat{\sigma}_{n+1}$$

- These forecast intervals will vary in width depending on the estimated conditional variance.
  – Wider in periods of high volatility
  – More narrow in periods of low volatility

# ARCH(p) model

- Allow p lags of squared errors

$$y_t = \mu + e_t$$
$$\sigma_t^2 = \omega + \alpha_1 e_{t-1}^2 + \alpha_2 e_{t-2}^2 + \cdots + \alpha_p e_{t-p}^2$$

- Similar to AR(p) in squares
- Estimation: ARCH(8)
  - **.arch r, arch(1/8)**
  - ARCH model with lags 1 through 8

# ARCH(8) Estimates

- **.arch r, arch(1/8)**

```
ARCH family regression

Sample: 1950w2 - 2010w5                         Number of obs   =      3124
Distribution: Gaussian                          Wald chi2(.)    =         .
Log likelihood = -6368.552                      Prob > chi2     =         .
```

| r | Coef. | OPG Std. Err. | z | P>\|z\| | [95% Conf. | Interval] |
|---|---|---|---|---|---|---|
| r | | | | | | |
| _cons | .2027503 | .0290226 | 6.99 | 0.000 | .1458671 | .2596335 |
| ARCH | | | | | | |
| arch | | | | | | |
| L1. | .1867283 | .0163376 | 11.43 | 0.000 | .1547071 | .2187495 |
| L2. | .1099957 | .0203355 | 5.41 | 0.000 | .0701388 | .1498526 |
| L3. | .1541191 | .0225999 | 6.82 | 0.000 | .1098241 | .1984142 |
| L4. | .0912413 | .0192753 | 4.73 | 0.000 | .0534625 | .1290202 |
| L5. | .0284588 | .0171987 | 1.65 | 0.098 | -.0052501 | .0621677 |
| L6. | .0811242 | .0192012 | 4.22 | 0.000 | .0434906 | .1187578 |
| L7. | .041083 | .0147083 | 2.79 | 0.005 | .0122553 | .0699107 |
| L8. | .0706622 | .0171741 | 4.11 | 0.000 | .0370015 | .1043229 |
| _cons | 1.144063 | .1001062 | 11.43 | 0.000 | .9478582 | 1.340267 |

# ARCH needs many lags

- Notice that we included 8 lags, and all appeared significant.
- This is commonly observed in estimated ARCH models
  - The conditional variance appears to be a function of many lagged past squares

# GARCH Model

- Tim Bollerslev (1986)
  - A student of Engle
  - Current faculty at Duke

proposed the GARCH model to simplify this problem

$$\sigma_t^2 = \omega + \beta\sigma_{t-1}^2 + \alpha e_{t-1}^2$$

$$\beta > 0$$

$$\omega > 0$$

$$\alpha \geq 0$$

# GARCH(1,1)

- This makes the variance a function of all past lags:

$$\sigma_t^2 = \omega + \beta\sigma_{t-1}^2 + \alpha e_{t-1}^2$$

$$= \sum_{j=0}^{\infty} \beta^j \left(\omega + \alpha e_{t-1-j}^2\right)$$

- It is also smoother than an ARCH model with a small number of lags

# GARCH(p,q)

- p lags of squared error
- q lags of conditional variance

$$\sigma_t^2 = \omega + \beta_1 \sigma_{t-1}^2 + \cdots + \beta_q \sigma_{t-q}^2 + \alpha_1 e_{t-1}^2 + \cdots + \alpha_p e_{t-p}^2$$

- GARCH(1,1):
  - **.arch r, arch(1) garch(1)**
- GARCH(3,2):
  - **.arch r, arch(1/3) garch(1/2)**

# GARCH(1,1)

```
ARCH family regression

Sample: 1950w2 - 2010w5                           Number of obs   =      3124
Distribution: Gaussian                            Wald chi2(.)    =         .
Log likelihood = -6359.118                        Prob > chi2     =         .
```

| r | Coef. | OPG Std. Err. | z | P>\|z\| | [95% Conf. | Interval] |
|---|---|---|---|---|---|---|
| **r** | | | | | | |
| _cons | .1935287 | .0296365 | 6.53 | 0.000 | .1354421 | .2516152 |
| **ARCH** | | | | | | |
| arch L1. | .1317621 | .0094385 | 13.96 | 0.000 | .113263 | .1502613 |
| garch L1. | .8444868 | .0117076 | 72.13 | 0.000 | .8215404 | .8674333 |
| _cons | .1207574 | .0221764 | 5.45 | 0.000 | .0772924 | .1642223 |

- Common GARCH features
  - Lagged variance has large coefficient
  - Sum of two coefficients very close to (but less than) one

# GARCH(2,2) for Stock Returns

```
ARCH family regression

Sample: 1950w2 - 2010w5                        Number of obs   =      3124
Distribution: Gaussian                         Wald chi2(.)    =         .
Log likelihood = -6356.166                     Prob > chi2     =         .
```

| r | Coef. | OPG Std. Err. | z | P>\|z\| | [95% Conf. | Interval] |
|---|---|---|---|---|---|---|
| **r** | | | | | | |
| _cons | .1913999 | .0295859 | 6.47 | 0.000 | .1334126 | .2493872 |
| **ARCH** | | | | | | |
| arch | | | | | | |
| L1. | .1658834 | .0149416 | 11.10 | 0.000 | .1365984 | .1951684 |
| L2. | -.0277704 | .042739 | -0.65 | 0.516 | -.1115373 | .0559964 |
| garch | | | | | | |
| L1. | .5991681 | .2941188 | 2.04 | 0.042 | .0227059 | 1.17563 |
| L2. | .2373325 | .2478461 | 0.96 | 0.338 | -.248437 | .7231021 |
| _cons | .1233949 | .0428309 | 2.88 | 0.004 | .0394478 | .207342 |

# GARCH(1,1)

- The GARCH(1,1) often fits well, and is a useful benchmark.
  - Daily, weekly, or monthly asset returns, exchange rates, or interest rates

# Extensions

- There are many extensions of the basic GARCH model, developed to handle a variety of situations
  - Asymmetric Response
  - Garch-in-mean
  - Explanatory variables in variance
  - Non-normal errors

# Asymetric GARCH

- Threshold GARCH

$$\sigma_t^2 = \omega + \beta\sigma_{t-1}^2 + \alpha e_{t-1}^2 + \gamma e_{t-1}^2 1(e_{t-1} > 0)$$

- The last tern is dummy variable for positive lagged errors
- This model specifies that the ARCH effect depends on whether the error was positive or negative
  - If the error is negative, the effect is $\alpha$
  - If the error is positive, the full effect is $\alpha+\gamma$

# TARCH estimation

- **.arch r, arch(1) tarch(1) garch(1)**
- Negative errors have coefficient of 0.19
- Positive errors have coefficient of 0.05
- Negative returns increase volatility much more than positive returns

```
ARCH family regression

Sample: 1950w2 - 2010w5                         Number of obs   =      3124
Distribution: Gaussian                          Wald chi2(.)    =         .
Log likelihood = -6332.433                      Prob > chi2     =         .
```

| r | Coef. | OPG Std. Err. | z | P>\|z\| | [95% Conf. | Interval] |
|---|---|---|---|---|---|---|
| **r** | | | | | | |
| _cons | .1474826 | .0313132 | 4.71 | 0.000 | .0861099 | .2088552 |
| **ARCH** | | | | | | |
| arch L1. | .1879679 | .0154078 | 12.20 | 0.000 | .1577692 | .2181665 |
| tarch L1. | -.1408097 | .0160892 | -8.75 | 0.000 | -.1723439 | -.1092754 |
| garch L1. | .8437111 | .0132294 | 63.78 | 0.000 | .817782 | .8696403 |
| _cons | .1540714 | .0219836 | 7.01 | 0.000 | .1109843 | .1971585 |

# Leverage Effect

- This model describes what is called the "leverage effect"
  - A negative shock to equity increases the ratio debt/equity of investors
  - This increases the *leverage* of their portfolios
  - This increases risk, and the conditional variance
  - Negative shocks have stronger effect on variance than positive shocks

# GARCH-in-mean

- If investors are risk averse, risky assets will earn higher returns (a risk premium) in market equilibrium
- If assets have varying volatility (risk), their expected return will vary with this volatility
  - Expected return should be positively correlated with volatility

# GARCH-M model

$$y = \beta_1 + \beta_1 \sigma_{t-1}^2 + e_t$$

$$\sigma_t^2 = \omega + \beta \sigma_{t-1}^2 + \alpha e_{t-1}^2$$

- **.arch arch(1) garch(1) archm**

# GARCH-M for Stock Returns

- Marginally positive effect

```
ARCH family regression

Sample: 1950w2 - 2010w5                         Number of obs   =      3124
Distribution: Gaussian                          Wald chi2(1)    =      2.96
Log likelihood = -6357.259                      Prob > chi2     =    0.0853
```

| r | Coef. | OPG Std. Err. | z | P>\|z\| | [95% Conf. | Interval] |
|---|---|---|---|---|---|---|
| **r** | | | | | | |
| _cons | .1211625 | .0512489 | 2.36 | 0.018 | .0207165 | .2216085 |
| **ARCHM** | | | | | | |
| sigma2 | .024739 | .0143783 | 1.72 | 0.085 | -.003442 | .0529199 |
| **ARCH** | | | | | | |
| arch L1. | .1315334 | .0096454 | 13.64 | 0.000 | .1126287 | .1504381 |
| garch L1. | .8450762 | .0118319 | 71.42 | 0.000 | .8218862 | .8682662 |
| _cons | .1193442 | .022376 | 5.33 | 0.000 | .075488 | .1632004 |

# TARCH and GARCH-M

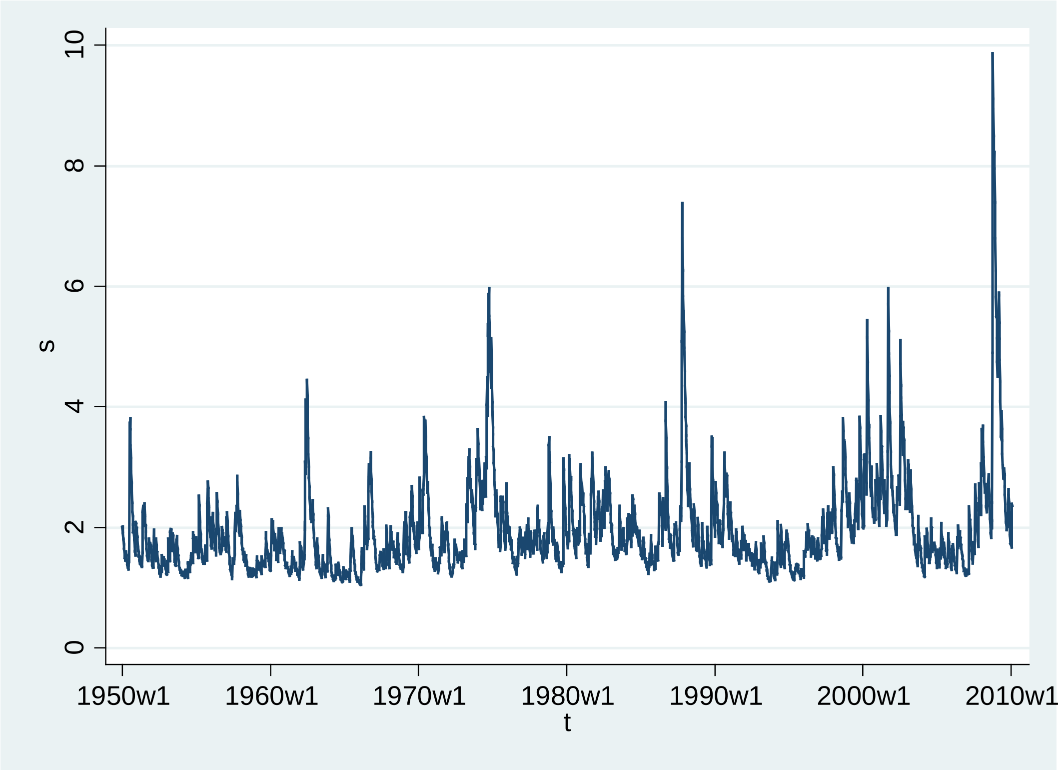
- **.arch arch(1) tarch(1) garch(1) archm**
- archm term appears insignificant

```
ARCH family regression

Sample: 1950w2 - 2010w5                         Number of obs   =      3124
Distribution: Gaussian                          Wald chi2(1)    =      0.17
Log likelihood = -6332.324                      Prob > chi2     =    0.6776
```

| r | Coef. | OPG Std. Err. | z | P>\|z\| | [95% Conf. | Interval] |
|---|---|---|---|---|---|---|
| **r** | | | | | | |
| _cons | .1309694 | .0512904 | 2.55 | 0.011 | .030442 | .2314967 |
| **ARCHM** | | | | | | |
| sigma2 | .0059058 | .0142042 | 0.42 | 0.678 | -.021934 | .0337456 |
| **ARCH** | | | | | | |
| arch L1. | .1872044 | .0155283 | 12.06 | 0.000 | .1567694 | .2176393 |
| tarch L1. | -.1391533 | .0161185 | -8.63 | 0.000 | -.1707449 | -.1075617 |
| garch L1. | .8425617 | .0137719 | 61.18 | 0.000 | .8155693 | .8695542 |
| _cons | .1565728 | .0228305 | 6.86 | 0.000 | .1118259 | .2013197 |

# Estimated standard deviation

- Estimated TARCH model
- **.predict  v,  variance**
- **.gen  s=sqrt(v)**
- Unconditional variance is 2.1



# S&P, returns, and standard deviation 2006-2010