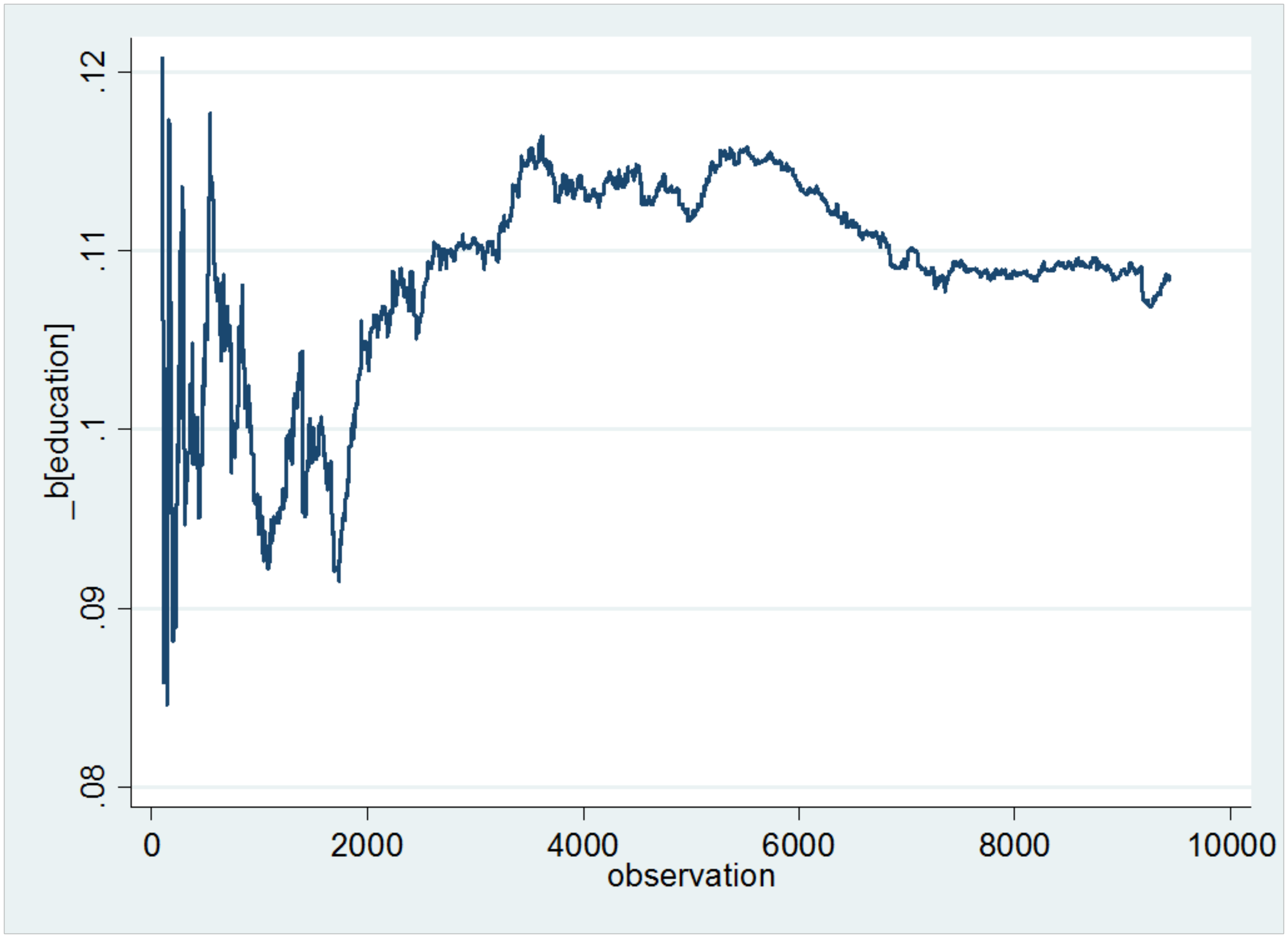
Distribution of Estimates

- From Econometrics (410)
- Linear Regression Model $$y_t = \alpha + \beta x_t + e_t$$
  - Assume (y_t, x_t) is iid and $E(x_t e_t) = 0$
- Estimation Consistency
  - The estimates approach the true values as the sample size increases
  - Estimation variance decreases as the sample size increases

Illustration of Consistency

- Take random sample of U.S. white men
- Estimate linear regression of log(wages) on education
- Total sample = 2089
- Start with 100 observations, sequentially increase to 2089

Sequence of Slope Coefficients



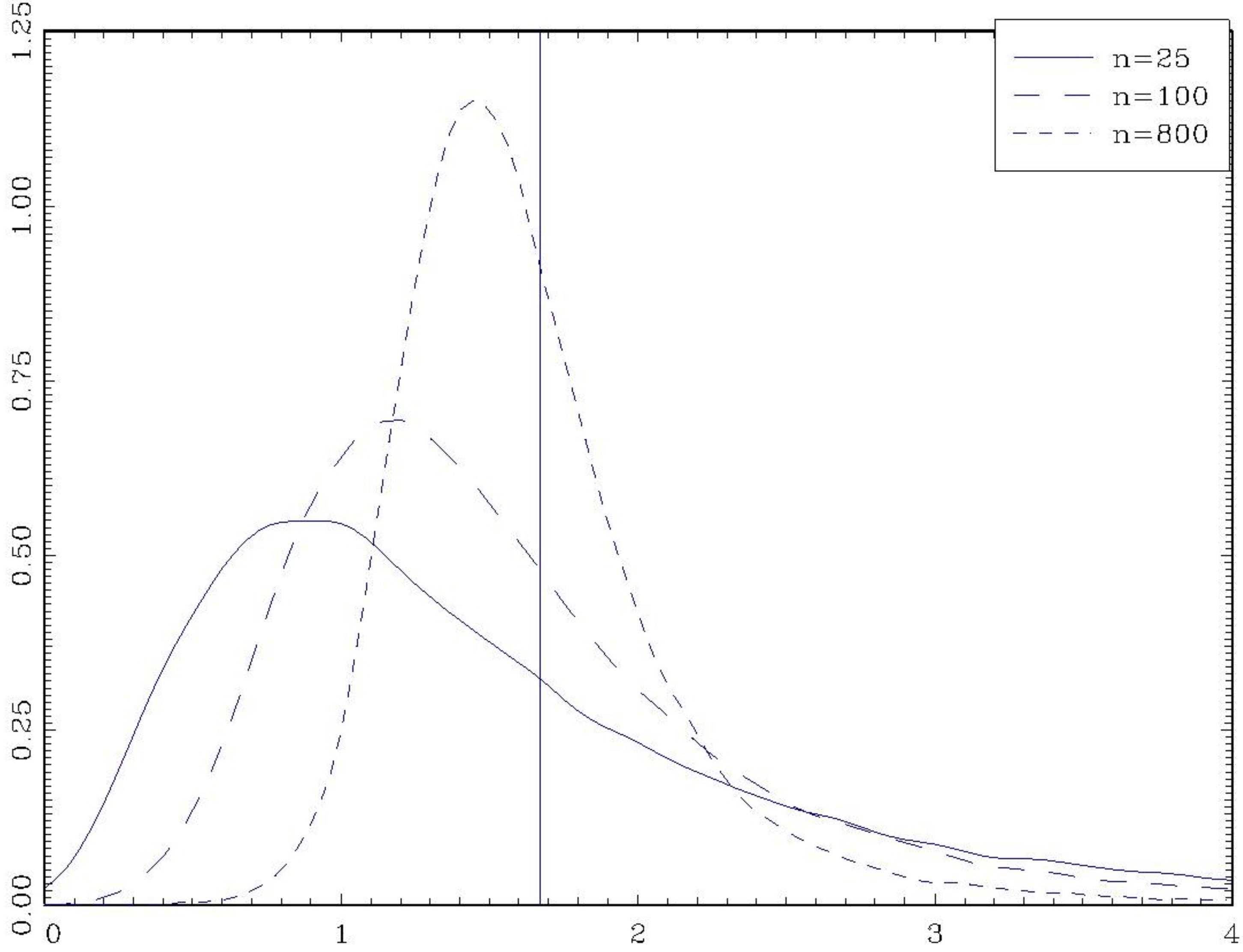
Asymptotic Normality

$$y_t = \alpha + \beta x_t + e_t$$

$$\hat{\beta} \overset{a}{\sim} N\left(\beta, \sigma_{\hat{\beta}}^2\right)$$

$$\sigma_{\hat{\beta}}^2 = \frac{1}{T} \frac{\operatorname{var}(x_t e_t)}{\left[\operatorname{var}(x_t)\right]^2}$$

Illustration of Asymptotic Normality



Time Series

- Do these results apply to time-series data?
  - Consistency
  - Asymptotic normality
  - Variance formula
- Time-series models
  - AR models, i.e., $x_t = y_{t-1}$
  - Trend and seasonal models
  - One-step and multi-step forecasting

Derivation of Variance Formula

- For simplicity
  - Assume the variables have zero mean
  - The regression has no intercept
- Model with no intercept:

$$y_t = \beta x_t + e_t$$

- Model with no intercept

$$y_t = \beta x_t + e_t$$

- OLS minimizes the sum of squares

$$\sum_{t=1}^{T} (y_t - \beta x_t)^2 = \sum_{t=1}^{T} y_t^2 - 2\beta \sum_{t=1}^{T} x_t y_t + \beta^2 \sum_{t=1}^{T} x_t^2$$

- The first-order condition is

$$0 = -2\sum_{t=1}^{T} x_t y_t + 2\hat{\beta} \sum_{t=1}^{T} x_t^2$$

- Solution

$$\hat{\beta} = \frac{\sum_{t=1}^{T} x_t y_t}{\sum_{t=1}^{T} x_t^2} = \frac{\frac{1}{T}\sum_{t=1}^{T} x_t y_t}{\frac{1}{T}\sum_{t=1}^{T} x_t^2}$$

- Now substitute $y_t = \beta x_t + e_t$

$$\hat{\beta} = \frac{\frac{1}{T}\sum_{t=1}^{T} x_t\left(x_t\beta + e_t\right)}{\frac{1}{T}\sum_{t=1}^{T} x_t^2} = \beta + \frac{\frac{1}{T}\sum_{t=1}^{T} x_t e_t}{\frac{1}{T}\sum_{t=1}^{T} x_t^2}$$

- We have

$$\hat{\beta} = \beta + \frac{\frac{1}{T}\sum_{t=1}^{T} x_t e_t}{\frac{1}{T}\sum_{t=1}^{T} x_t^2}$$

- The denominator is the sample variance (when x has mean zero), so

$$\frac{1}{T}\sum_{t=1}^{T} x_t^2 \overset{a}{\sim} \mathrm{var}(x_t)$$

- Then
$$\hat{\beta} \overset{a}{\sim} \beta + \frac{\sum_{t=1}^{T} v_t}{T \operatorname{var}(x_t)}$$

where $v_t = x_t e_t$

- Since $E(v_t) = E(x_t e_t) = 0$

then
$$\operatorname{var}(\hat{\beta}) \overset{a}{\sim} \frac{\operatorname{var}\left(\sum_{t=1}^{T} v_t\right)}{\left[T \operatorname{var}(x_t)\right]^2}$$

- From the covariance formula

$$\operatorname{var}\left(\sum_{t=1}^{T} v_t\right) = \sum_{t=1}^{T} \operatorname{var}(v_t) + \sum_{j \neq t}^{T} \operatorname{cov}(v_t, v_j)$$

- When the observations are independent, the covariances are zero.
- And since $\operatorname{var}(v_t) = \operatorname{var}(x_t e_t)$

we obtain $\operatorname{var}\left(\sum_{t=1}^{T} v_t\right) = T \operatorname{var}(x_t e_t)$

- We have found

$$\operatorname{var}\left(\hat{\beta}\right) \overset{a}{\sim} \frac{T \operatorname{var}\left(x_t e_t\right)}{\left[T \operatorname{var}\left(x_t\right)\right]^2} = \frac{\operatorname{var}\left(x_t e_t\right)}{T\left[\operatorname{var}\left(x_t\right)\right]^2}$$

as stated at the beginning

Extension to Time-Series

- The only place in this argument where we used the assumption of the *independence* of observations was to show that $v_t = x_t e_t$ has zero covariance with $v_j = x_j e_j$
- This is saying that v_t is not autocorrelated.
- When does this happen in time-series?

Unforecastable one-step errors

- **Claim**: In one-step-ahead forecasting, if the regression error is **unforecastable** then v_t is not autocorrelated
- In this case, the variance formula for the least-squares estimate is the same as in regression

$$\mathrm{var}\left(\hat{\beta}\right) \overset{a}{\sim} \frac{\mathrm{var}\left(x_t e_t\right)}{T\left[\mathrm{var}\left(x_t\right)\right]^2}$$

- Why is the claim true?
- The error is unforecastable if $E(e_t \mid \Omega_{t-1})=0$
- For simplicity suppose $x_t=1$
- Then for $t \neq j$

$$\operatorname{cov}(v_t, v_j) = E\left(e_t e_j\right) = 0$$

Summary

- In one-step-ahead time-series models, if the error is unforecastable, then least-squares estimates satisfy the asymptotic (approximate) distribution

$$\hat{\beta} \overset{a}{\sim} N\left(\beta, \sigma_{\hat{\beta}}^2\right)$$

$$\sigma_{\hat{\beta}}^2 = \frac{1}{T} \frac{\mathrm{var}(x_t e_t)}{\left[\mathrm{var}(x_t)\right]^2}$$

- As the sample size T is in the demoninator, the variance **decreases** as the sample size **increases**.
- This means that least-squares is consistent

Variance Formula

- The variance formula for the least-squares estimate takes the form

$$\sigma_{\hat{\beta}}^2 = \frac{1}{T}\frac{\mathrm{var}(x_t e_t)}{\left[\mathrm{var}(x_t)\right]^2}$$

- This formula is valid in time-series regression when the error is unforcastable

Classical Variance Formula

If we make the simplifying assumption

$$\operatorname{var}(x_t e_t) = \operatorname{var}(x_t)\operatorname{var}(e_t)$$

then

$$\sigma_{\hat{\beta}}^2 = \frac{1}{T}\frac{\operatorname{var}(x_t e_t)}{\left[\operatorname{var}(x_t)\right]^2} = \frac{1}{T}\frac{\operatorname{var}(e_t)}{\operatorname{var}(x_t)}$$

This can be a useful simplification

Homoskedasticity

- The variance simplication is valid under "conditional homoskedasticity"

$$E\left(e_t \mid \Omega_{t-1}\right) = 0$$
$$E\left(e_t^2 \mid \Omega_{t-1}\right) = \sigma^2$$

- This is a simplifying assumption made to make calculations easier, and is a conventional *assumption* in introductory econometrics courses
- It is not used in serious econometrics

Variance Formula : AR(1) Model

- Take the AR(1) model with unforecastable homoskedastic errors

$$y_t = \alpha + \beta y_{t-1} + e_t$$
$$E(e_t \mid \Omega_{t-1}) = 0$$
$$E(e_t^2 \mid \Omega_{t-1}) = \sigma^2$$

- Then the variance of the OLS estimate is

$$\sigma_{\hat{\beta}}^2 = \frac{1}{T}\frac{\mathrm{var}(e_t)}{\mathrm{var}(x_t)} = \frac{1}{T}\frac{\mathrm{var}(e_t)}{\mathrm{var}(y_{t-1})}$$

since $x_t = y_{t-1}$ in this model

AR(1) Asymptotic Variance

- We know that

$$\operatorname{var}(y_{t-1}) = \frac{\operatorname{var}(e_t)}{1-\beta^2}$$

- So

$$\sigma_{\hat{\beta}}^2 = \frac{1}{T}\frac{\operatorname{var}(e_t)}{\operatorname{var}(y_{t-1})} = \frac{1-\beta^2}{T}$$

- The asymptotic distribution is very simple

$$\hat{\beta} \overset{a}{\sim} N\left(\beta, \frac{1-\beta^2}{T}\right)$$

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- The variance is a function of the unknown true value of β
- As |β| increases, the variance decreases, so the OLS estimate is actually more precise