Integration

- Orders of Integration Terminology
  - A series with a unit root (a random walk) is said to be **integrated of order one**, or ***I*(1)**
  - A stationary series without a trend is said to be **integrated of order 0**, or ***I*(0)**
  - An ***I*(1)** series is differenced once to be ***I*(0)**
  - In general, we say that a series is ***I*(d)** if its d'th difference is stationary.

Integrated of order d

- A series is I(d) if

$$(1-L)^d y_t = z_t$$

  is stationary and without trend.
- Examples
  - I(0): $y_t = z_t$
  - I(1): $(1-L)y_t = z_t$
  - I(2): $(1-L)^2 y_t = z_t$
- Possible I(2) series are price levels and money supply

Fractional Integration

- **Advanced side note!**
- We said a series is I(d) if

$$(1-L)^d y_t = z_t$$

- We did not require d to be an integer
- We say that y is **fractionally integrated** if 0<d<1 or -1<d<0
- A fractionally integrated series is in between I(0) and I(1)
- Strong dependence, slow autocorrelation decay
- Popular model for asset return volatility.

Fractional Differencing

- The fractional differencing operator is an infinite series

$$
\begin{aligned}
(1-B)^d &= \sum_{k=0}^{\infty} \binom{d}{k} (-B)^k \\
&= \sum_{k=0}^{\infty} \frac{\prod_{a=0}^{k-1}(d-a)\ (-B)^k}{k!} \\
&= 1 - dB + \frac{d(d-1)}{2} B^2 - \cdots .
\end{aligned}
$$

Co-Integration

- We say that two series are co-integrated if a linear combination has a lower level of integration
- If y and x are each I(1), yet $z=y-\theta x$ is I(0)
- Example: Term Structure
  - We saw before that T3 appears to have a unit root
  - But the spread T12-T3 was stationary
  - T3 and T12 are co-integrated!

Common Co-Integration Relations

- Interest Rates of different maturities
- Stock prices and dividends
  - (Campbell and Shiller)
- Aggregate consumption and income
  - (Campbell and Shiller)
- Aggregate output, consumption, and investment
  - King, Plosser, Stock and Watson

Cointegrating Equation

- We said that y and x are cointegrated if

$$z_t = y_t - \theta x_t$$

  is stationary
- This is called the cointegrating equation
- θ is the cointegrating coefficient
- In some cases, θ is known from theory
  - often θ=1

Great Ratios

- If the aggregate variables Y and X are proportional in the long run, then

$$Z_t = \frac{Y_t}{X_t}$$

  is stationary.
- Then $\log(Z_t) = \log(Y_t) - \log(X_t)$

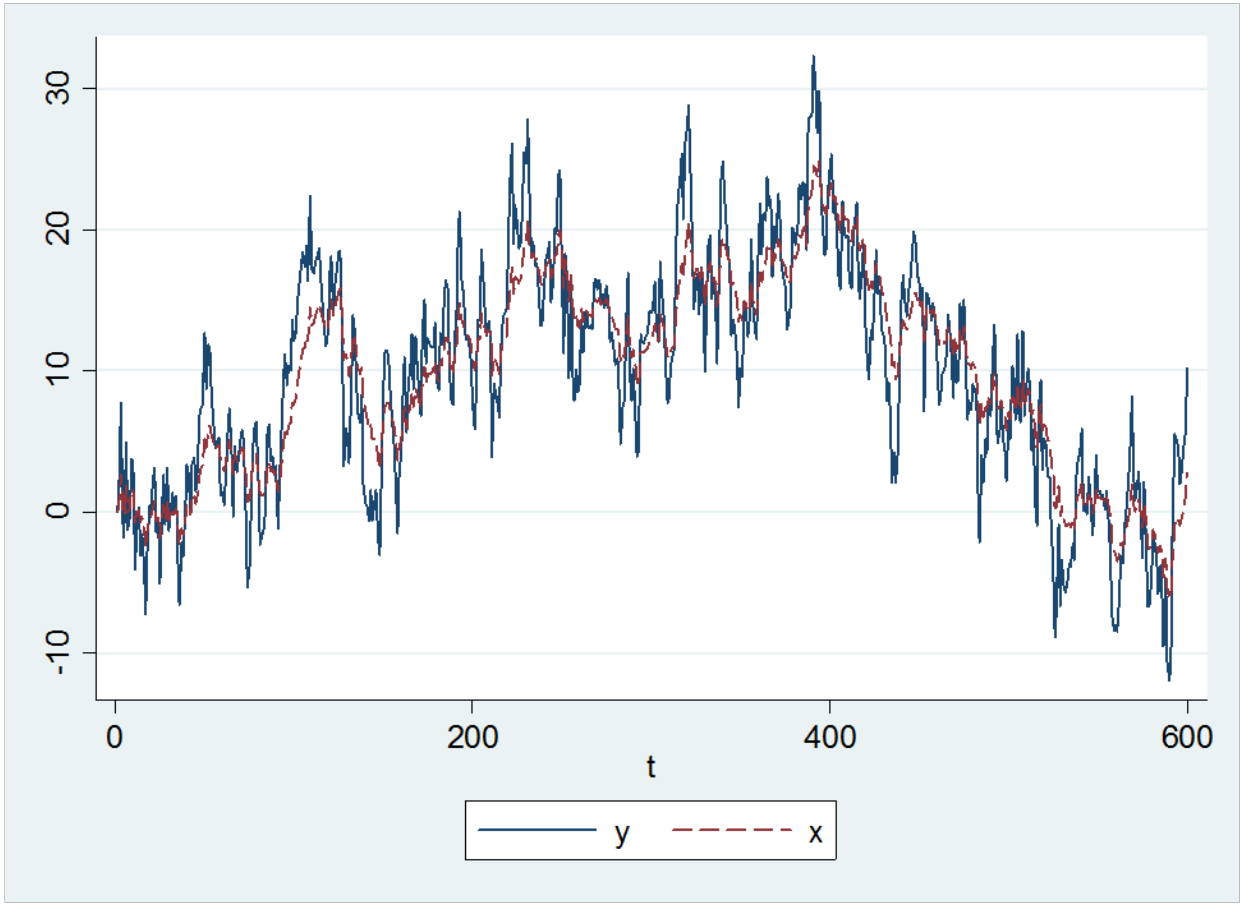
  and $z_t = y_t - x_t$

  where y=log(Y) and x=log(X)
- In this case, the logs y and x are cointegrated with coefficient 1.

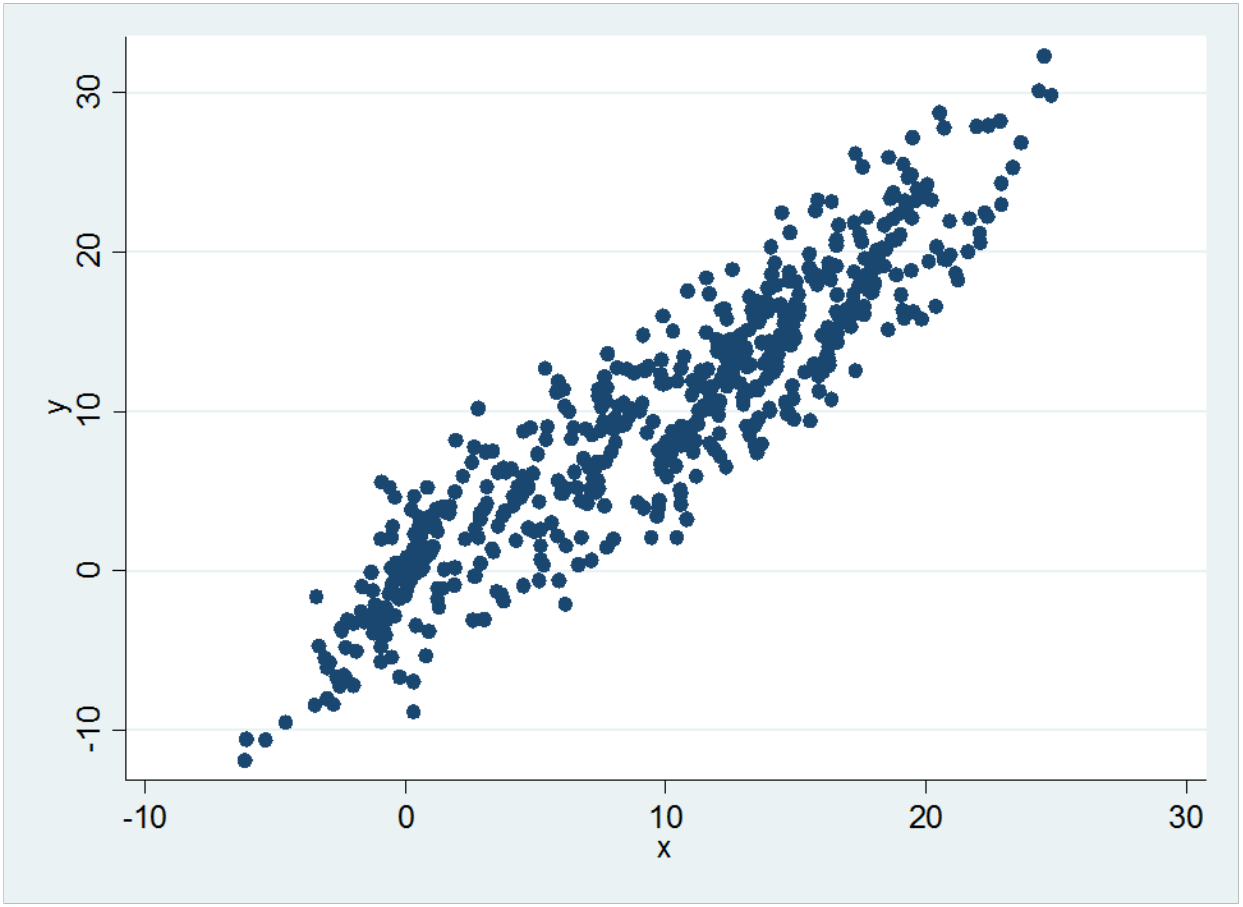
Equilibrium Error

- The difference $z_t = y_t - \theta x_t$
  is sometimes called the equilibrium error, as it measures the deviation of y and x from the long-term cointegrating relationship

Simulated Example



Scatter plot
Variables stay close to cointegration line



Granger Representation Theory

- If y and x are I(1) and cointegrated, then the optimal regression for y takes the form

$$\Delta y_t = \mu + \gamma z_{t-1} + \alpha_1 \Delta y_{t-1} + \cdots + \alpha_p \Delta y_{t-p}$$
$$+ \beta_1 \Delta x_{t-1} + \cdots + \beta_q \Delta x_{t-q} + e_t$$

$$z_{t-1} = y_{t-1} - \theta x_{t-1}$$

- A dynamic regression in first differences, plus the error correction term z.

Answer to spurious regression

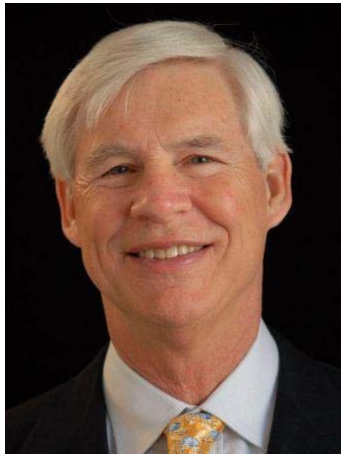
- The reaction to spurious regression was:
  - If the series are I(1), then do regressions in differences
- Cointegration says:
  - Add the error correction z!
- The difference is critical
  - The variable z measures if y is high or low relative to x
  - The error-correction coefficient γ pushes y back towards the cointegration relationship

Origin of Cointegration

- British econometricians
  - Davidson, Hendry, Srba and Yeo (1978)
  - Suggested $\ln(C_t)$-$\ln(Y_t)$ was a valuable predictor of consumption growth $\Delta\ln(C_t)$
  - This puzzled Clive Granger, as he knew that the variables were I(1), so should not be in a regression

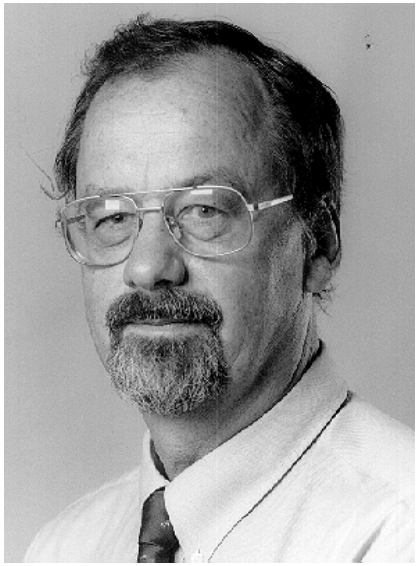
Theory of Cointegration

- This led Clive Granger to develop the theory of cointegration and the Granger Representation Theorem
- The most influential statement was a co-authored paper with Robert Engle (1987)
- Granger and Engle shared the Nobel Prize in economics in 2003



Cointegration Development

- Much of the statistical theory was developed by Peter Phillips and his students at Yale
- A multivariate statistical method was developed by the statistician Soren Johansen (U Copenhagen)
- Some jointly with the economist Katarina Juselius (Copenhagen)
- Their methods are programmed in STATA as VECM (vector error-correction models)



Example: Term Structure

- Regress change in 3-month T-bill on lagged spread, lagged changes in 3-month and 10-year
- Positive error correction coefficient
- Short rate increases when long rate exceeds short

```
. reg d.t3 L.spread120 L(1/12).d.t3 L(1/12).d.t120,r

Linear regression                                      Number of obs =     671
                                                       F( 25,   645) =    4.42
                                                       Prob > F      =  0.0000
                                                       R-squared     =  0.3032
                                                       Root MSE      =  .37838

------------------------------------------------------------------------------
             |               Robust
        D.t3 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
   spread120 |
         L1. |   .0304828   .0186479     1.63   0.103    -.0061351    .0671007
```

Regression for Long Rate

- Long Rate decreases when long rate exceeds short

```
. reg d.t120 L.spread120 L(1/12).d.t3 L(1/12).d.t120,r

Linear regression                                      Number of obs =     671
                                                       F( 25,   645) =    5.04
                                                       Prob > F      =  0.0000
                                                       R-squared     =  0.2550
                                                       Root MSE      =  .24577

------------------------------------------------------------------------------
             |               Robust
     D.t120  |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
   spread120 |
         L1. |   -.021608   .0114718    -1.88   0.060    -.0441345    .0009185
------------------------------------------------------------------------------
```

Unknown Cointegrating Coefficient

- If the cointegrating coefficient is unknown, it can be estimated
- Simplest estimator
  - Least squares of y on x
  - Consistent (Stock, 1987), but inefficient
  - Standard errors meaningless

```
. reg t3 t120

      Source |       SS       df       MS              Number of obs =     684
-------------+------------------------------           F(  1,   682) = 3240.27
       Model |  4663.61118     1  4663.61118           Prob > F      =  0.0000
    Residual |  981.579369   682  1.43926594           R-squared     =  0.8261
-------------+------------------------------           Adj R-squared =  0.8259
       Total |  5645.19055   683   8.2652863           Root MSE      =  1.1997

------------------------------------------------------------------------------
          t3 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        t120 |   .9708703   .0170557    56.92   0.000     .9373823    1.004358
       _cons |  -1.220776   .1175367   -10.39   0.000    -1.451554   -.9899991
------------------------------------------------------------------------------
```

Dynamic OLS

- Stock and Watson (1994) proposed a simple efficient estimator called dynamic OLS (DOLS)
- Regress y on x and leads and lags of Dx
- Use Newey-West standard errors
  - Lag $M=.75*T^{1/3}$
- **newey t3 t120 L(-12/12).d.t120, lag(6)**

Interest Rate Cointegration

```
. newey t3 t120 L(-12/12).d.t120, lag(6)

Regression with Newey-West standard errors          Number of obs   =       659
maximum lag: 6                                      F( 26,    632)  =     36.23
                                                    Prob > F        =    0.0000

------------------------------------------------------------------------------
             |             Newey-West
          t3 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        t120 |
         --. |   .9514564   .0377427    25.21   0.000     .8773402    1.025573
------------------------------------------------------------------------------
```

- The estimated cointegrating coefficient is 0.95
- The confidence interval contains our expected value of 1
- So in this case using the value 1 is recommended.

Estimated Cointegrating Coefficient

- Otherwise, the regression can use the estimated equilibrium error

$$z_{t-1} = y_{t-1} - \hat{\theta} x_{t-1}$$

Johansen VECM Method

- Alternatively, you can estimate the full VECM
- **vec t3 t120, trend(constant) lags(12)**
- This estimates a Vector Error Correction model with the variables T3 and T120, including a constant, and 12 lags of the variables
- This estimates equations for both variables, plus the cointegrating coefficient

Cointegrating Estimate

beta	Coef.	Std. Err.	z	P>|z|	[95% Conf.	Interval]
_ce1						
t3	1	.	.	.	.	.
t120	-.9629871	.0747933	-12.88	0.000	-1.109579	-.8163949
_cons	1.198716	.	.	.	.	.

- The estimate is .96, similar to DOLS (.95)
- The DOLS method is simpler, but many econometricians prefer the VECM estimate.

Evaluating Forecasts

- Are our forecasts good?
- How do we know?
- How do we assess a historical forecast?
- How do we compare competing forecasts?

Properties of Forecasts

- What are the properties of a good forecast?
- We start by examining optimal forecasts.

Linear Representation

- The Wold representation for y, h steps out, is

$$y_{n+h} = \mu + e_{n+h} + b_1 e_{n+h-1} + b_2 e_{n+h-2} + \cdots$$

- The h-step-ahead optimal forecast is

$$y_{n+h|n} = \mu + b_h e_n + b_{h+1} e_{n-1} + b_{h+2} e_{n-2} + \cdots$$

- The h-step-ahead optimal forecast error is

$$e_{n+h|n} = e_{n+h} + b_1 e_{n+h-1} + b_2 e_{n+h-2} + \cdots + b_{h-1} e_{n+1}$$

Optimal Forecast is Unbiased

- The forecast error is

$$e_{n+h|n} = e_{n+h} + b_1 e_{n+h-1} + b_2 e_{n+h-2} + \cdots + b_{h-1} e_{n+1}$$

- It has expectation

$$E\left(e_{n+h|n}\right) = 0$$

- And thus the optimal forecast is unbiased

One-Step Errors are White Noise

- The one-step forecast error is

$$e_{n+|n} = e_{n+h}$$

- Which is unforecastable white noise
- Thus the optimal one-step-ahead forecast error is white noise and unforecastable

h-step-ahead errors are MA(h-1)

- The h-step forecast error is

$$e_{n+h|n} = e_{n+h} + b_1 e_{n+h-1} + b_2 e_{n+h-2} + \cdots + b_{h-1} e_{n+1}$$

- This is a MA(h-1)
- Thus optimal h-step-ahead forecast errors are correlated, but at most a MA(h-1)

Forecast Variance

- The h-step forecast error is

$$e_{n+h|n} = e_{n+h} + b_1 e_{n+h-1} + b_2 e_{n+h-2} + \cdots + b_{h-1} e_{n+1}$$

- Its variance is the forecast variance, and is

$$\mathrm{var}\left(e_{n+h|n}\right) = \left(1 + b_1^2 + b_2^2 + \cdots + b_{h-1}^2\right)\sigma^2$$

- This is increasing in the forecast horizon *h*
- The variance of optimal forecasts increases with the forecast horizon

Unforecastable Errors

- The forecast errors should be unforecastable from all information available at the time of the forecast
- Not even the optimal forecast
- The coefficients should be zero in the regression

$$e_{n+h|n} = \alpha + \beta y_{n+h|n} + \varepsilon_{n+h}$$

$$\alpha = 0, \beta = 0$$

Formal Comparison

- Since $$e_{n+h|n} = y_{n+h} - y_{n+h|n}$$

  this implies

$$y_{n+h} = \alpha + \beta y_{n+h|n} + e_{n+h|h}$$

$$\alpha = 0, \beta = 1$$

- The regression of the actual value on the ex-ante forecast should have a zero intercept and a coefficient of 1

Mincer-Zarnowitz Regression

- This is called a "Mincer-Zarnowitz" regression, proposed in a paper
  - "The evaluation of economic forecasts"
- Jacob Mincer (1922-2006)
  - Father of modern labor economics
- Victor Zarnowitz (1919-2009)
  - Leading figure in business cycle dating

Mincer-Zarnowitz Test

- Estimate the simple regression

$$y_{n+h} = \alpha + \beta y_{n+h|n} + e_{n+h|h}$$

- Test the joint hypothesis

$$\alpha = 0, \beta = 1$$

- If the coefficients are different, it indicates systematic bias in the historical forecasts

Summary:
Properties of Optimal Forecasts

- Unbiased
- 1-step-ahead errors are white noise
- h-step-ahead errors are at most MA(h-1)
- Variance of h-step-ahead error is increasing in h
- Forecast errors should be unforecastable

Forecasting Average Growth

- When we are forecasting future growth, we may be interested in total future growth out to h periods
- For example, the growth rate of GDP during 2014
- This is the average of the growth rates during the four quarters 2014Q1, …, 2014Q4

Average Growth

- If y_t is the growth rate in period t, then the average future h-step growth is

$$y_{n+1:n+h} = \frac{y_{n+1} + \cdots + y_{n+h}}{h}$$

- The forecast of the average growth is

$$y_{n+1:n+h|n} = \frac{y_{n+1|n} + \cdots + y_{n+h|n}}{h}$$

- What are its properties?

Average Forecast Error

- The error of the average forecast is

$$
\begin{aligned}
e_{n+1:n+h|n} &= y_{n+1:n+h|n} - y_{n+1:n+h} \\
&= \frac{y_{n+1|n} + \cdots + y_{n+h|n}}{h} - \frac{y_{n+1} + \cdots + y_{n+h}}{h} \\
&= \frac{\left(y_{n+1|n} - y_{n+1}\right) + \cdots + \left(y_{n+h|n} - y_{n+h}\right)}{h} \\
&= \frac{e_{n+1|n} + \cdots + e_{n+h|n}}{h}
\end{aligned}
$$

- Which is the average of the 1-step through h-step errors

Average Forecast Error Variance

- Since the average forecast error is the **average** of forecast errors, it has a smaller variance than the h-step variance

$$\mathrm{var}\left(e_{n+1:n+h|n}\right) = \mathrm{var}\left(\frac{e_{n+1|n} + \cdots + e_{n+h|n}}{h}\right) \leq \mathrm{var}\left(e_{n+h|n}\right)$$

- So multi-period growth rate forecasts will have smaller variance than h-step ahead growth forecasts
  - The forecasted average growth rate for 2010 has a smaller variance than the forecasted growth rate for 2010Q4

Evaluating Forecasts

- Suppose we have a sequence of real forecasts
- Perhaps they are our own forecasts
- How can we evaluate the forecasts?

Measures of Forecast Performance

- Form the historical sequence of forecasts and actual values.
- Construct the forecast error as the difference

Example

- CBO's Economic Forecasting Record: 2009 Update
- Economic forecasts made by
  - Congressional budget office (CBO)
  - U.S. Administration
  - Private forecasters
    - Blue Chip average
  - CBO regularly assesses their forecasts

CBO Comparison

- Real Output
- Nominal Output
- Inflation
- 3-month T-Bill rate
- 10-year Treasury note rate
- Difference between CPI and GDP inflation
- Both 2-year and 5-year forecasts

Table 4.

Comparison of CBO's, *Blue Chip's*, and the Administration's Forecasts of Two-Year Average Growth Rates for Nominal Output

(Percent, by calendar year)

		CBO		*Blue Chip*[b]		Administration	
	Actual	Forecast	Error[a]	Forecast	Error[a]	Forecast	Error[a]
GNP							
1976–1977	11.5	13.1	1.7	*	*	12.3	0.8
1977–1978	12.1	10.8	-1.3	*	*	11.2	-1.0
1978–1979	12.5	10.9	-1.6	*	*	11.2	-1.3
1979–1980	10.4	11.0	0.5	*	*	10.4	-0.1
1980–1981	10.4	9.7	-0.7	*	*	9.5	-0.8
1981–1982	8.0	12.1	4.1	*	*	11.9	4.0
1982–1983	6.3	9.7	3.4	9.5	3.2	9.8	3.5
1983–1984	9.8	8.2	-1.6	9.0	-0.9	8.0	-1.8
1984–1985	9.0	9.9	0.9	9.6	0.6	9.6	0.6
1985–1986	6.2	7.6	1.3	7.4	1.2	8.2	1.9
1986–1987	5.8	7.1	1.3	6.7	0.9	7.7	1.8
1987–1988	7.0	6.5	-0.5	6.4	-0.5	6.9	-0.1
1988–1989	7.6	6.3	-1.3	6.1	-1.5	6.8	-0.9
1989–1990	6.7	6.8	0.1	6.6	-0.1	7.1	0.4
1990–1991	4.6	6.1	1.5	6.0	1.4	7.1	2.5
1991–1992	4.4	5.7	1.3	5.2	0.8	5.6	1.2

Comparison

- By showing the actual, forecasts, and forecast errors side-by-side, we can informally see which forecast performs better

Formal Comparison

- The forecasts can be compared by estimating the **bias** and **risk** (expected loss) of the forecasts
- They are estimated from R forecast errors:
  - Bias, Mean Absolute Error, Root Mean Squared Error

$$Bias = \frac{1}{R}\sum_{n=1}^{R} e_{n+h|n}$$

$$MAE = \frac{1}{R}\sum_{n=1}^{R} \left| e_{n+h|n} \right|$$

$$RMSE = \left(\frac{1}{R}\sum_{n=1}^{R} e_{n+h|n}^{2} \right)^{1/2}$$

CBO Comparison

Table 1.

Summary Measures of Performance for Two-Year Average Forecasts

(Percentage points)

	CBO	*Blue Chip*[a]	Administratio
Growth Rate for Real Output (1982-2007)			
Mean error	-0.3	-0.3	-0.1
Mean absolute error	1.0	1.0	1.0
Root-mean-square error	1.2	1.2	1.3
Growth Rate for Nominal Output (1982-2007)			
Mean error	0.1	0.1	0.2
Mean absolute error	1.0	0.9	1.1
Root-mean-square error	1.2	1.1	1.4
Inflation in the Consumer Price Index (1982-2007)			
Mean error	0.2	0.2	0.1
Mean absolute error	0.7	0.7	0.8
Root-mean-square error	0.9	0.9	0.9

Table 2.

Summary Measures of Performance for Five-Year Average Projections

(Percentage points)

	CBO	*Blue Chip*[a]	Administratio
Growth Rate for Real Output (1979-2004)			
Mean error	-0.2	-0.2	0.1
Mean absolute error	0.6	0.6	0.8
Root-mean-square error	0.9	0.8	0.9
Growth Rate for Nominal Output (1982-2004)			
Mean error	0.3	0.5	0.4
Mean absolute error	0.7	0.7	0.8
Root-mean-square error	0.9	0.9	1.0
Difference Between Inflation in the CPI and the GDP Price Index (1983-2004)			
Mean error	-0.2	-0.3	-0.3
Mean absolute error	0.4	0.4	0.5
Root-mean-square error	0.4	0.5	0.5

Data Revision

- A major difficulty with forecast evaluation is that for many series, there are serious data revisions
- The data used for forecasting, and the series published today, are different
- The series forecasted, and the series reported today, are different
- Price series, and real series based on price levels, are rebased every few years
- These rebasing are not scale transformations, because the construction of real output is done at a disaggregate level, and then aggregated.

Real Output

	Actual									
	1972 Dollars[a]	1982 Dollars[b]	1987 Dollars[c]	Chain-Type Annual-Weighted Index	CBO		*Blue Chip*[e]		Administration	
					Forecast	Error[d]	Forecast	Error[d]	Forecast	Error[d]
Real GNP										
1976–1977	6.7	4.8	4.8	5.1	6.2	1.1	*	*	5.9	0.9
1977–1978	5.2	5.0	4.7	5.1	5.5	0.4	*	*	5.1	0.1
1978–1979	3.9	3.9	3.8	4.5	4.7	0.3	*	*	4.7	0.3
1979–1980	1.3	1.1	1.1	1.6	2.7	1.2	*	*	2.9	1.3
1980–1981	1.1	0.9	0.5	1.0	0.5	-0.5	*	*	0.5	-0.5
1981–1982	0.2	-0.3	-0.4	0.2	2.1	1.9	*	*	2.6	2.4
1982–1983	0.7	0.5	0.7	1.2	2.1	0.9	2.0	0.8	2.7	1.4
1983–1984	5.2	5.2	4.9	5.7	3.4	-2.3	3.5	-2.2	2.6	-3.1
1984–1985	*	5.1	4.4	5.4	4.7	-0.7	4.3	-1.1	4.7	-0.7
1985–1986	*	3.0	2.8	3.5	3.3	-0.2	3.2	-0.3	3.9	0.4
1986–1987	*	3.1	2.9	3.3	3.1	-0.1	3.0	-0.3	3.7	0.4
1987–1988	*	3.9	3.5	3.8	2.9	-0.9	2.8	-0.9	3.3	-0.5
1988–1989	*	3.5	3.3	3.9	2.4	-1.4	2.1	-1.7	3.0	-0.9
1989–1990	*	1.7	2.0	2.8	2.5	-0.3	2.2	-0.6	3.2	0.4
1990–1991	*	*	0.3	0.9	2.0	1.2	1.9	1.1	2.8	1.9
1991–1992	*	*	0.7	1.5	1.6	0.1	1.2	-0.3	1.4	-0.1

Meese-Rogoff Puzzle

- The most influential paper using the method of forecast model comparison is
  - "Empirical exchange rate models of the seventies"
  - Richard Meese and Kenneth Rogoff
  - *Journal of International Economics*, 1983

Meese-Rogoff

- Ken Rogoff (currently Harvard)
  - Recent book
  - *This Time is Different: Eight Centuries of Financial Folly*
- Dick Meese (formerly Berkeley, now Barclay Global Investors)
  - 1978 UW Ph.D.
  - Economics Dept Advisory Board

Meese-Rogoff paper

- They compare the RMSE and bias of 1-month, 6-month and 12-month forecasts of a set of exchange rates, using structural models
- They compare the performance of the economic models with the performance of a random walk
- They found the random walk beat the economic models
- Very influential paper

Root mean square forecast errors.[a]

Exchange rate	Model: / Horizon	Random walk	Forward rate	Univariate autoregression	Vector autoregression	Frenkel–Bilson[b]	Dornbusch–Frankel[b]	Hooper–Morton[b]
$/mark	1 month	3.72	3.20	3.51	5.40	3.17	3.65	3.50
	6 months	8.71	9.03	12.40	11.83	9.64	12.03	9.95
	12 months	12.98	12.60	22.53	15.06	16.12	18.87	15.69
$/yen	1 month	3.68	3.72	4.46	7.76	4.11	4.40	4.20
	6 months	11.58	11.93	22.04	18.90	13.38	13.94	11.94
	12 months	18.31	18.95	52.18	22.98	18.55	20.41	19.20
$/pound	1 month	2.56	2.67	2.79	5.56	2.82	2.90	3.03
	6 months	6.45	7.23	7.27	12.97	8.90	8.88	9.08
	12 months	9.96	11.62	13.35	21.28	14.62	13.66	14.57
Trade-weighted dollar	1 month	1.99	N.A.	2.72	4.10	2.40	2.50	2.74
	6 months	6.09	N.A.	6.82	8.91	7.07	6.49	7.11
	12 months	8.65	14.24	11.14	10.96	11.40	9.80	10.35

Summary

- Evaluation of forecasts achieve by comparing the bias, MAE and RMSE of forecast errors
- Most influential paper is Meese-Rogoff, because they showed that naïve random walk model has lower forecast risk than structural economic models
- This established a challenge for economic modeling and forecasting.
  - Can we beat simple naïve models?!