Autoregressive Processes

- The first-order autoregressive process, AR(1) is

$$y_t = \beta y_{t-1} + e_t$$

where e_t is WN(0, σ^2)

- Using the lag operator, we can write

$$(1 - \beta L) y_t = e_t$$

- If $\beta > 0$, y_{t-1} and y_t are positively correlated
- If $\beta < 0$, y_{t-1} and y_t are negatively correlated

Inversion

- By back-substitution

$$\begin{aligned} y_t &= \beta y_{t-1} + e_t \\ &= e_t + \beta\left(\beta y_{t-2} + e_{t-1}\right) \\ &= e_t + \beta e_{t-1} + \beta^2 e_{t-2} + \cdots \\ &= \sum_{i=0}^{\infty} \beta^i e_{t-i} \end{aligned}$$

  a general linear process with geometrically declining coefficients

  – This inversion requires that $|\beta|<1$

  – $|\beta|<1$ is required for stationarity

Importance of |β|<1

- If β=1 then

$$y_t = e_t + e_{t-1} + e_{t-2} + \cdots$$

does not converge, so the sum is not defined.

Mean and Variance

- By the formula for the unconditional mean and variance of a general linear process

$$E(y_t) = E(\sum_{i=0}^{\infty} \beta^i e_{t-i}) = 0$$

$$\begin{aligned} \text{var}(y_t) &= \text{var}(\sum_{i=0}^{\infty} \beta^i e_{t-i}) \\ &= \left(\sum_{i=0}^{\infty} \beta^{2i} \right) \sigma^2 \\ &= \frac{\sigma^2}{1-\beta^2} \end{aligned}$$

Another Variance Calculation

- Take variance of both sides of

$$y_t = \beta y_{t-1} + e_t$$

- Thus

$$\begin{aligned} \operatorname{var}(y_t) &= \operatorname{var}(\beta y_{t-1} + e_t) \\ &= \operatorname{var}(\beta y_{t-1}) + \operatorname{var}(e_t) \\ &= \beta^2 \operatorname{var}(y_{t-1}) + \sigma^2 \end{aligned}$$

- If y is variance stationary, we solve and find

$$\operatorname{var}(y_t) = \operatorname{var}(y_{t-1}) = \frac{\sigma^2}{1-\beta^2}$$

|β|<1

- If $|\beta|=1$ then

$$\operatorname{var}(y_t) = \frac{\sigma^2}{1-\beta^2}$$

is infinite

$|\beta|=1$

- We calculated that

$$\text{var}(y_t) = \beta^2 \text{ var}(y_{t-1}) + \sigma^2$$

- When $|\beta|=1$, then

$$\text{var}(y_t) = \text{var}(y_{t-1}) + \sigma^2 > \text{var}(y_{t-1})$$

  so the variance is increasing with t
- $|\beta|=1$ is inconsistent with variance stationarity.
- $|\beta|<1$ is necessary for stationarity.

Random Walk

- An AR(1) with β=1 is known as a random walk or unit root process

$$y_t = y_{t-1} + e_t$$

- By back-substitution

$$y_t = y_0 + \sum_{i=0}^{t} e_{t-i}$$

- The past never disappears. Shocks have permanent effects

Unit Root

- The random walk is called a **unit root** process because the lag operator 1-L has a "root" (intersection with the x-axis) at L=1
- It is called a **random walk** because it tends to wander without mean-reversion.
- If y_t is an AR(1) with a unit root (β=1) then its first difference $\Delta y_t = y_t - y_{t-1}$ is white noise

Conditional Mean and Variance of AR(1)

- Conditional mean:

$$E(y_t \mid \Omega_{t-1}) = E(\beta y_{t-1} + e_t \mid \Omega_{t-1}) = \beta y_{t-1}$$

- Conditional variance:

$$\begin{aligned}\operatorname{var}(y_t \mid \Omega_{t-1}) &= \operatorname{var}(y_t - E(y_t \mid \Omega_{t-1}) \mid \Omega_{t-1}) \\ &= \operatorname{var}(e_t \mid \Omega_{t-1}) \\ &= \sigma^2\end{aligned}$$

Autocovariance of AR(1)

- Take the equation

$$y_t = \beta y_{t-1} + e_t$$

- And then multiply both sides by y_{t-k}

$$y_{t-k} y_t = \beta y_{t-k} y_{t-1} + y_{t-k} e_t$$

- Then take expectations. Since e_t is white noise, it is uncorrelated with

$$E(y_{t-k} y_t) = \beta E(y_{t-k} y_{t-1}) + E(y_{t-k} e_t)$$

or

$$\gamma(k) = \beta \gamma(k-1)$$

Autocorrelation of AR(1)

- Dividing by the variance, this implies

$$\rho(k) = \beta\rho(k-1)$$

- We know

$$\rho(0) = 1$$

- Then

$$\rho(1) = \beta\rho(0) = \beta$$

$$\rho(2) = \beta\rho(1) = \beta^2$$

$$\vdots$$

$$\rho(k) = \beta^k$$

Autocorrelation of AR(1)

- We have derived

$$\rho(k) = \beta^k$$

- The autocorrelation of the stationary AR(1) is a simple geometric decay ($|\beta|<1$)
- If β is small, the autocorrelations decay rapidly to zero with k
- If β is large (close to 1) then the autocorrelations decay moderately
- The AR(1) parameter describes the persistence in the time series

One-Step-Ahead Forecast

- As we showed earlier

$$E(y_t \mid \Omega_{t-1}) = \beta y_{t-1}$$

- Thus

$$E(y_{T+1} \mid \Omega_T) = \beta y_T$$

- The optimal one-step-ahead forecast is a linear function of the final observed value

2-step-ahead forecast

- By back-substitution

$$\begin{aligned} y_t &= \beta y_{t-1} + e_t \\ &= e_t + \beta\left(\beta y_{t-2} + e_{t-1}\right) \\ &= \beta^2 y_{t-2} + e_t + \beta e_{t-1} \end{aligned}$$

- Thus

$$\begin{aligned} E(y_t \mid \Omega_{t-2}) &= E(\beta^2 y_{t-2} + e_t + \beta e_{t-1} \mid \Omega_{t-2}) \\ &= \beta^2 y_{t-2} \end{aligned}$$

- and

$$E(y_{T+2} \mid \Omega_T) = \beta^2 y_T$$

2-step-ahead forecast

- This shows that the optimal 2-step-ahead forecast is also a linear function of the final observed value, but with the coefficient β^2.

$$E(y_{T+2} \mid \Omega_T) = \beta^2 y_T$$

h-step-ahead forecast

- Similarly

$$y_t = \beta^h y_{t-h} + e_t + \beta e_{t-1} + \cdots + \beta^{h-1} e_{t-h+1}$$

- So

$$E(y_t \mid \Omega_{t-h}) = E(\beta^h y_{t-h} + e_t + \beta e_{t-1} + \cdots + \beta^{h-1} e_{t-h+1} \mid \Omega_{t-2})$$

$$= \beta^h y_{t-h}$$

- Optimal forecast:

$$E(y_{T+h} \mid \Omega_T) = \beta^h y_T$$

Inversion of AR(1)

- By inverting the lag operator

$$
\begin{aligned}
(1-\beta L)y_t &= e_t \\
y_t &= (1-\beta L)^{-1} e_t \\
&= \left(\sum_{i=0}^{\infty} \beta^i L^i \right) e_t \\
&= \sum_{i=0}^{\infty} \beta^i e_{t-i}
\end{aligned}
$$

- Which is the same as found by back substitution

Condition for Invertibility

- The operator $(1-\beta L)$ is invertible when $|\beta|<1$
- This is the same as for the MA(1) model
- β is the inverse of the root of the polynomial $1-\beta L$
- The root of a function is the value where it crosses the x-axis
- The root of $1-\beta L$ is $1/\beta$, the inverse of the root is β
- Invertibility requires that the inverse of the root be less than one

AR(1) with Intercept

- An AR(1) with intercept is

$$y_t = \alpha + \beta y_{t-1} + e_t$$

Taking expectations

$$E(y_t) = \alpha + \beta E(y_{t-1}) + E(e_t)$$

- Thus

$$\mu = \alpha + \beta\mu$$

- and

$$\mu = \frac{\alpha}{1-\beta}$$

Best Linear Predictor

- A linear predictor of y_t given y_{t-1} is

$$\alpha + \beta y_{t-1}$$

- The forecast error is

$$e_t = y_t - \alpha - \beta y_{t-1}$$

- The linear predictor which minimizes the expected squared forecast error solves

$$\min_{\alpha,\beta} E(y_t - \alpha - \beta y_{t-1})^2$$

Least-Squares

- The estimate of the expected squared linear forecast error is the sum of squared errors
- The least squares estimate

$$y_t = \hat{\alpha} + \hat{\beta} y_{t-1} + \hat{e}_t$$

minimizes the sum of squared errors, so is the estimate of the best linear predictor

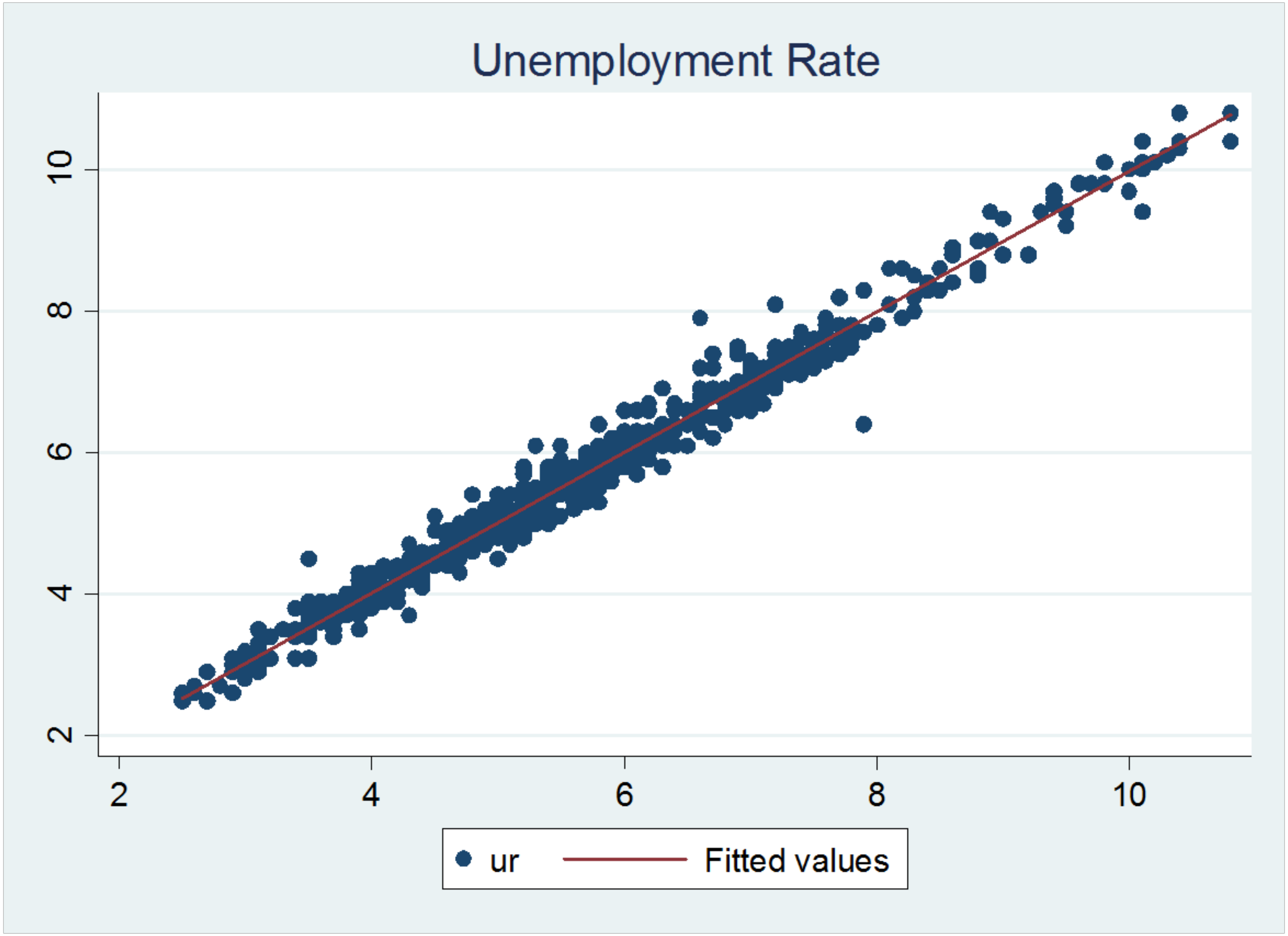
- This is a linear regression, treating y_{t-1} as a regressor.

Unemployment Rate

```
. regress ur L.ur

      Source |       SS       df       MS              Number of obs =     744
-------------+------------------------------           F(  1,   742) =37941.25
       Model |  1775.33245     1  1775.33245           Prob > F      =  0.0000
    Residual |  34.7193756   742  .046791611           R-squared     =  0.9808
-------------+------------------------------           Adj R-squared =  0.9808
       Total |  1810.05182   743  2.43613974           Root MSE      =  .21631

------------------------------------------------------------------------------
          ur |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
          ur |
         L1. |   .9934454   .0051002   194.79   0.000     .9834329    1.003458
             |
       _cons |    .045538   .0299153     1.52   0.128    -.0131907    .1042667
------------------------------------------------------------------------------
```



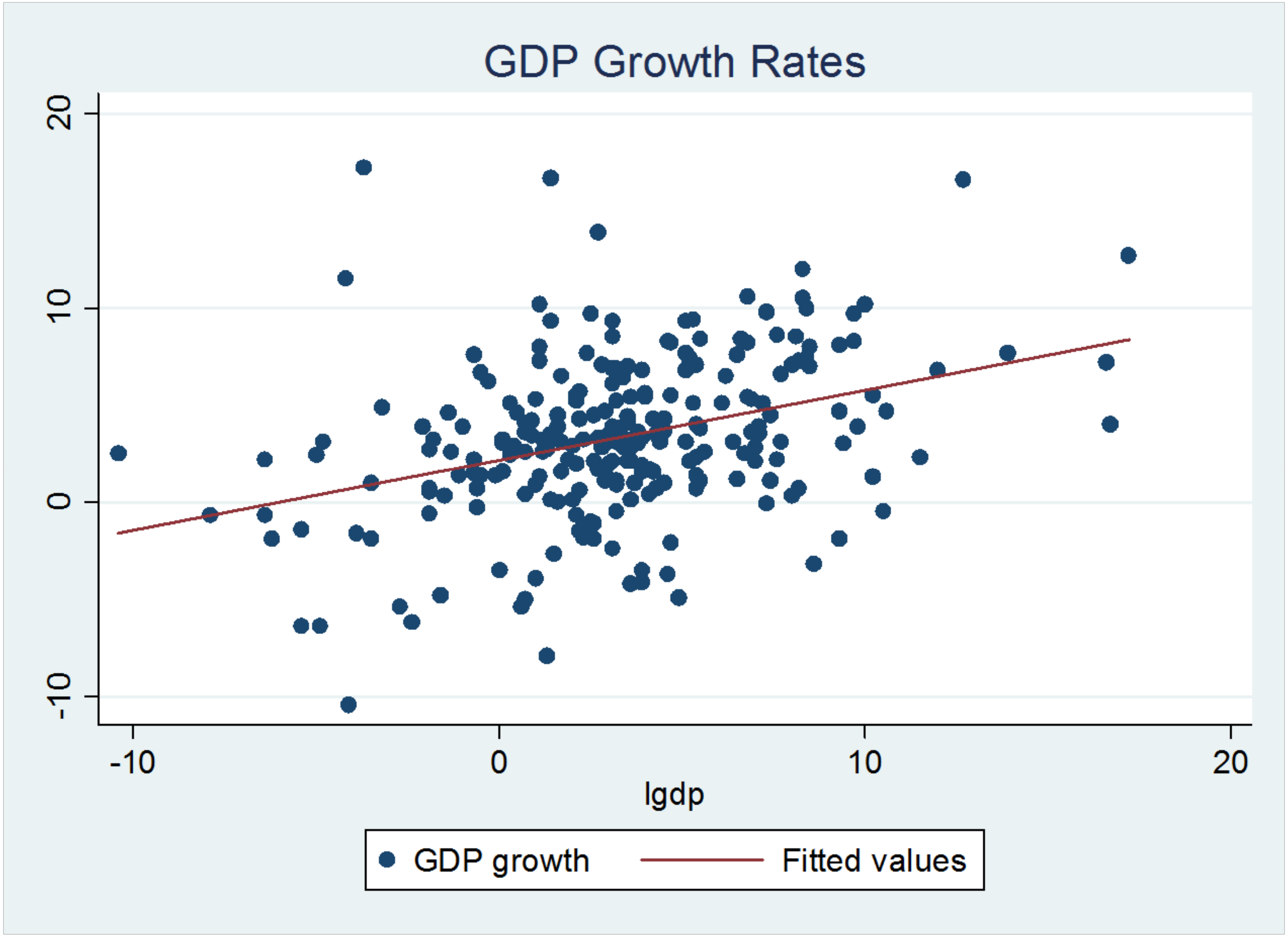
Unemployment Rate
10
8
6
4
2
2
4
6
8
10
ur
Fitted values

GDP Growth Rates

```
. reg gdp L.gdp

      Source |       SS       df       MS              Number of obs =     250
-------------+------------------------------           F(  1,   248) =   37.15
       Model |  548.684959     1  548.684959           Prob > F      =  0.0000
    Residual |  3662.76711   248  14.7692222           R-squared     =  0.1303
-------------+------------------------------           Adj R-squared =  0.1268
       Total |  4211.45207   249  16.9134621           Root MSE      =  3.8431

------------------------------------------------------------------------------
         gdp |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         gdp |
         L1. |   .3605283   .0591503     6.10   0.000     .2440274    .4770292
             |
       _cons |   2.146711   .3123873     6.87   0.000     1.531441    2.761982
------------------------------------------------------------------------------
```



GDP Growth Rates
20
10
0
-10
-10
0
10
20
lgdp
GDP growth
Fitted values

One-Step-Ahead Forecast

- The optimal forecast for T+1 given T is

$$\hat{y}_{T+1|T} = \alpha + \beta y_T$$

- The forecast using the estimates is

$$\hat{y}_{T+1|T} = \hat{\alpha} + \hat{\beta} y_T$$

Example – Unemployment Rate

- The estimates were

ur					
L1.	.9934454	.0051002	194.79	0.000	.9834329
_cons	.045538	.0299153	1.52	0.128	-.0131907

$$y_t = 0.0455 + 0.993 y_{t-1} + \hat{e}_t$$

- The value for Jan 2010 is 9.7%, so

$$\hat{y}_{2010:2} = 0.0455 + 0.993 \times 9.7 = 9.68$$

	y_t	y_{t-1}	fitted value
Jan-09	7.7	7.4	7.40
Feb-09	8.2	7.7	7.70
Mar-09	8.6	8.2	8.19
Apr-09	8.9	8.6	8.59
May-09	9.4	8.9	8.89
Jun-09	9.5	9.4	9.38
Jul-09	9.4	9.5	9.48
Aug-09	9.7	9.4	9.38
Sep-09	9.8	9.7	9.68
Oct-09	10.1	9.8	9.78
Nov-09	10	10.1	10.08
Dec-09	10	10	9.98
Jan-10	9.7	10	9.98
Feb-10	?	9.7	**9.68**

Example – GDP Growth

- The estimates were

gdp	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
gdp						
L1.	.3605283	.0591503	6.10	0.000	.2440274	.4770292
_cons	2.146711	.3123873	6.87	0.000	1.531441	2.761982

$$y_t = 2.14 + 0.361 y_{t-1} + \hat{e}_t$$

- The value for 4th quarter 2009 is 5.7%, so

$$\hat{y}_{2010:1} = 2.14 + 0.361 \times 5.7 = 4.2\%$$

GDP Growth

	y_t	y_{t-1}	fitted
2008q4	-5.4	-2.7	1.2
2009q1	-6.4	-5.4	0.2
2009q2	-0.7	-6.4	-0.2
2009q3	2.2	-0.7	1.9
2009q4	5.7	2.2	2.9
2010q1	?	5.7	**4.2**

One-Step-Ahead Forecast Error

- The forecast error is

$$
\begin{aligned}
y_{T+1} - \hat{y}_{T+1|T} &= \alpha + \beta y_T + e_{T+1} - \left(\alpha + \beta y_T\right) \\
&= e_{T+1}
\end{aligned}
$$

- The forecast variance is

$$
\text{var}(y_{T+1} - \hat{y}_{T+1|T}) = \text{var}(e_{T+1}) = \sigma^2
$$

Forecast variance estimation

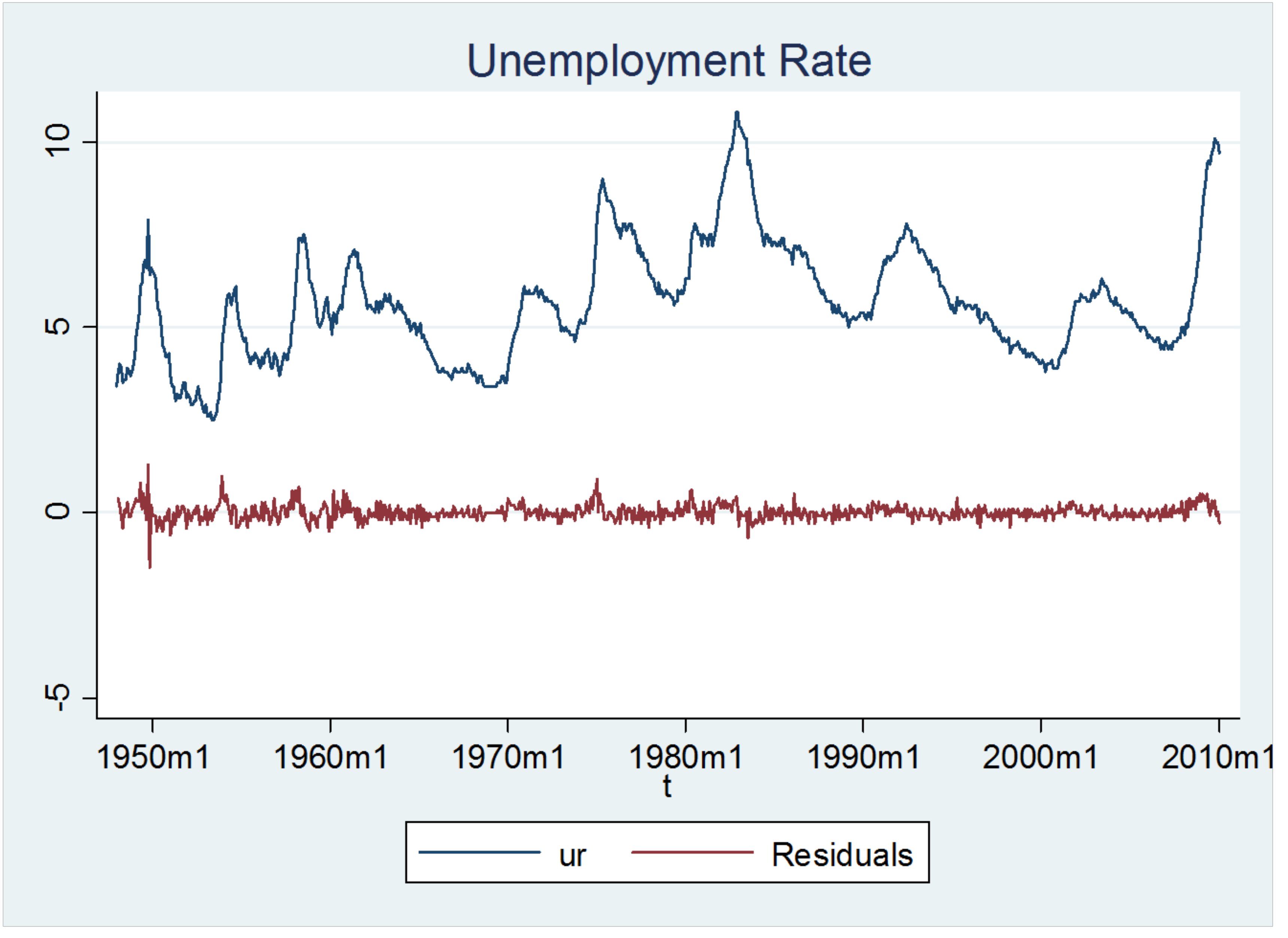
- Average of squared residuals

$$\hat{\sigma}^2 = \frac{1}{T}\sum_{t=1}^{T}\hat{e}_t^2$$

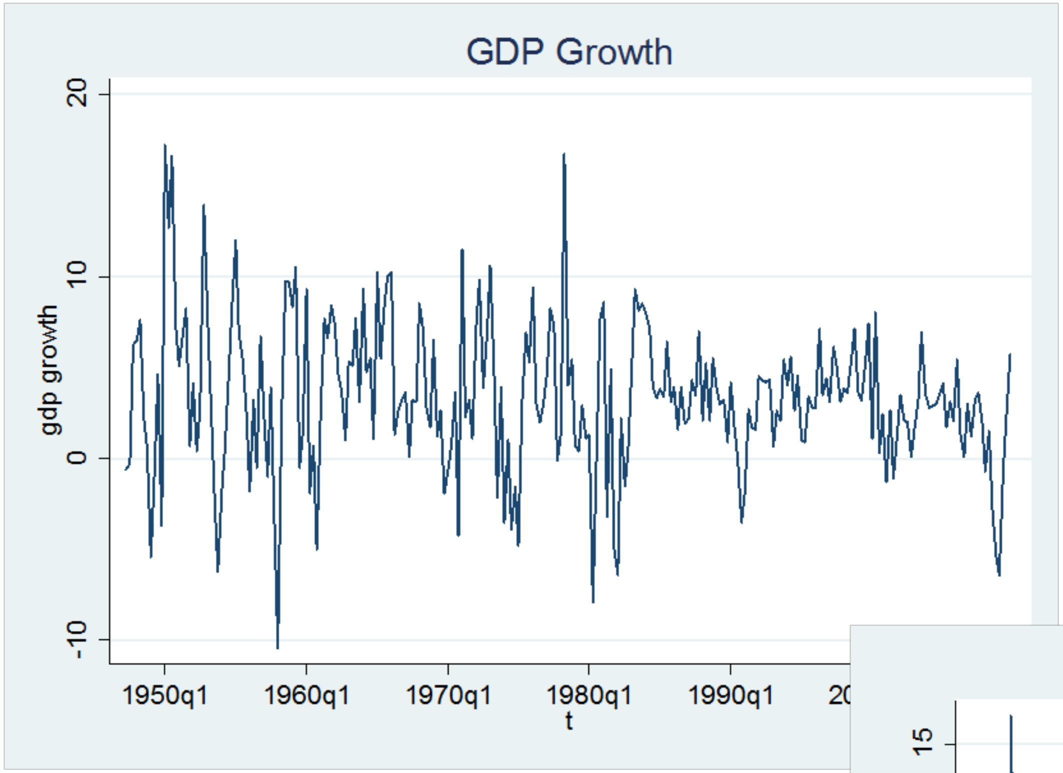
$$\hat{\sigma} = \sqrt{\hat{\sigma}^2}$$

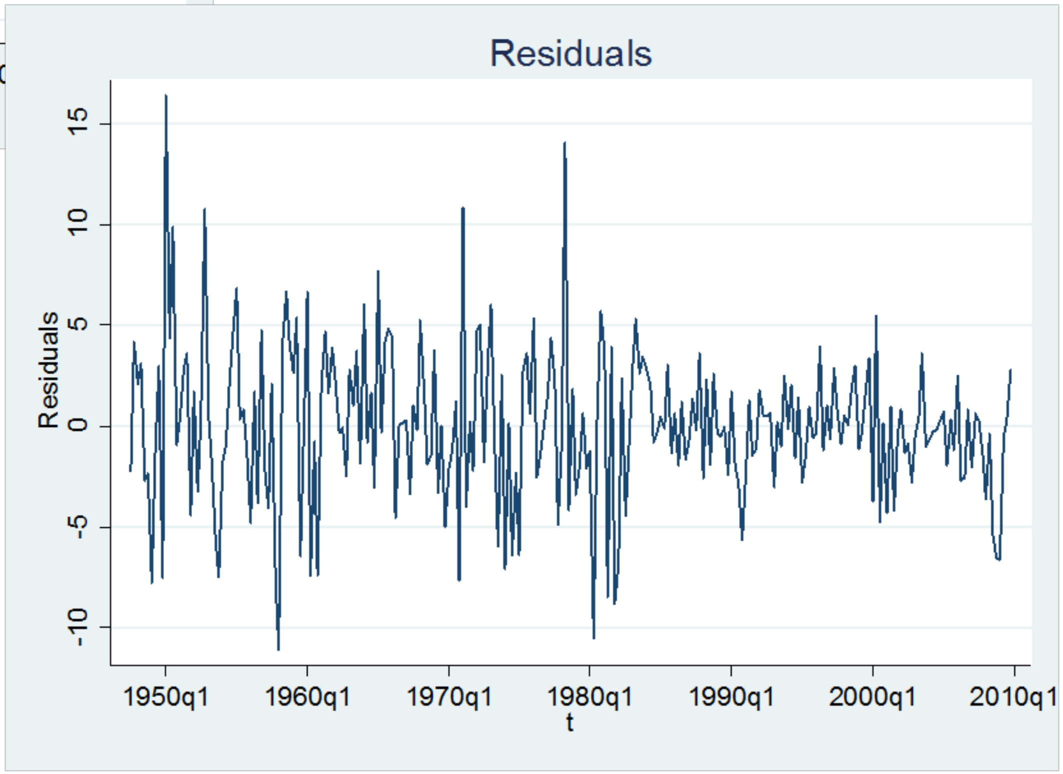
where the least-squares residuals are

$$\hat{e}_t = y_t - \hat{\alpha} - \hat{\beta} y_{t-1}$$



Unemployment Rate
10
5
0
-5
1950m1
1960m1
1970m1
1980m1
1990m1
2000m1
2010m1
t
ur
Residuals


GDP Growth
gdp growth
20
10
0
-10
1950q1
1960q1
1970q1
1980q1
1990q1
t


Residuals
Residuals
15
10
5
0
-5
-10
1950q1
1960q1
1970q1
1980q1
1990q1
2000q1
2010q1
t

One-Step-Ahead Intervals

- Normal Method
  - Assume forecast error is normally distributed
  - Forecast interval is point estimate, plus and minus the estimated standard deviation multiplied by a normal quantile
  - For a 95% interval:

$$\hat{y}_{T+1|T} \pm \hat{\sigma} \cdot z_{.025} = \hat{y}_{T+1|T} \pm \hat{\sigma} \cdot 1.96$$

  - For a 90% interval

$$\hat{y}_{T+1|T} \pm \hat{\sigma} \cdot z_{.05} = \hat{y}_{T+1|T} \pm \hat{\sigma} \cdot 1.645$$

Estimating Forecast Variance

- The estimated variance is 16.9
- The estimated st. dev. is 3.84

```
. reg gdp L.gdp

      Source |       SS       df       MS              Number of obs =     250
-------------+------------------------------           F(  1,   248) =   37.15
       Model |  548.684959     1  548.684959           Prob > F      =  0.0000
    Residual |  3662.76711   248  14.7692222           R-squared     =  0.1303
-------------+------------------------------           Adj R-squared =  0.1268
       Total |  4211.45207   249  16.9134621           Root MSE      =  3.8431

------------------------------------------------------------------------------
         gdp |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         gdp |
         L1. |   .3605283   .0591503     6.10   0.000     .2440274    .4770292
             |
       _cons |   2.146711   .3123873     6.87   0.000     1.531441    2.761982
------------------------------------------------------------------------------
```

Standard Dev Estimation

- In STATA, the estimate of σ or "root mean squared error" is saved after you estimate a regression in "e(rmse)"
- Better, the forecast standard deviation is stdf, through the command

predict s, stdf

Forecast Interval Construction

```
. tsappend, add(1)

. predict p if t>tq(2009q4)
(option xb assumed; fitted values)
(251 missing values generated)

. gen p1=p-1.645*e(rmse)
(251 missing values generated)

. gen p2=p+1.645*e(rmse)
(251 missing values generated)
```

- Point estimate = 4.2%
- 90% Interval = [-2.1%, 10.5%]