Components Model

- Remember that we said that it was useful to think about the components representation

$$y_t = T_t + S_t + C_t$$

- Suppose that C_t is an AR(p) process
- What model does this imply for y_t?

Trend+Cycle Model

- For simplicity, we start with the trend-cycle model

$$y_t = T_t + C_t$$

- And specify the cycle as an AR(1)

$$C_t = \beta C_{t-1} + e_t$$

Intercept only

- Suppose the trend is just an intercept

$$T_t = \mu$$

- The model is

$$y_t = \mu + C_t$$

$$C_t = \beta C_{t-1} + e_t$$

Partial Differencing

- Lag the first equation, multiply by β and subtract

$$y_t = \mu + C_t$$

$$y_{t-1} = \mu + C_{t-1}$$

$$y_t - \beta y_{t-1} = (1 - \beta)\mu + C_t - \beta C_{t-1}$$

Then use

$$C_t = \beta C_{t-1} + e_t$$

to find

$$y_t = (1 - \beta)\mu + \beta y_{t-1} + e_t$$

Equivalence with AR(1)

- Thus

$$y_t = \mu + C_t$$

implies

$$C_t = \beta C_{t-1} + e_t$$

$$y_t = \alpha + \beta y_{t-1} + e_t$$

with

$$\alpha = (1 - \beta)\mu$$

- The model is just an AR(1) with intercept

Linear Trend

- Suppose the trend is a linear time trend

$$T_t = \mu_1 + \mu_2 t$$

- Then

$$y_t = \mu_1 + \mu_2 t + C_t$$

$$C_t = \beta C_{t-1} + e_t$$

Partial Differencing

- Lag the first equation, multiply by β and subtract, and use AR(1) equation

$$y_t = \mu_1 + \mu_2 t + C_t$$

$$y_{t-1} = \mu_1 + \mu_2 (t-1) + C_{t-1}$$

$$y_t = \beta y_{t-1} + (1-\beta)\mu_1 + \beta\mu_2 + (1-\beta)\mu_2 t + e_t$$

- We find

$$y_t = \alpha + \beta_1 y_{t-1} + \beta_2 t + e_t$$

Summary : Trend+AR(1) Cycle

- When the trend is an intercept or a time trend
- The components model

$$y_t = T_t + C_t$$

$$C_t = \beta C_{t-1} + e_t$$

is equivalent with

$$y_t = T_t + \beta y_{t-1} + e_t$$

- The components model is equivalent with a regression on the trend variables and the lag

AR(p)+Trend

- A linear trend plus AR(p)

$$y_t = \mu_1 + \mu_2 t + C_t$$

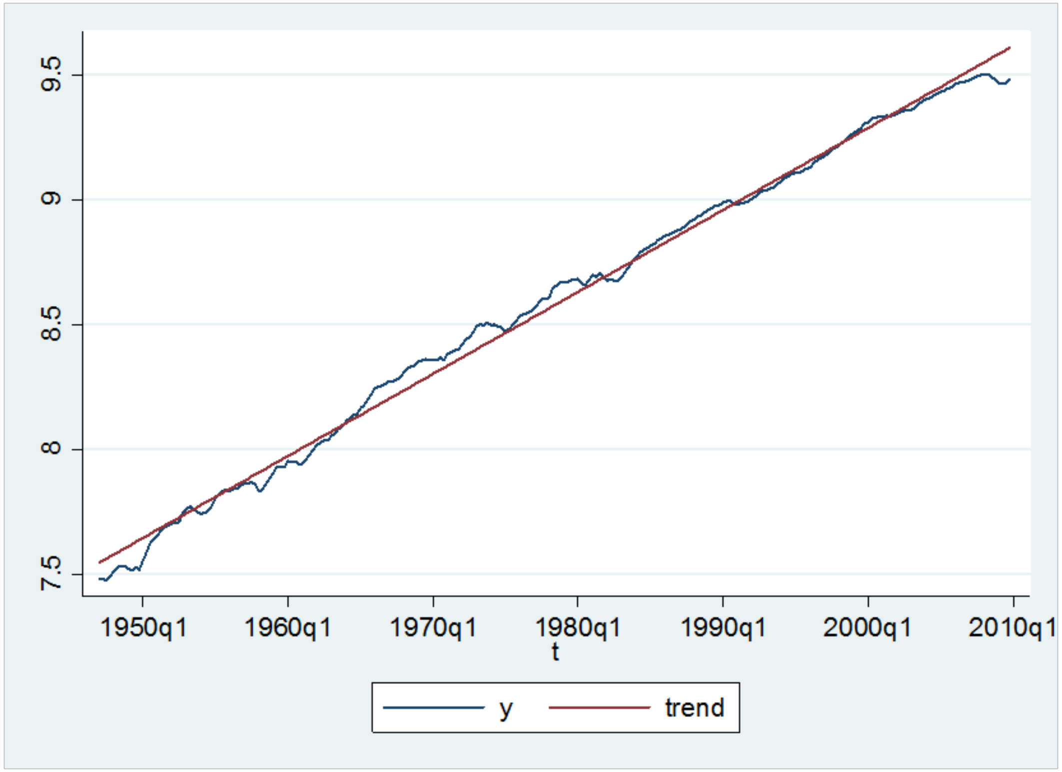
$$C_t = \beta_1 C_{t-1} + \cdots + \beta_p C_{t-p} + e_t$$

is equivalent to

$$y_t = \alpha + \gamma t + \beta_1 y_{t-1} + \cdots + \beta_p y_{t-p} + e_t$$

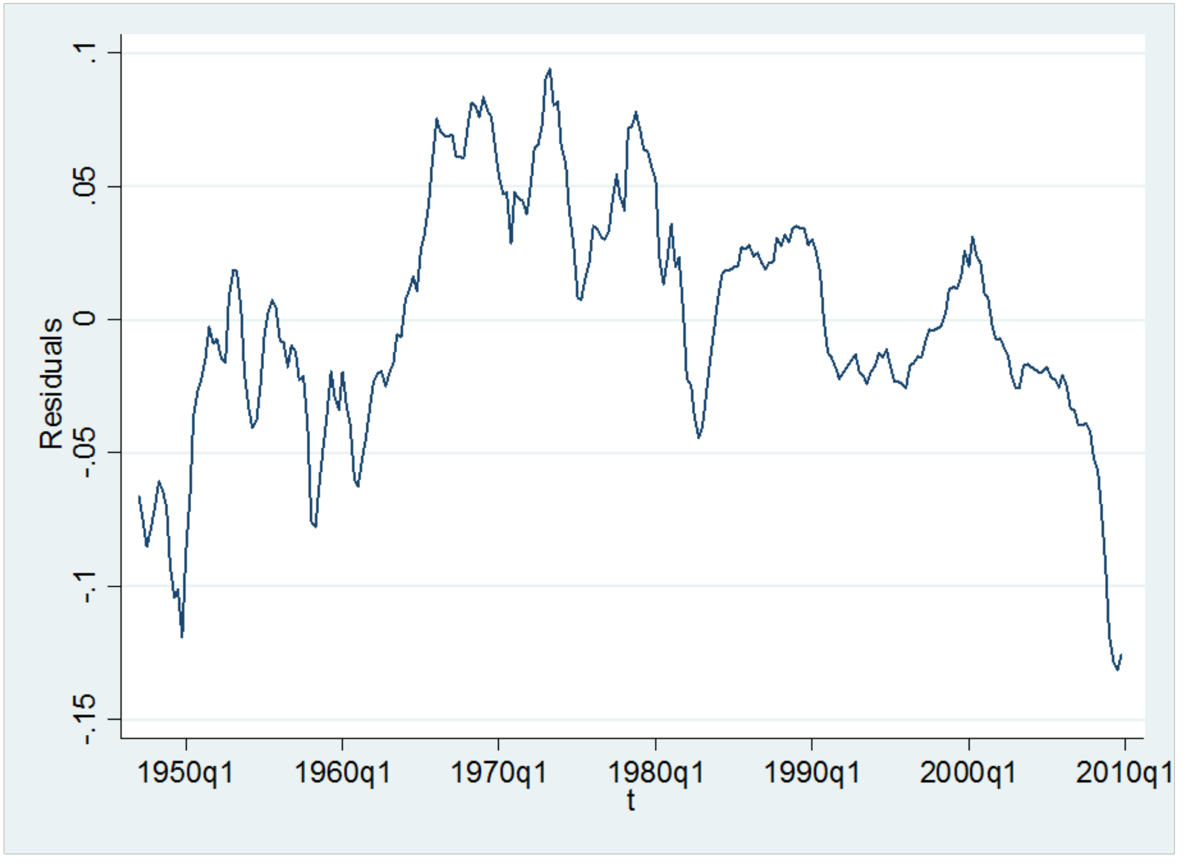
- A regression on a time trend plus *p* lags of *y*

Example: Real GDP



- Ln(rgdp) and linear trend

Residuals from Linear Trend



AR(4) on residuals

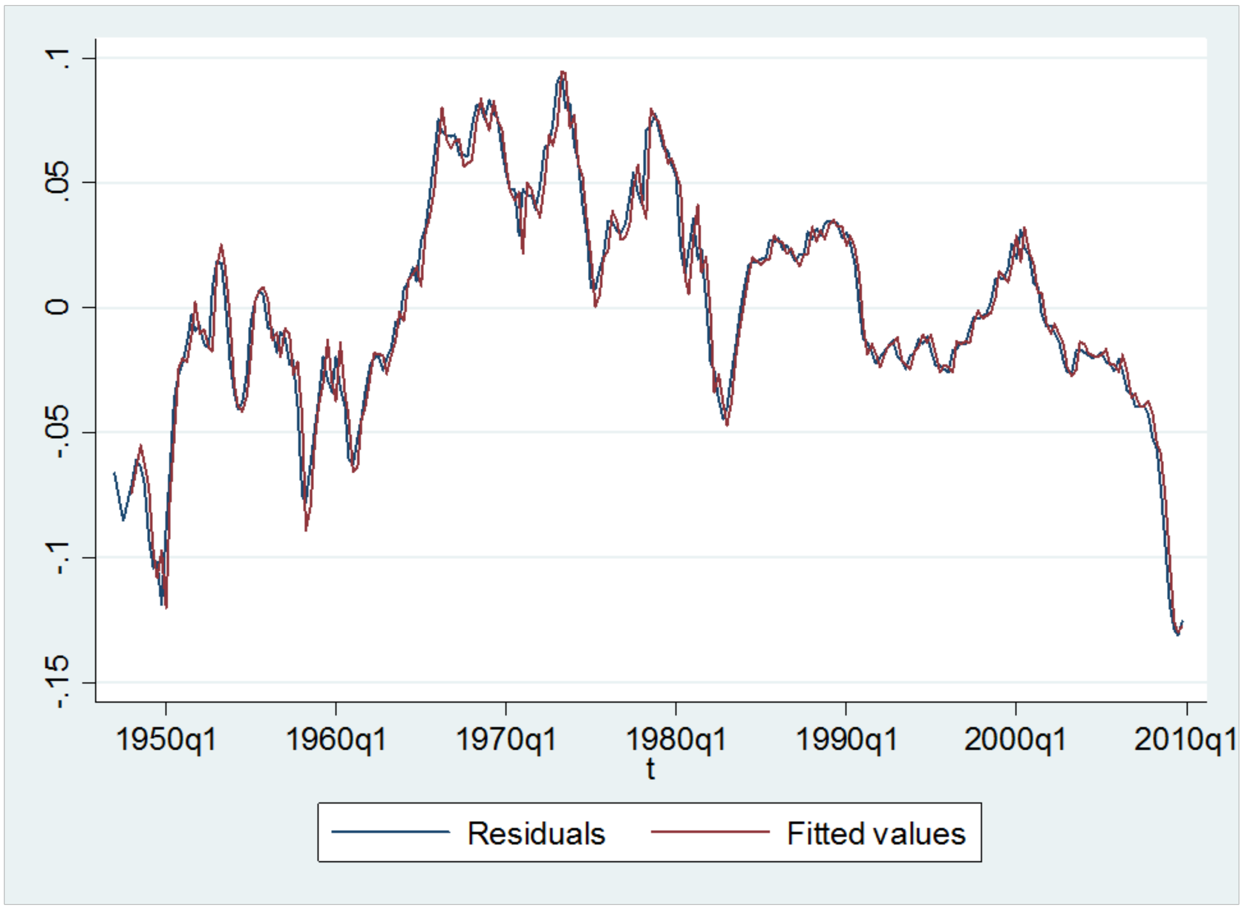
```
. reg u L(1/4).u
```

Source	SS	df	MS
Model	.460299839	4	.11507496
Residual	.020396897	243	.000083938
Total	.480696735	247	.001946141

Number of obs	= 248
F(4, 243)	= 1370.95
Prob > F	= 0.0000
R-squared	= 0.9576
Adj R-squared	= 0.9569
Root MSE	= .00916

u	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
u						
L1.	1.322292	.0637735	20.73	0.000	1.196673	1.447911
L2.	-.2053751	.1047122	-1.96	0.051	-.4116345	.0008844
L3.	-.2435104	.1047857	-2.32	0.021	-.4499145	-.0371063
L4.	.0958713	.0644404	1.49	0.138	-.0310618	.2228043
_cons	-.0000699	.0005826	-0.12	0.905	-.0012174	.0010776

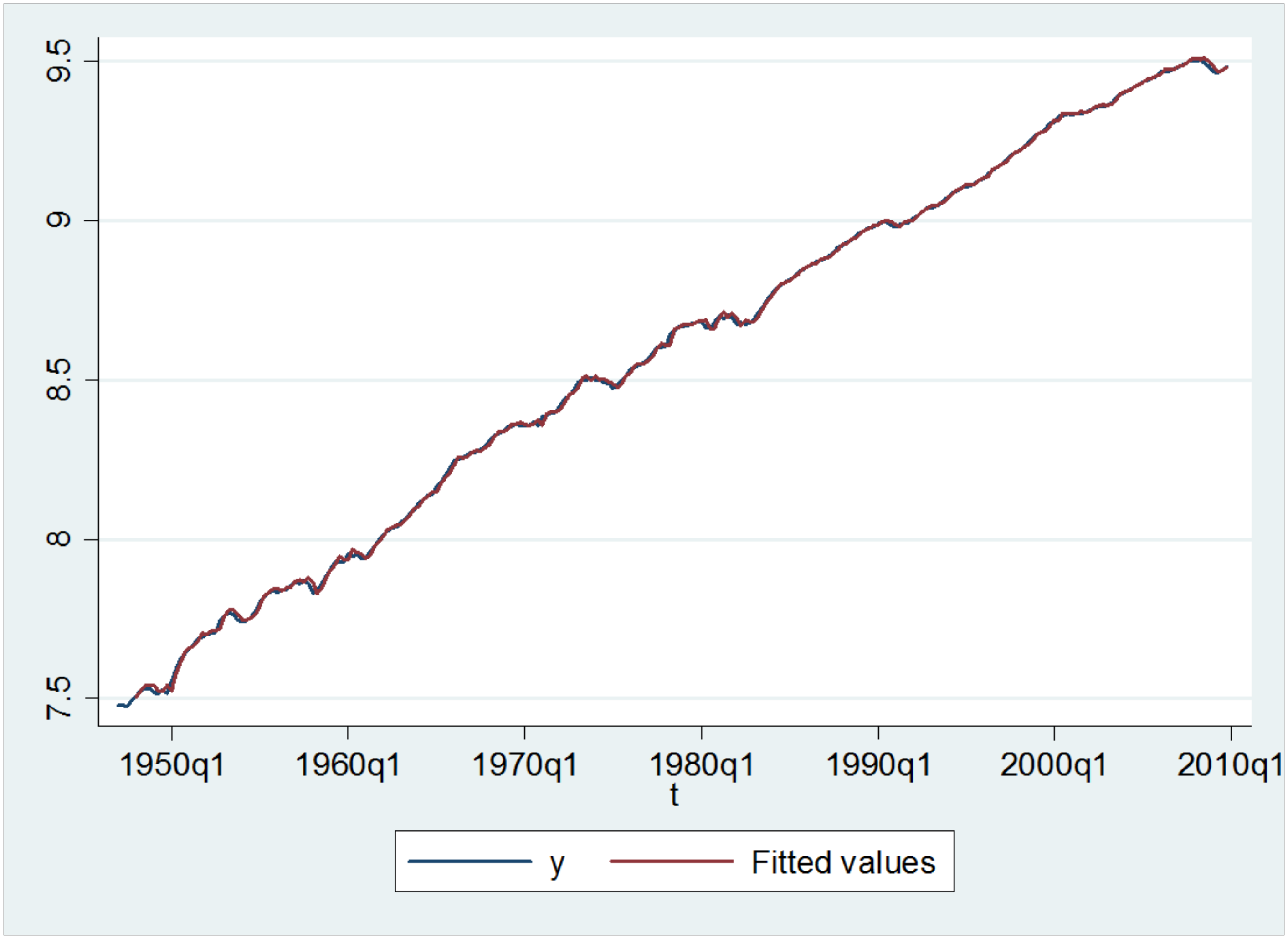
Fitted Values



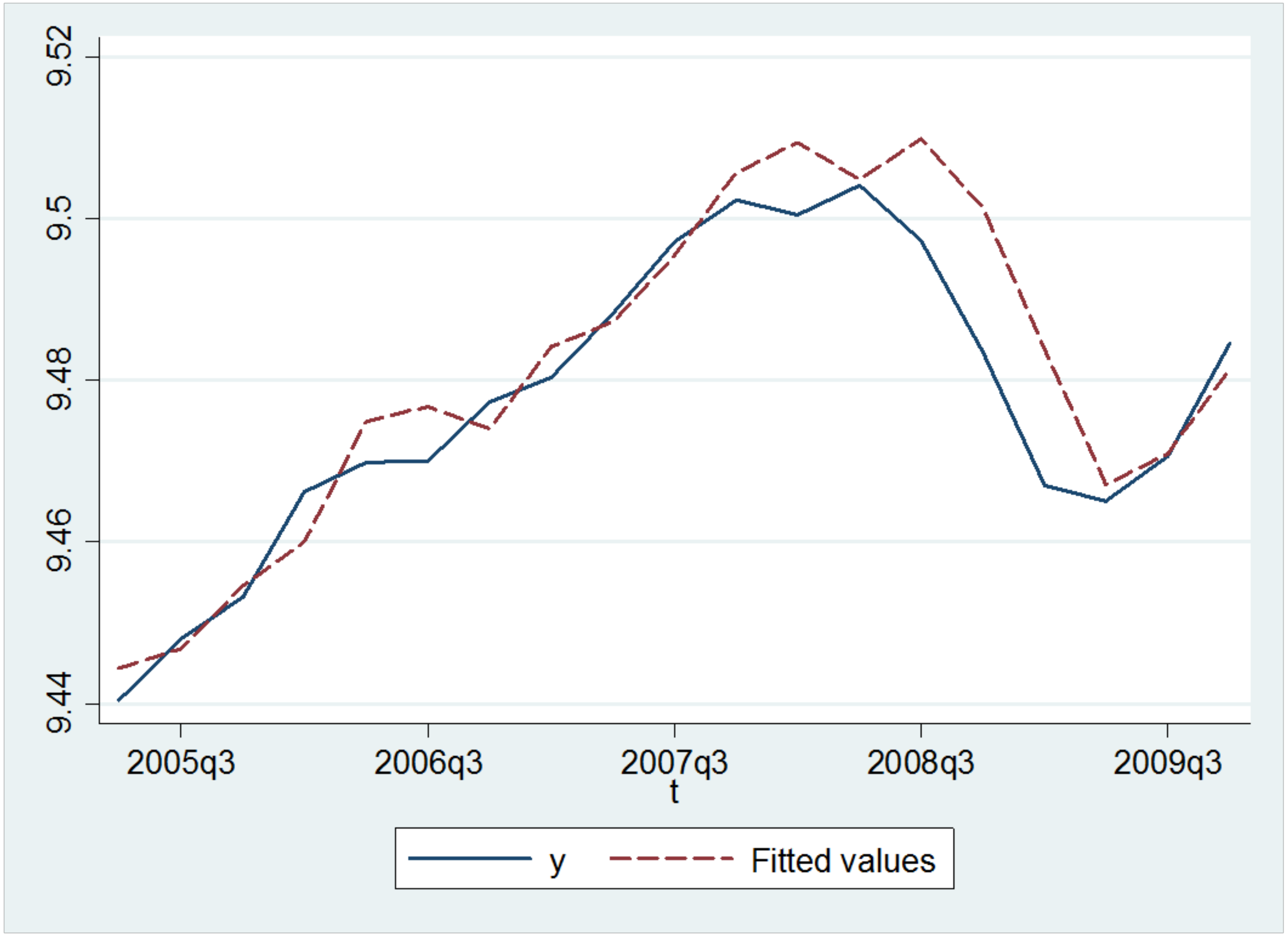
AR(4) with trend

```
. reg y t L(1/4).y

      Source |       SS       df       MS              Number of obs =     248
-------------+------------------------------           F(  5,   242) =       .
       Model |  85.7584421     5  17.1516884           Prob > F      =  0.0000
    Residual |  .020216073   242  .000083537           R-squared     =  0.9998
-------------+------------------------------           Adj R-squared =  0.9998
       Total |  85.7786581   247  .347282017           Root MSE      =  .00914

------------------------------------------------------------------------------
           y |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           t |   .0002323    .000117     1.99   0.048     1.93e-06    .0004628
             |
           y |
         L1. |   1.313422   .0639062    20.55   0.000     1.187539    1.439305
         L2. |  -.2028161   .1044767    -1.94   0.053    -.4086158    .0029836
         L3. |  -.2443394    .104537    -2.34   0.020    -.4502579   -.0384209
         L4. |   .1039944   .0645232     1.61   0.108    -.0231044    .2310931
             |
       _cons |   .2428384   .1124364     2.16   0.032     .0213595    .4643172
------------------------------------------------------------------------------
```

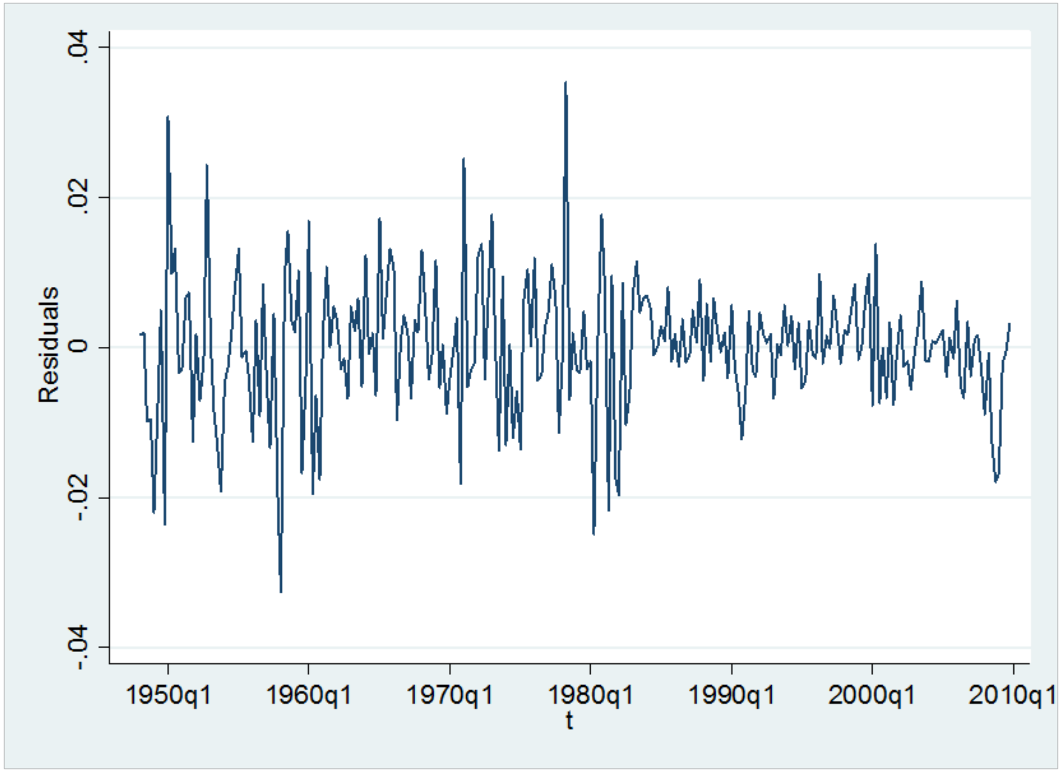
Fitted Values from AR(4) with Trend



Last 5 years


9.52
9.5
9.48
9.46
9.44
2005q3
2006q3
2007q3
2008q3
2009q3
t
y
Fitted values

Residuals



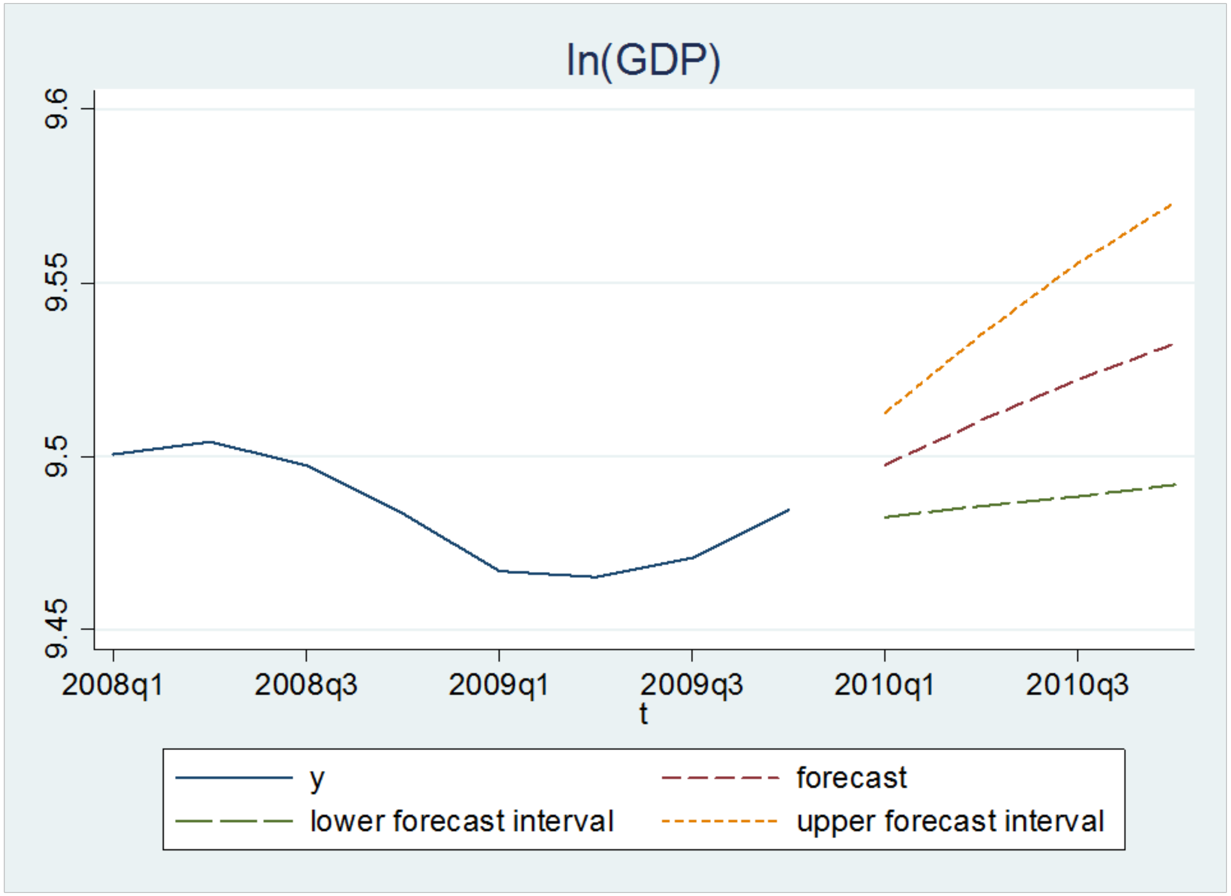
.04
.02
0
-.02
-.04
Residuals
1950q1
1960q1
1970q1
1980q1
1990q1
2000q1
2010q1
t

Forecasts

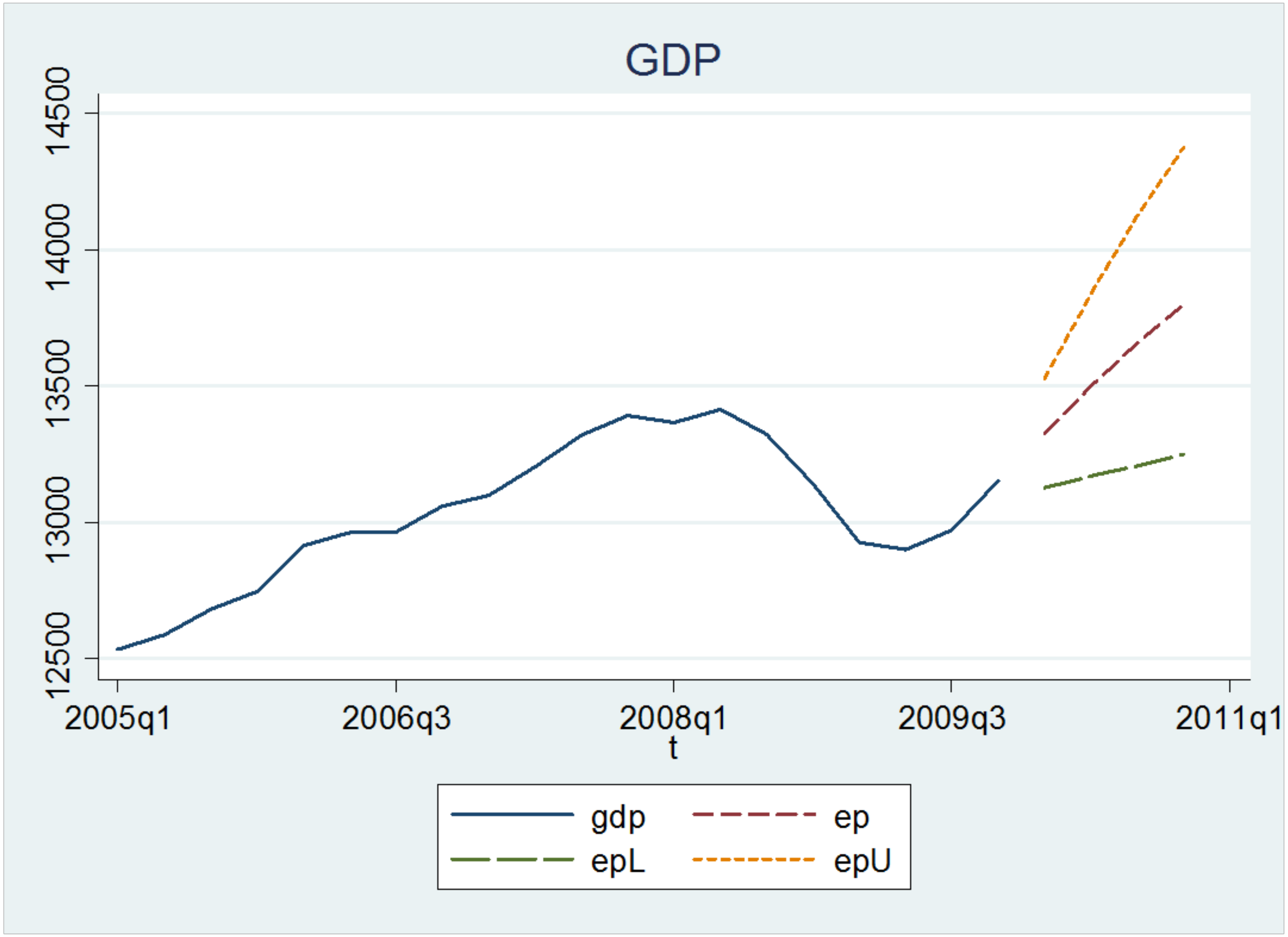
- Same as for AR(p) models, but include time trend as a regressor
- h-step forecast based on

$$y_t = \alpha + \gamma t + \beta_1 y_{t-h} + \cdots + \beta_p y_{t-h-p+1} + e_t$$

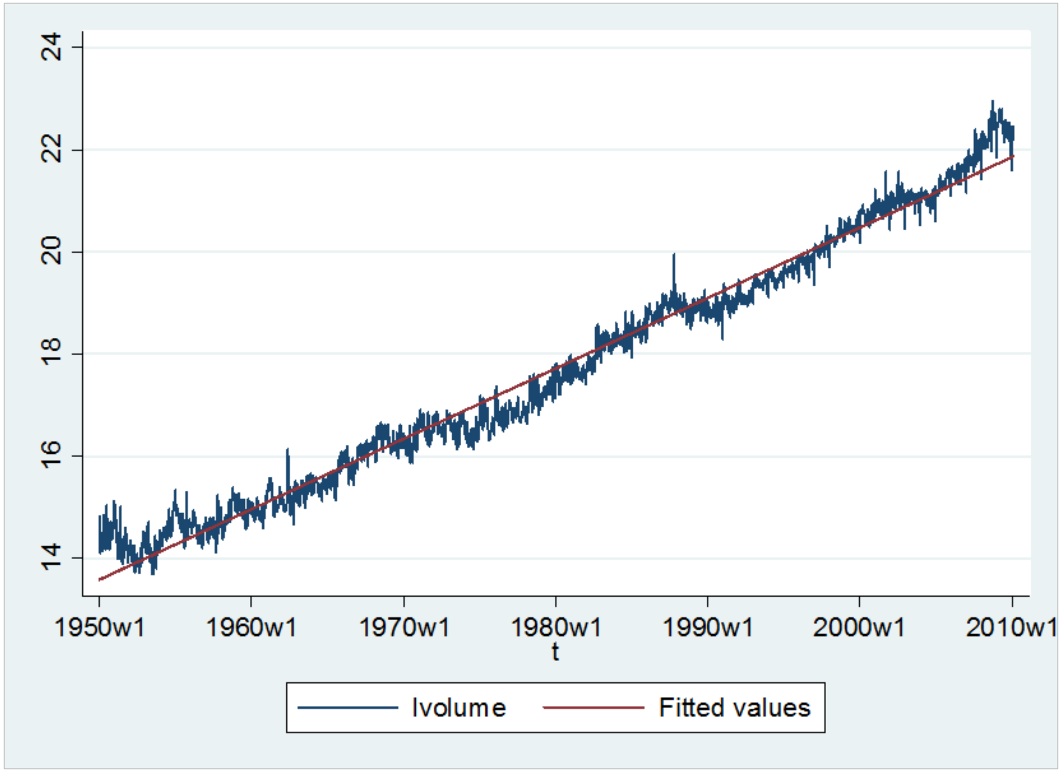
Forecast for ln(GDP) using AR(4)+trend



Forecast for GDP (use exponential)

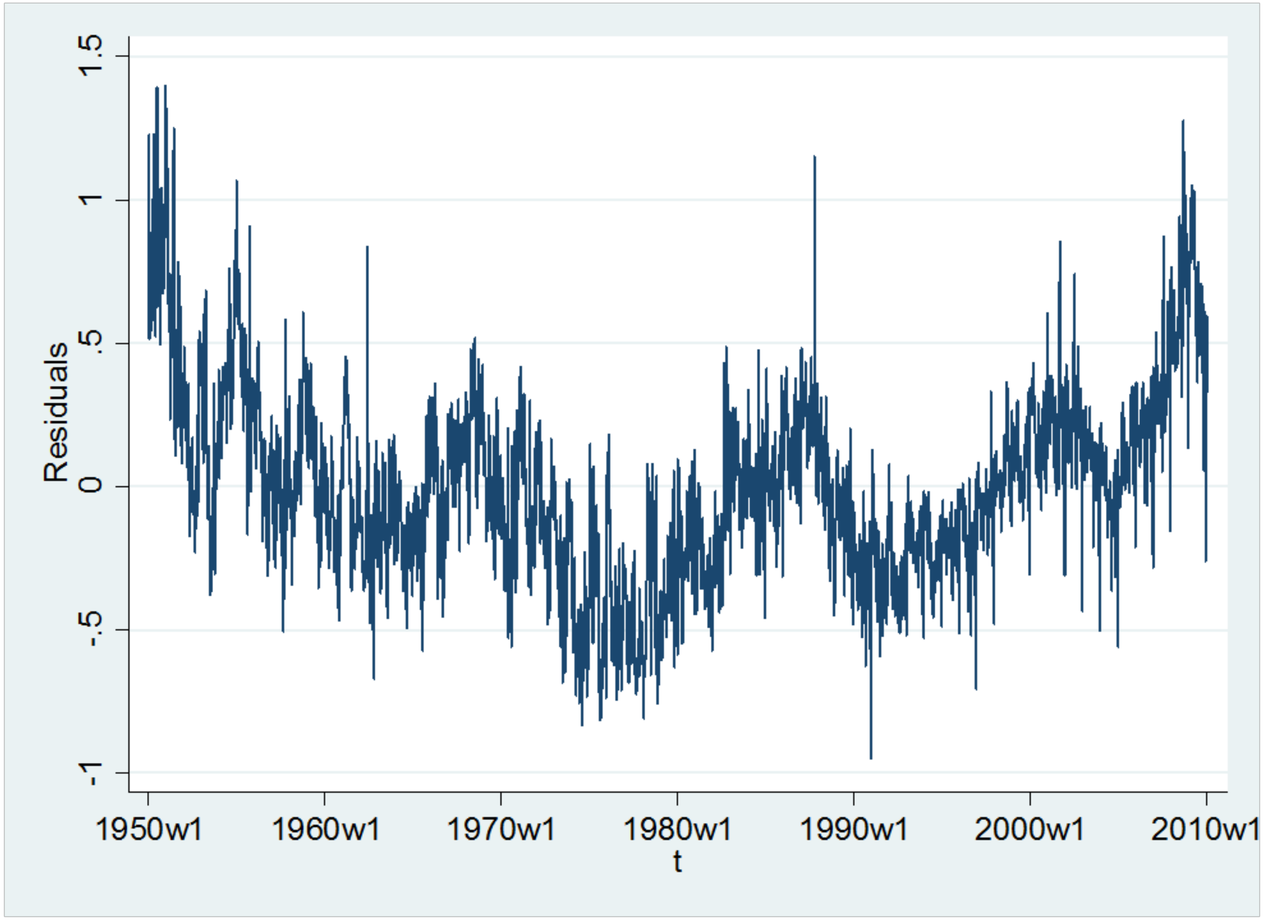


Example: Log Stock Volume



- Log Volume and Linear Trend

Residuals

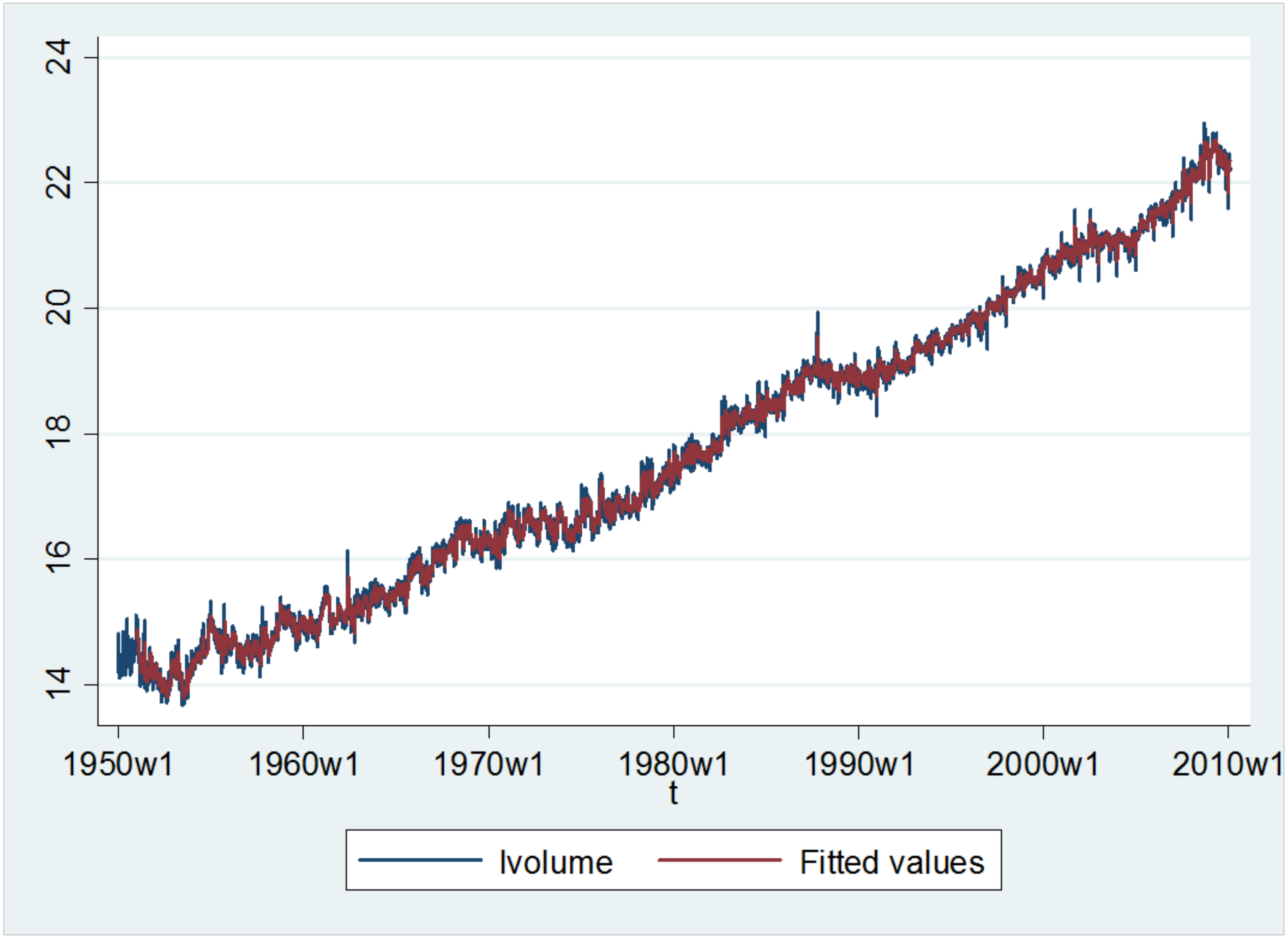


Residuals
1.5
1
.5
0
-.5
-1
1950w1
1960w1
1970w1
1980w1
1990w1
2000w1
2010w1
t

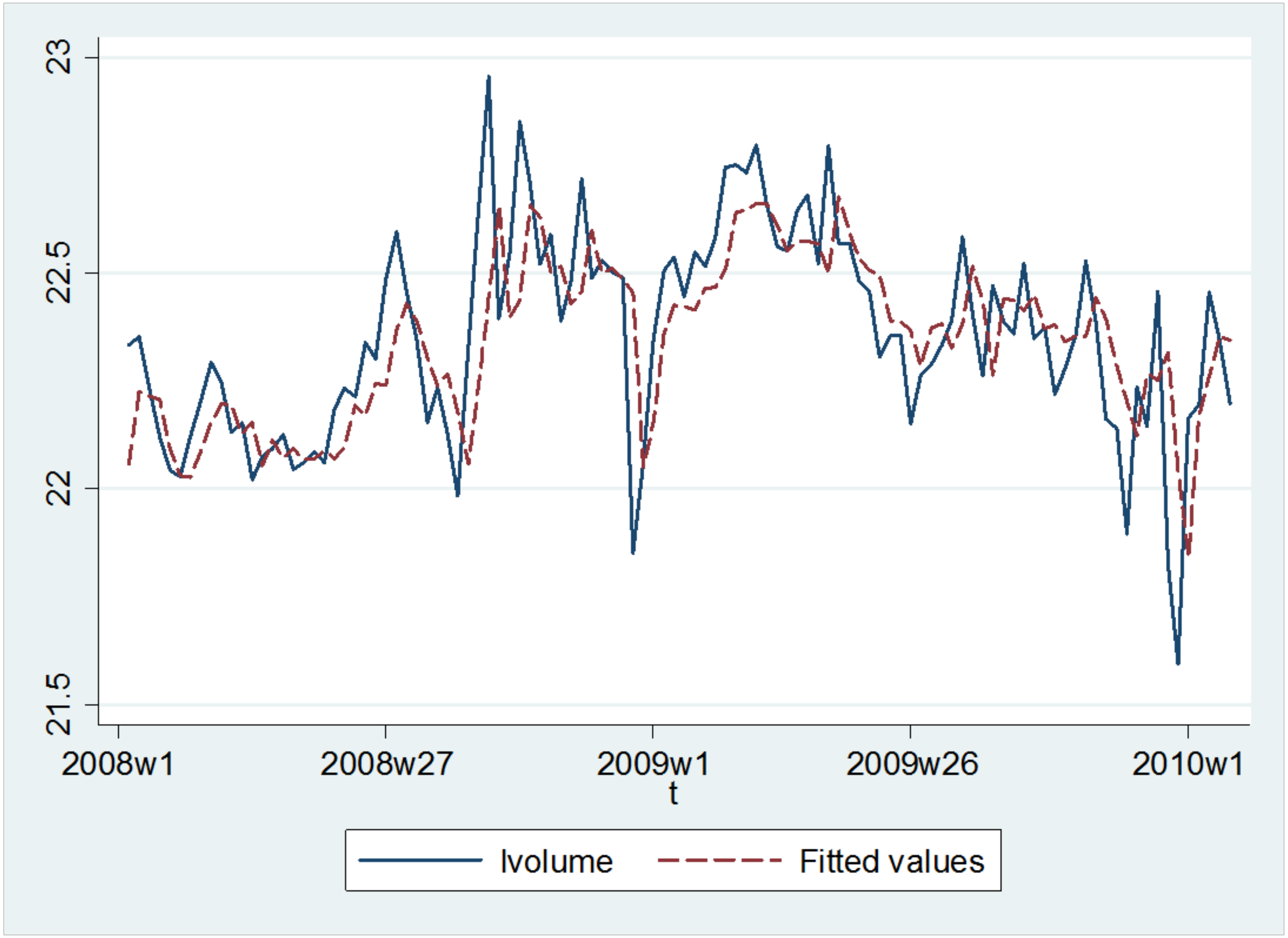
Model

- Weekly Data
- 3082 observations
- Fit AR(52)+trend

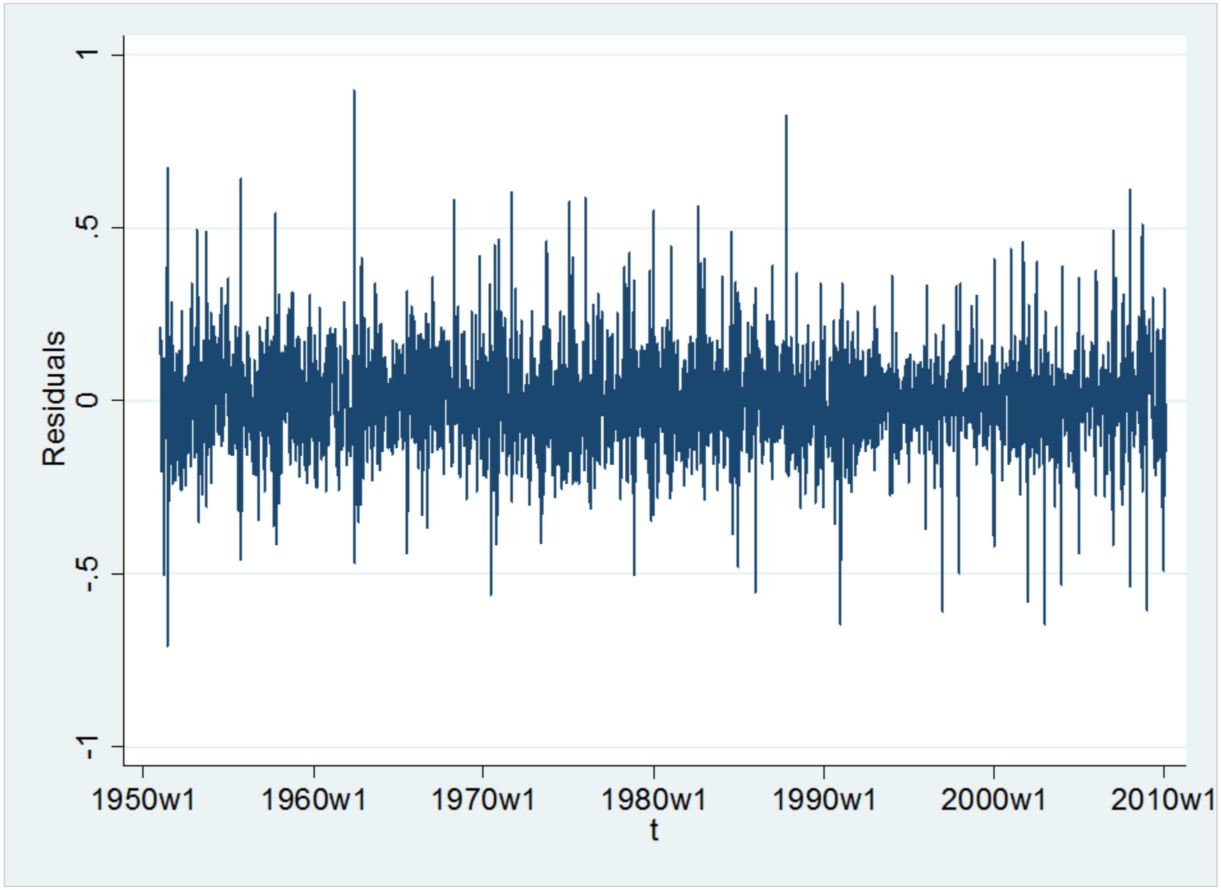
Data and Fitted



Last Two Years



Residuals



Residuals
1
.5
0
-.5
-1
1950w1
1960w1
1970w1
1980w1
1990w1
2000w1
2010w1
t

Trend Omission

- Suppose the truth is that the data have a trend, but you fit an AR model without a trend.
- What happens?
- Suppose

$$y_t = \mu_1 + \mu_2 t$$

- Then

$$y_t = y_{t-1} + \mu_2$$

Example

- Since

$$y_t = y_{t-1} + \mu_2$$

- If you estimate an AR(1), you obtain

$$y_t = \hat{\alpha} + \hat{\beta} y_{t-1} + \hat{e}_t$$

$$= \mu_2 + y_{t-1}$$

$$\hat{\alpha} = \mu_2$$

$$\hat{\beta} = 1$$

- You estimate a unit coefficient on the AR lag

General Effect of Trend Omission

- If the truth is

$$y_t = \mu_1 + \mu_2 t + \beta y_{t-1} + e_t$$

- But you estimate an AR(1) **without** a trend

$$y_t = \hat{\alpha} + \hat{\beta} y_{t-1} + \hat{e}_t$$

- Then you tend to find

$$\hat{\beta} \approx 1$$

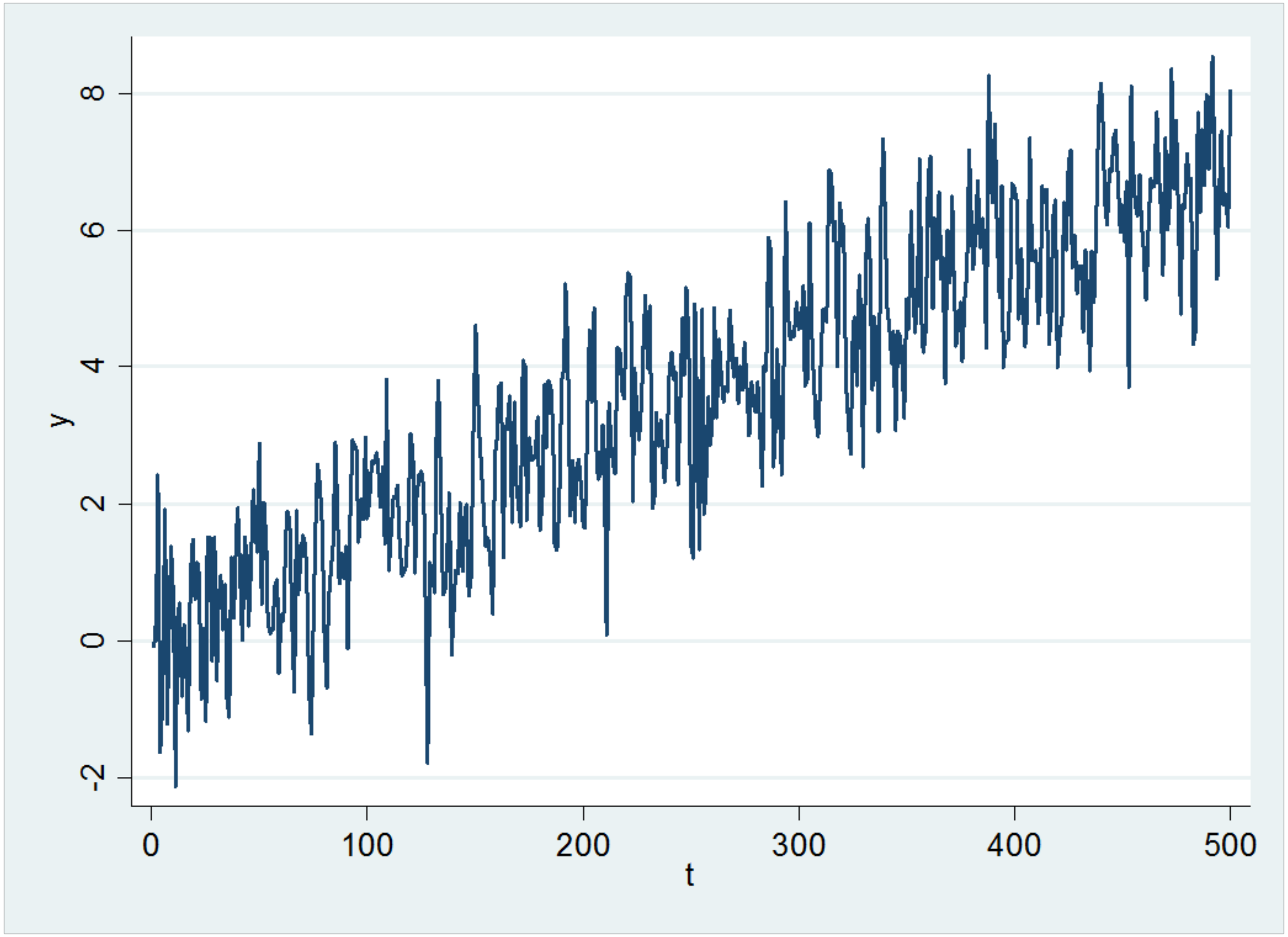
- This is due to model misspecification

Simualted Example

$$y_t = .01t + .3y_{t-1} + e_t$$

```
. gen e=rnormal(0)

. gen y=e

. replace y=.01*t+.3*L.y+e if t>1
(499 real changes made)
```

Simulated Process



Estimate AR(1) without Trend

```
. reg y L.y

      Source |       SS       df       MS              Number of obs =     499
-------------+------------------------------           F(  1,   497) = 1212.27
       Model |  2031.45086     1  2031.45086           Prob > F      =  0.0000
    Residual |  832.842361   497  1.67573916           R-squared     =  0.7092
-------------+------------------------------           Adj R-squared =  0.7086
       Total |  2864.29322   498  5.75159281           Root MSE      =  1.2945

------------------------------------------------------------------------------
           y |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           y |
         L1. |   .8423331   .0241927    34.82   0.000     .7948006    .8898657
             |
       _cons |   .5786825   .1039191     5.57   0.000     .3745076    .7828573
------------------------------------------------------------------------------
```

- The estimated AR(1) coefficient is 0.84, much too large (true value was 0.3)

Estimate AR(1) with Trend

```
. reg y t L.y

      Source |       SS       df       MS              Number of obs =     499
-------------+------------------------------           F(  2,   496) = 1032.23
       Model |  2309.43596     2  1154.71798           Prob > F      =  0.0000
    Residual |  554.857262   496  1.11866383           R-squared     =  0.8063
-------------+------------------------------           Adj R-squared =  0.8055
       Total |  2864.29322   498  5.75159281           Root MSE      =  1.0577

------------------------------------------------------------------------------
           y |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           t |   .0114629   .0007272    15.76   0.000     .0100342    .0128916
             |
           y |
         L1. |   .2274365   .0437293     5.20   0.000      .141519     .313354
             |
       _cons |  -.1060309    .095372    -1.11   0.267    -.2934137     .081352
------------------------------------------------------------------------------
```

- The estimated AR coef is 0.23, close to the true 0.3
- The estimated trend coef is 0.11, close to the true 0.10
- The root MSE decreases from 1.29 to 1.06