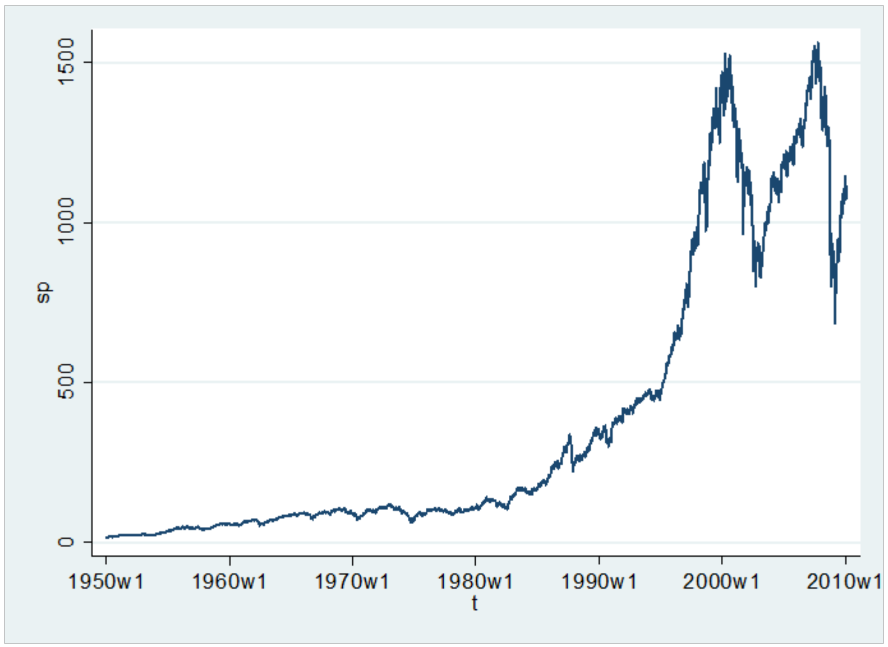
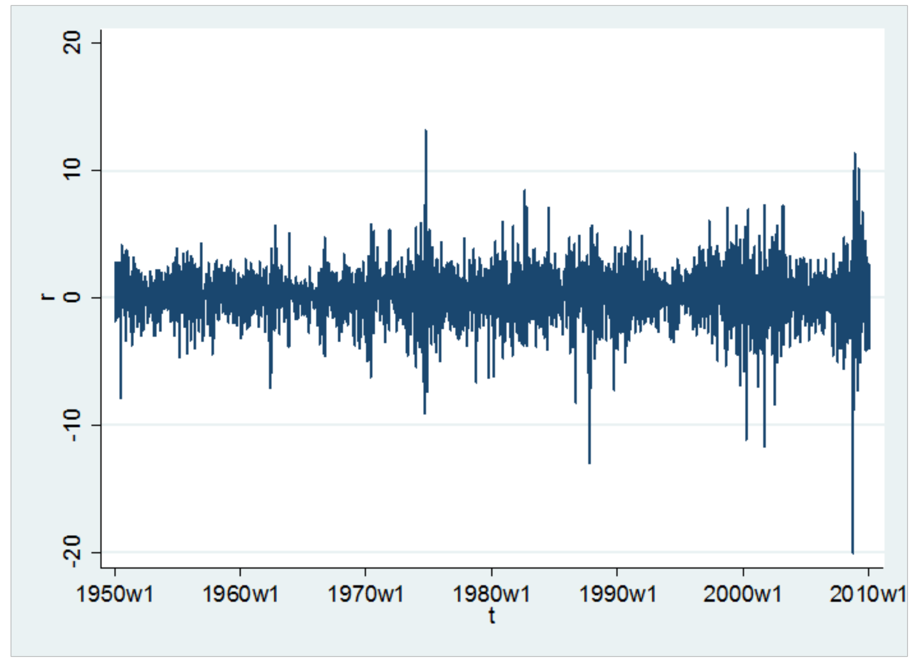
Volatility

- Many economic series, and most financial series, display conditional volatility
  - The conditional variance changes over time
  - There are periods of high volatility
    - When large changes frequently occur
  - And periods of low volatility
    - When large changes are less frequent

Weekly Stock Prices
Levels and Returns





Conditional Mean

- The conditional mean of y is

$$E\left(y_t \mid \Omega_{t-1}\right)$$

- The regression error is mean zero and unforecastable

$$E\left(e_t \mid \Omega_{t-1}\right) = 0$$

Conditional Variance

- The conditional variance of y is

$$\operatorname{var}(y_t \mid \Omega_{t-1}) = E\left(\left(y_t - E(y_t \mid \Omega_{t-1})\right)^2 \mid \Omega_{t-1}\right)$$
$$= E\left(e_t^2 \mid \Omega_{t-1}\right)$$

- The squared regression error can be forecastable

Forecastable Conditional Variance

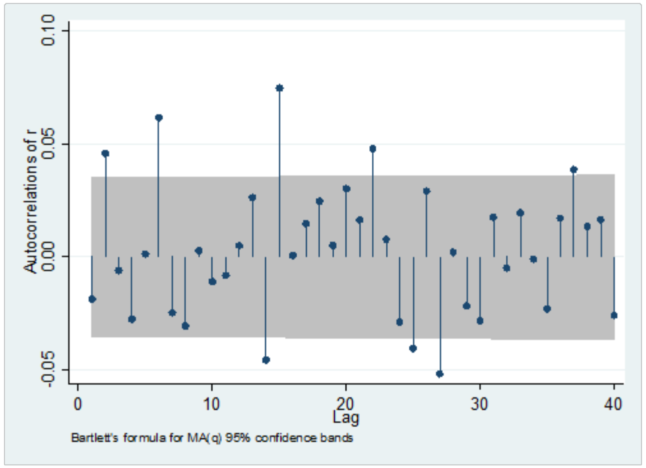
- If the squared error is forecastable, then the conditional variance is time-varying and correlated.
  - The magnitude of changes is predictable
  - The sign is not predictable

Stock returns are unpredictable

```
. reg r L(1/4).r,r

Linear regression                                      Number of obs =    3120
                                                       F(  4,  3115) =    1.22
                                                       Prob > F      =  0.3009
                                                       R-squared     =  0.0033
                                                       Root MSE      =  2.0793
```

r	Coef.	Robust Std. Err.	t	P>|t|	[95% Conf.	Interval]
r						
L1.	-.01737	.032557	-0.53	0.594	-.0812053	.0464653
L2.	.0466548	.0273551	1.71	0.088	-.0069811	.1002907
L3.	-.0044898	.0282283	-0.16	0.874	-.0598378	.0508581
L4.	-.0298138	.0264335	-1.13	0.259	-.0816427	.0220151
_cons	.1337402	.0410395	3.26	0.001	.053273	.2142074

```
. testparm L(1/4).r

 ( 1)  L.r = 0
 ( 2)  L2.r = 0
 ( 3)  L3.r = 0
 ( 4)  L4.r = 0

       F(  4,  3115) =    1.22
            Prob > F =    0.3009
```



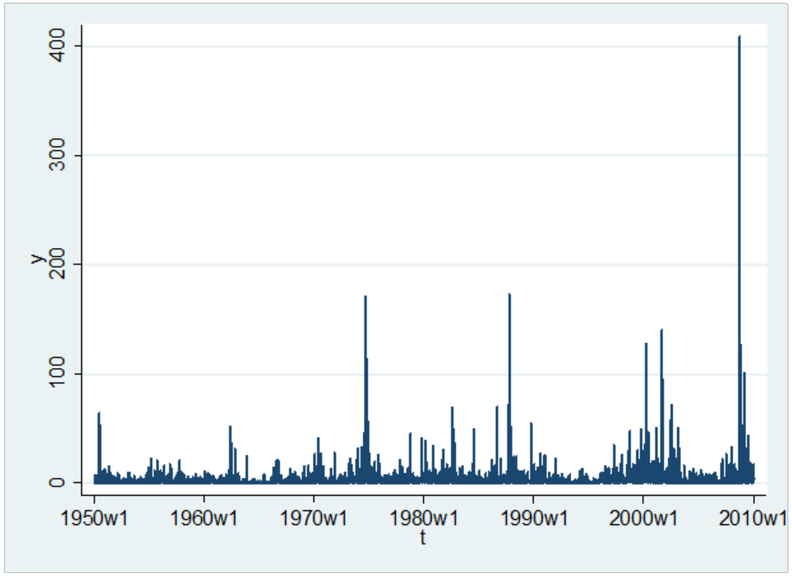
Squared Returns are predictable

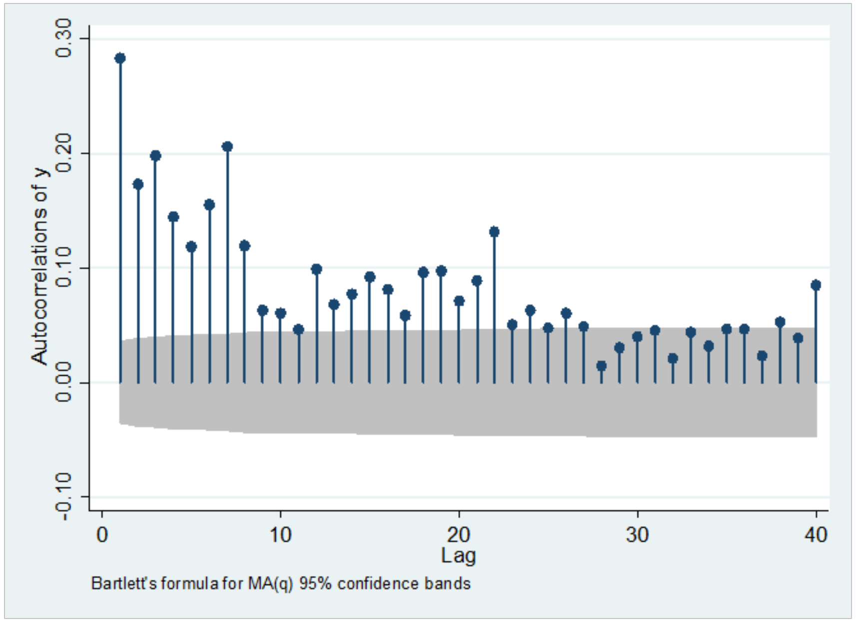
```
. gen y=(r-.1334364)^2
(1 missing value generated)

. reg y L(1/4).y,r

Linear regression                               Number of obs =     3120
                                                F(  4,  3115) =     9.72
                                                Prob > F      =   0.0000
                                                R-squared     =   0.1092
                                                Root MSE      =   11.618
```

y	Coef.	Robust Std. Err.	t	P>|t|	[95% Conf.	Interval]
y						
L1.	.2332184	.1248813	1.87	0.062	-.0116395	.4780763
L2.	.0627729	.0308486	2.03	0.042	.0022873	.1232586
L3.	.125441	.0364307	3.44	0.001	.0540103	.1968717
L4.	.0517234	.0506949	1.02	0.308	-.0476755	.1511222
_cons	2.282243	.3480455	6.56	0.000	1.599821	2.964665

```
. testparm L(1/4).y

 ( 1)  L.y = 0
 ( 2)  L2.y = 0
 ( 3)  L3.y = 0
 ( 4)  L4.y = 0

       F(  4,  3115) =    9.72
            Prob > F =    0.0000
```

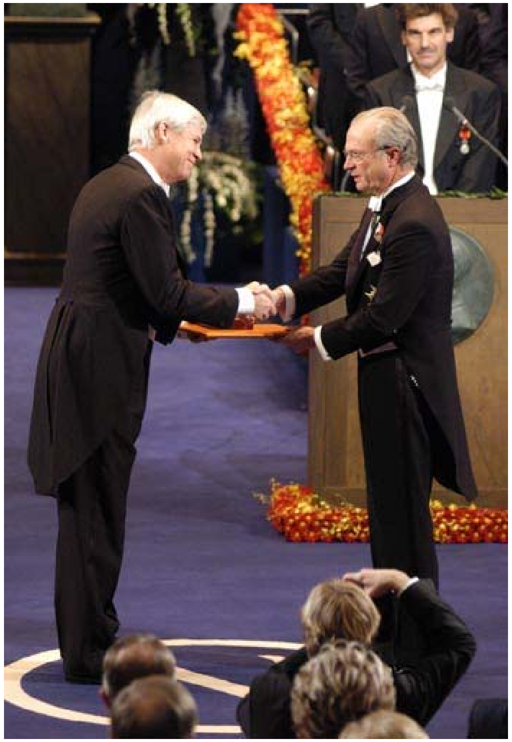
Squared Returns





ARCH

- Robert Engle (1982) proposed a model for the conditional variance
  - AutoRegressive Conditional Heteroskedasticity
  - "ARCH" now describes volatility models
- Nobel Prize 2003



ARCH(1) Model

$$y_t = \mu + e_t$$

$$\sigma_t^2 = \text{var}\left(e_t \mid \Omega_{t-1}\right) = \omega + \alpha e_{t-1}^2$$

$$\omega > 0$$

$$\alpha \geq 0$$

- α>0 means that the conditional variance is high when the lagged squared error is high
- Large errors (either sign) today mean high expected errors (in magnitude) tomorrow.
- Small magnitude errors forecast next period small magnitude errors.

Unconditional variance

- A property of expectations is that expected (average) conditional expectations are unconditional expectations.
- So the average conditional variance is the average variance – the variance of the regression error.

$$\sigma^2 = E\left(\sigma_t^2\right) = \omega + \alpha E\left(e_{t-1}^2\right) = \omega + \alpha\sigma^2$$

- Solving for the variance: $\sigma^2 = \dfrac{\omega}{1-\alpha}$

- Rewriting, this implies

$$\omega = \sigma^2 (1 - \alpha)$$

- Substituting into ARCH(1) equation

$$\sigma_t^2 = (1 - \alpha)\sigma^2 + \alpha e_{t-1}^2$$

or

$$\sigma_t^2 = \sigma^2 + \alpha\left(e_{t-1}^2 - \sigma^2\right)$$

- This shows that the conditional variance is a combination of the unconditional variance, and the deviation of the squared error from its average value.

ARCH(1) as AR(1) in squares

- The model

$$\mathrm{var}(e_t \mid \Omega_{t-1}) = E(e_t^2 \mid \Omega_{t-1}) = \omega + \alpha e_{t-1}^2$$

implies the regression

$$e_t^2 = \omega + \alpha e_{t-1}^2 + u_t$$

where u is white noise

- Thus e-squared is an AR(1)

Estimation

- **.arch r, arch(1)**

```
ARCH family regression

Sample: 1950w2 - 2010w5                         Number of obs   =      3124
Distribution: Gaussian                          Wald chi2(.)    =         .
Log likelihood = -6525.268                      Prob > chi2     =         .
```

r	Coef.	OPG Std. Err.	z	P>|z|	[95% Conf.	Interval]
r						
_cons	.1996426	.0314495	6.35	0.000	.1380027	.2612825
ARCH						
arch L1.	.3006982	.0216209	13.91	0.000	.2583219	.3430745
_cons	2.926873	.0686021	42.66	0.000	2.792416	3.061331

Variance Forecast

- Given the parameter estimates, the estimated conditional variance for period t is

$$\hat{\sigma}_t^2 = \hat{\omega} + \hat{\alpha}\hat{e}_{t-1}^2 = \hat{\omega} + \hat{\alpha}\left(y_{t-1} - \hat{\mu}\right)^2$$

- The forecasted out-of-sample variance is

$$\hat{\sigma}_{n+1}^2 = \hat{\omega} + \hat{\alpha}\left(y_n - \hat{\mu}\right)^2$$

Forecast Interval for the mean

- You can use the estimated conditional standard deviation to obtain forecast intervals for the mean

$$\hat{y}_{n+1|n} \pm Z_{\alpha/2}\hat{\sigma}_{n+1}$$

- These forecast intervals will vary in width depending on the estimated conditional variance.
  - Wider in periods of high volatility
  - More narrow in periods of low volatility

ARCH(p) model

- Allow p lags of squared errors

$$y_t = \mu + e_t$$
$$\sigma_t^2 = \omega + \alpha_1 e_{t-1}^2 + \alpha_2 e_{t-2}^2 + \cdots + \alpha_p e_{t-p}^2$$

- Similar to AR(p) in squares
- Estimation: ARCH(8)
  - **.arch r, arch(1/8)**
  - ARCH model with lags 1 through 8

ARCH(8) Estimates

- **.arch r, arch(1/8)**

```
ARCH family regression

Sample: 1950w2 - 2010w5                         Number of obs   =      3124
Distribution: Gaussian                          Wald chi2(.)    =         .
Log likelihood = -6368.552                      Prob > chi2     =         .

------------------------------------------------------------------------------
             |                 OPG
           r |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
r            |
       _cons |   .2027503   .0290226     6.99   0.000     .1458671    .2596335
-------------+----------------------------------------------------------------
ARCH         |
        arch |
         L1. |   .1867283   .0163376    11.43   0.000     .1547071    .2187495
         L2. |   .1099957   .0203355     5.41   0.000     .0701388    .1498526
         L3. |   .1541191   .0225999     6.82   0.000     .1098241    .1984142
         L4. |   .0912413   .0192753     4.73   0.000     .0534625    .1290202
         L5. |   .0284588   .0171987     1.65   0.098    -.0052501    .0621677
         L6. |   .0811242   .0192012     4.22   0.000     .0434906    .1187578
         L7. |    .041083   .0147083     2.79   0.005     .0122553    .0699107
         L8. |   .0706622   .0171741     4.11   0.000     .0370015    .1043229
             |
       _cons |   1.144063   .1001062    11.43   0.000     .9478582    1.340267
------------------------------------------------------------------------------
```

ARCH needs many lags

- Notice that we included 8 lags, and all appeared significant.
- This is commonly observed in estimated ARCH models
  - The conditional variance appears to be a function of many lagged past squares

GARCH Model

- Tim Bollerslev (1986)
  - A student of Engle
  - Current faculty at Duke

proposed the GARCH model to simplify this problem

$$\sigma_t^2 = \omega + \beta\sigma_{t-1}^2 + \alpha e_{t-1}^2$$

$$\beta > 0$$

$$\omega > 0$$

$$\alpha \geq 0$$

GARCH(1,1)

- This makes the variance a function of all past lags:

$$\sigma_t^2 = \omega + \beta\sigma_{t-1}^2 + \alpha e_{t-1}^2$$

$$= \sum_{j=0}^{\infty} \beta^j \left(\omega + \alpha e_{t-1-j}^2\right)$$

- It is also smoother than an ARCH model with a small number of lags

GARCH(p,q)

- p lags of squared error
- q lags of conditional variance

$$\sigma_t^2 = \omega + \beta_1 \sigma_{t-1}^2 + \cdots + \beta_q \sigma_{t-q}^2 + \alpha_1 e_{t-1}^2 + \cdots + \alpha_p e_{t-p}^2$$

- GARCH(1,1):
  - **.arch r, arch(1) garch(1)**
- GARCH(3,2):
  - **.arch r, arch(1/3) garch(1/2)**

GARCH(1,1)

ARCH family regression

Sample: 1950w2 - 2010w5
Distribution: Gaussian
Log likelihood = -6359.118

Number of obs = 3124
Wald chi2(.) = .
Prob > chi2 = .

r	Coef.	OPG Std. Err.	z	P>|z|	[95% Conf.	Interval]
r						
_cons	.1935287	.0296365	6.53	0.000	.1354421	.2516152
ARCH						
arch L1.	.1317621	.0094385	13.96	0.000	.113263	.1502613
garch L1.	.8444868	.0117076	72.13	0.000	.8215404	.8674333
_cons	.1207574	.0221764	5.45	0.000	.0772924	.1642223

- Common GARCH features
  - Lagged variance has large coefficient
  - Sum of two coefficients very close to (but less than) one

GARCH(2,2) for Stock Returns

ARCH family regression

Sample: 1950w2 - 2010w5
Distribution: Gaussian
Log likelihood = -6356.166

Number of obs = 3124
Wald chi2(.) = .
Prob > chi2 = .

r	Coef.	OPG Std. Err.	z	P>|z|	[95% Conf.	Interval]
r						
_cons	.1913999	.0295859	6.47	0.000	.1334126	.2493872
ARCH						
arch						
L1.	.1658834	.0149416	11.10	0.000	.1365984	.1951684
L2.	-.0277704	.042739	-0.65	0.516	-.1115373	.0559964
garch						
L1.	.5991681	.2941188	2.04	0.042	.0227059	1.17563
L2.	.2373325	.2478461	0.96	0.338	-.248437	.7231021
_cons	.1233949	.0428309	2.88	0.004	.0394478	.207342

GARCH(1,1)

- The GARCH(1,1) often fits well, and is a useful benchmark.
  - Daily, weekly, or monthly asset returns, exchange rates, or interest rates

Extensions

- There are many extensions of the basic GARCH model, developed to handle a variety of situations
  - Asymmetric Response
  - Garch-in-mean
  - Explanatory variables in variance
  - Non-normal errors

Asymetric GARCH

- Threshold GARCH

$$\sigma_t^2 = \omega + \beta\sigma_{t-1}^2 + \alpha e_{t-1}^2 + \gamma e_{t-1}^2 1(e_{t-1} > 0)$$

- The last tern is dummy variable for positive lagged errors
- This model specifies that the ARCH effect depends on whether the error was positive or negative
  - If the error is negative, the effect is α
  - If the error is positive, the full effect is α+γ

TARCH estimation

- **.arch r, arch(1) tarch(1) garch(1)**
- Negative errors have coefficient of 0.19
- Positive errors have coefficient of 0.05
- Negative returns increase volatility much more than positive returns

ARCH family regression

Sample: 1950w2 - 2010w5 Number of obs = 3124
Distribution: Gaussian Wald chi2(.) = .
Log likelihood = -6332.433 Prob > chi2 = .

r	Coef.	OPG Std. Err.	z	P>|z|	[95% Conf.	Interval]
r						
_cons	.1474826	.0313132	4.71	0.000	.0861099	.2088552
ARCH						
arch L1.	.1879679	.0154078	12.20	0.000	.1577692	.2181665
tarch L1.	-.1408097	.0160892	-8.75	0.000	-.1723439	-.1092754
garch L1.	.8437111	.0132294	63.78	0.000	.817782	.8696403
_cons	.1540714	.0219836	7.01	0.000	.1109843	.1971585

Leverage Effect

- This model describes what is called the “leverage effect”
  - A negative shock to equity increases the ratio debt/equity of investors
  - This increases the *leverage* of their portfolios
  - This increases risk, and the conditional variance
  - Negative shocks have stronger effect on variance than positive shocks

GARCH-in-mean

- If investors are risk averse, risky assets will earn higher returns (a risk premium) in market equilibrium
- If assets have varying volatility (risk), their expected return will vary with this volatility
  - Expected return should be positively correlated with volatility

GARCH-M model

$$y = \beta_1 + \beta_1 \sigma_{t-1}^2 + e_t$$

$$\sigma_t^2 = \omega + \beta \sigma_{t-1}^2 + \alpha e_{t-1}^2$$

- **.arch arch(1) garch(1) archm**

GARCH-M for Stock Returns

- Marginally positive effect

```
ARCH family regression

Sample: 1950w2 - 2010w5                         Number of obs   =      3124
Distribution: Gaussian                          Wald chi2(1)    =      2.96
Log likelihood = -6357.259                      Prob > chi2     =    0.0853
```

r	Coef.	OPG Std. Err.	z	P>|z|	[95% Conf.	Interval]
r						
_cons	.1211625	.0512489	2.36	0.018	.0207165	.2216085
ARCHM						
sigma2	.024739	.0143783	1.72	0.085	-.003442	.0529199
ARCH						
arch L1.	.1315334	.0096454	13.64	0.000	.1126287	.1504381
garch L1.	.8450762	.0118319	71.42	0.000	.8218862	.8682662
_cons	.1193442	.022376	5.33	0.000	.075488	.1632004

TARCH and GARCH-M

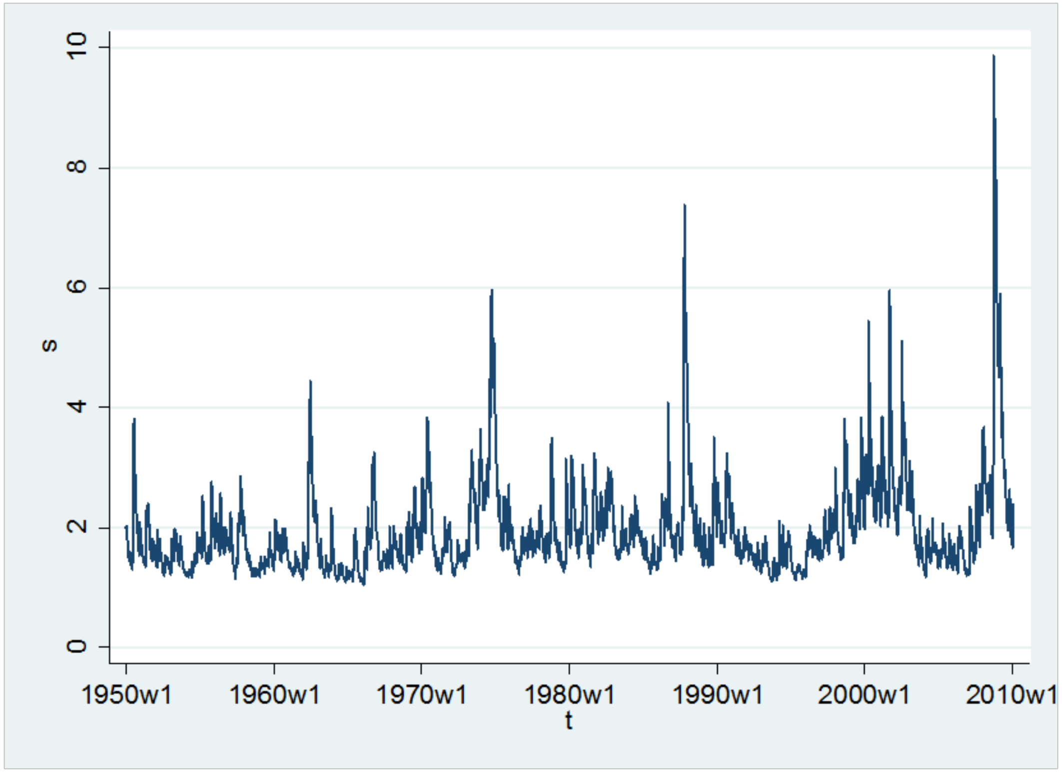
- **.arch arch(1) tarch(1) garch(1) archm**
- archm term appears insignificant

```
ARCH family regression

Sample: 1950w2 - 2010w5                         Number of obs   =      3124
Distribution: Gaussian                          Wald chi2(1)    =      0.17
Log likelihood = -6332.324                      Prob > chi2     =    0.6776
```

r	Coef.	OPG Std. Err.	z	P>|z|	[95% Conf.	Interval]
r						
_cons	.1309694	.0512904	2.55	0.011	.030442	.2314967
ARCHM						
sigma2	.0059058	.0142042	0.42	0.678	-.021934	.0337456
ARCH						
arch L1.	.1872044	.0155283	12.06	0.000	.1567694	.2176393
tarch L1.	-.1391533	.0161185	-8.63	0.000	-.1707449	-.1075617
garch L1.	.8425617	.0137719	61.18	0.000	.8155693	.8695542
_cons	.1565728	.0228305	6.86	0.000	.1118259	.2013197

Estimated standard deviation

- Estimated TARCH model
- **.predict v, variance**
- **.gen s=sqrt(v)**
- Unconditional variance is 2.1



S&P, returns, and standard deviation 2006-2010