AR(2) Process

- An autoregressive process of order 2, or AR(2) is

$$y_t = \beta_1 y_{t-1} + \beta_2 y_{t-2} + e_t$$

where e_t is WN(0, σ^2)

- Using the lag operator

$$\left(1 - \beta_1 L - \beta_2 L^2\right) y_t = e_t$$

AR(2) Process with Intercept

- An autoregressive process of order 2, or AR(2) is

$$y_t = \alpha + \beta_1 y_{t-1} + \beta_2 y_{t-2} + e_t$$

or

$$\left(1 - \beta_1 L - \beta_2 L^2\right) y_t = \alpha + e_t$$

Multiplier-Accelerator Model

- Due to Paul Samuelson
- Output *Y*, Consumption *C* and Investment *I*
- Aggregate Income $Y_t = C_t + I_t$
- Consumption Multiplier $C_t = a_0 + a_1 Y_{t-1}$
- Investment Accelerator $I_t = b\left(C_t - C_{t-1}\right) + e_t$
- Combine to find process for output

$$Y_t = a_0 + a_1\left(1 + b\right)Y_{t-1} - a_1 b Y_{t-2} + e_t$$

an AR(2) Process

Multiplier-Accelerator

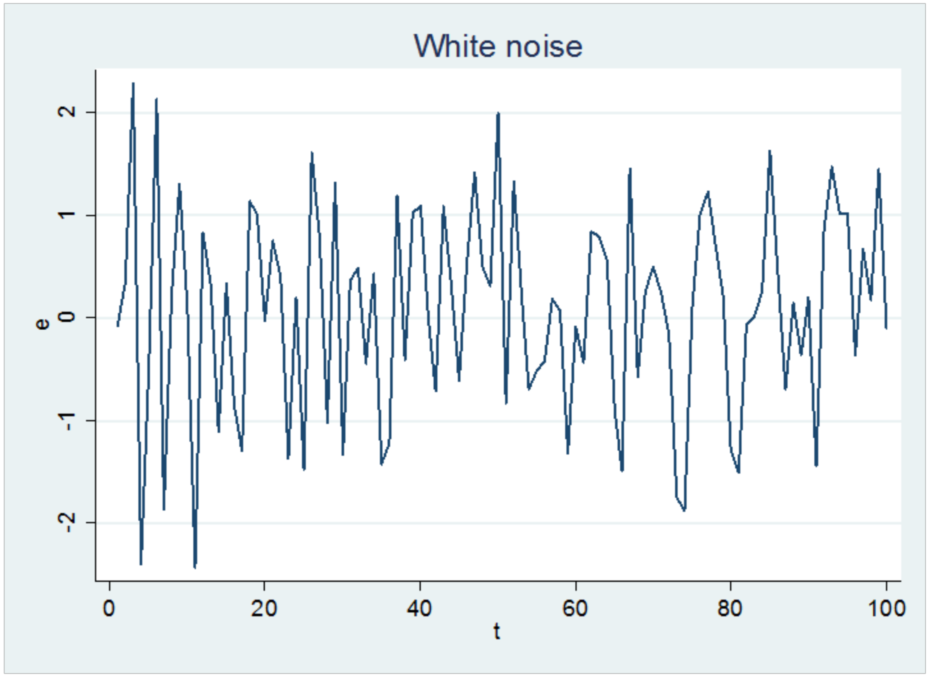
- Example

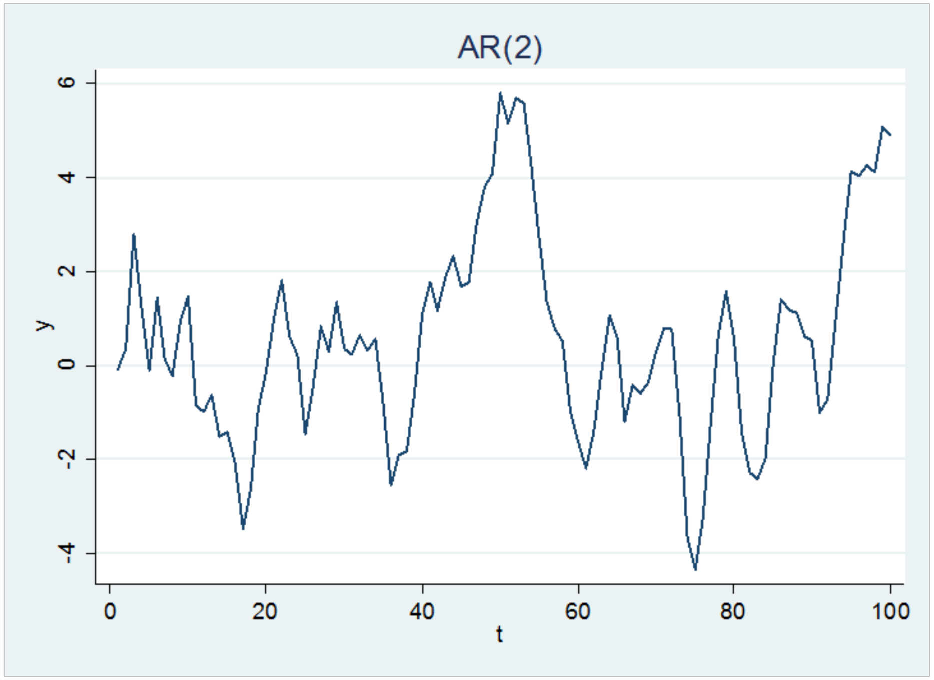
$$C_t = 0.9Y_{t-1}$$

$$I_t = .5(C_t - C_{t-1}) + e_t$$

$$Y_t = 1.35Y_{t-1} - 0.45Y_{t-2} + e_t$$

Simulated Example





Simulating AR processes in STATA

```
. set obs 100
obs was 0, now 100

. gen t=_n

. tsset t
        time variable:  t, 1 to 100
                delta:  1 unit

. gen e=rnormal()

. gen y=e

. replace y=1.35*L.y-.45*L2.y+e if t>2
(98 real changes made)
```

Stationarity of AR(2)

- The AR(2) process is stationary if we can invert the lag polynomial to write it as a general liner process, if

$$\left(1-\beta_1 L-\beta_2 L^2\right)y_t=e_t$$

$$y_t=\left(1-\beta_1 L-\beta_2 L^2\right)^{-1}e_t$$

- When is this valid?

Factors

- Write the polynomial $(1-\beta_1 L-\beta_2 L^2)$ as factors

$$1-\beta_1 L-\beta_2 L^2=(1-\lambda_1 L)(1-\lambda_2 L)$$

- Then

$$\begin{aligned} y_t &= \left(1-\beta_1 L-\beta_2 L^2\right)^{-1} e_t \\ &= (1-\lambda_1 L)^{-1}(1-\lambda_2 L)^{-1} e_t \end{aligned}$$

- This is valid if both $(1-\lambda_1 L)$ and $(1-\lambda_2 L)$ are invertible
- This is when $|\lambda_1|<1$ and $|\lambda_2|<1$

Stationary Roots

- λ_1 and λ_2 are the inverses of the roots of the polynomial $(1-\beta_1 L-\beta_2 L^2)$
- They can be real or complex
- If $|\lambda_1|<1$ and $|\lambda_2|<1$ we say they "are within the unit circle"
- The AR(2) is stationary if the inverse roots are within the unit circle (are less than one in absolute value)

Necessary Condition

- The polynomial $(1-\beta_1 L-\beta_2 L^2)$ has a unit root if it equals 1 when L=1
- In this case, $1-\beta_1-\beta_2=1$
- Or $\beta_1+\beta_2=1$
- This means that the AR(2) is nonstationary if the sum of the AR coefficients equals 1

Multiplier Example

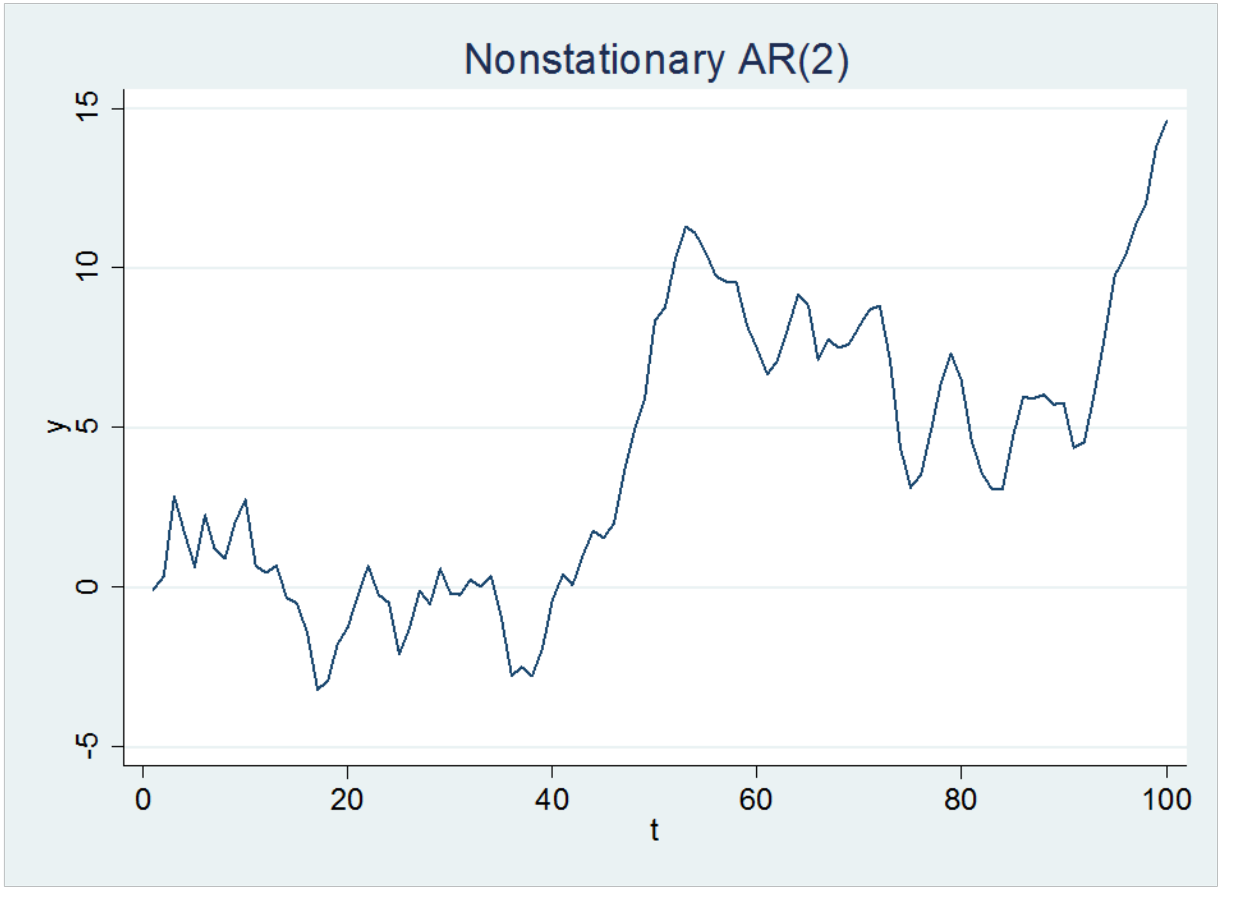
- In the model

$$Y_t = a_0 + a_1(1+b)Y_{t-1} - a_1 b Y_{t-2} + e_t$$

  the sum of the coefficients is a_1, the consumption coefficient.

- In this model, if a_1=1, then the output process has a unit root, it is nonstationary

Simulated example with a_1=1

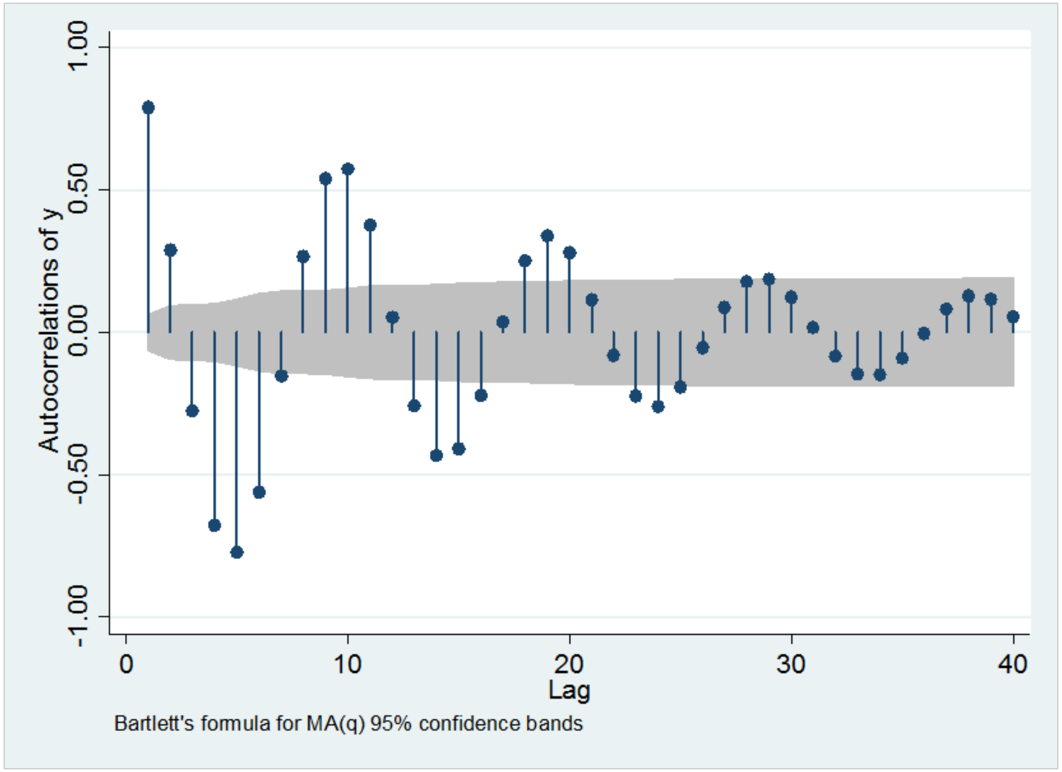


Autocorrelation of AR(2)

- The autocorrelations of an AR(2) can be much more complicated than that of an AR(1)
- Take the example

$$Y_t = 1.5Y_{t-1} - 0.9Y_{t-2} + e_t$$

Autocorrelation Function of AR(2)



Alternative expression

$$
\begin{aligned}
y_t &= \beta_1 y_{t-1} + \beta_2 y_{t-2} + e_t \\
&= (\beta_1 + \beta_2) y_{t-1} - \beta_2 (y_{t-1} - y_{t-2}) + e_t \\
&= (\beta_1 + \beta_2) y_{t-1} - \beta_2 \Delta y_{t-1} + e_t
\end{aligned}
$$

- The AR(2) can be written as a function of the lagged **value** and the lagged **change**
- These are equivalent expressions

Estimation of AR(2)

- Least Squares Regression

$$y_t = \hat{\alpha} + \hat{\beta}_1 y_{t-1} + \hat{\beta}_2 y_{t-2} + \hat{e}_t$$

Example: Unemployment Rate

```
. reg ur L.ur L2.ur

      Source |       SS       df       MS              Number of obs =     743
-------------+------------------------------           F(  2,   740) =19269.45
       Model |  1772.53707     2  886.268537           Prob > F      =  0.0000
    Residual |  34.0351553   740  .045993453           R-squared     =  0.9812
-------------+------------------------------           Adj R-squared =  0.9811
       Total |  1806.57223   742  2.43473346           Root MSE      =  .21446

------------------------------------------------------------------------------
          ur |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
          ur |
         L1. |   1.117757   .0364384    30.68   0.000     1.046222    1.189292
         L2. |  -.1255343   .0365794    -3.43   0.001    -.1973461   -.0537226
             |
       _cons |   .0508359   .0298172     1.70   0.089    -.0077005    .1093723
------------------------------------------------------------------------------
```

Example: GDP Growth

```
. reg gdp L.gdp L2.gdp

      Source |       SS       df       MS              Number of obs =     249
-------------+------------------------------           F(  2,   246) =   19.24
       Model |  567.993088     2  283.996544           Prob > F      =  0.0000
    Residual |  3631.11257   246  14.7606202           R-squared     =  0.1353
-------------+------------------------------           Adj R-squared =  0.1282
       Total |  4199.10566   248  16.9318776           Root MSE      =   3.842

------------------------------------------------------------------------------
         gdp |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         gdp |
         L1. |   .3268403    .063548     5.14   0.000     .2016727    .4520078
         L2. |   .0870349   .0634413     1.37   0.171    -.0379225    .2119923
             |
       _cons |   1.979628   .3409666     5.81   0.000     1.308041    2.651214
------------------------------------------------------------------------------
```

Alternative Command

```
. reg gdp L(1/2).gdp

      Source |       SS       df       MS              Number of obs =     249
-------------+------------------------------           F(  2,   246) =   19.24
       Model |  567.993088     2  283.996544           Prob > F      =  0.0000
    Residual |  3631.11257   246  14.7606202           R-squared     =  0.1353
-------------+------------------------------           Adj R-squared =  0.1282
       Total |  4199.10566   248  16.9318776           Root MSE      =   3.842

------------------------------------------------------------------------------
         gdp |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         gdp |
         L1. |   .3268403     .063548     5.14   0.000     .2016727    .4520078
         L2. |   .0870349    .0634413     1.37   0.171    -.0379225    .2119923
             |
       _cons |   1.979628    .3409666     5.81   0.000     1.308041    2.651214
------------------------------------------------------------------------------
```

- The “L(1/2).gdp” means
  - “regress on lags 1 through 2 of gdp”

One-Step-Ahead Forecast

- The optimal forecast for *T*+1 given *T* is

$$\hat{y}_{T+1|T} = \alpha + \beta_1 y_T + \beta_2 y_{T-1}$$

- The forecast using the estimates is

$$\hat{y}_{T+1|T} = \hat{\alpha} + \hat{\beta}_1 y_T + \hat{\beta}_2 y_{T-1}$$

Two-Step-Ahead Forecast

- The optimal two-step forecast is a linear function of two lags, with a MA(1) forecast error

$$
\begin{aligned}
y_t &= \alpha + \beta_1 y_{t-1} + \beta_2 y_{t-2} + e_t \\
&= \alpha + \beta_1 \left(\alpha + \beta_1 y_{t-2} + \beta_2 y_{t-3} + e_{t-1} \right) + \beta_2 y_{t-2} + e_t \\
&= \left(1 + \beta_1 \right) \alpha + \left(\beta_1^2 + \beta_2 \right) y_{t-2} + \beta_1 \beta_2 y_{t-3} + e_t + \beta_1 e_{t-1}
\end{aligned}
$$

Three-Step-Ahead Forecast

$$
\begin{aligned}
y_t &= (1+\beta_1)\alpha + (\alpha\beta_1 + \beta_1^2)y_{t-2} + \beta_1\beta_2 y_{t-3} + e_t + \beta_1 e_{t-1} \\
&= (1+\beta_1)\alpha + (\beta_1^2 + \beta_2)(\alpha + \beta_1 y_{t-3} + \beta_2 y_{t-4} + e_{t-2}) \\
&\quad + \beta_1\beta_2 y_{t-3} + e_t + \beta_1 e_{t-1} \\
&= (1+\beta_1 + \beta_1^2 + \beta_2)\alpha + (\beta_1^3 + 2\beta_1\beta_2)y_{t-3} + (\beta_1^2\beta_2 + \beta_2^2)y_{t-4} \\
&\quad + e_t + \beta_1 e_{t-1} + (\beta_1^2 + \beta_2)e_{t-2}
\end{aligned}
$$

Iterated Rule

$$\hat{y}_{T+1|T} = \hat{\alpha} + \hat{\beta}_1 y_T + \hat{\beta}_2 y_{T-1}$$

$$\hat{y}_{T+2|T} = \hat{\alpha} + \hat{\beta}_1 \hat{y}_{T+1|T} + \hat{\beta}_2 y_T$$

$$\hat{y}_{T+3|T} = \hat{\alpha} + \hat{\beta}_1 \hat{y}_{T+2|T} + \hat{\beta}_2 \hat{y}_{T+1|T}$$

$$\hat{y}_{T+h|T} = \hat{\alpha} + \hat{\beta}_1 \hat{y}_{T+h-1|T} + \hat{\beta}_2 \hat{y}_{T+h-1|T}$$

Direct Forecast

- Estimation is by least squares on two lags, h periods in past
- Forecast is least square prediction using final two observations

$$y_t = \hat{\alpha} + \hat{\beta}_1 y_{t-h} + \hat{\beta}_2 y_{t-h-1} + \hat{u}_t$$

$$\hat{y}_{T+h|T} = \hat{\alpha} + \hat{\beta}_1 y_T + \hat{\beta}_2 y_{T-1}$$

AR(p) Process

- An autoregressive process of order p, or AR(p) is

$$y_t = \alpha + \beta_1 y_{t-1} + \beta_2 y_{t-2} + \cdots + \beta_p y_{t-p} + e_t$$

or

$$\left(1 - \beta_1 L - \beta_2 L^2 - \cdots - \beta_p L^p\right) y_t = \alpha + e_t$$

Stationarity

- The process is stationary if the inverses of the roots of the polynomial

$$\left(1-\beta_1 L-\beta_2 L^2-\cdots-\beta_p L^p\right)$$

are less than one (in absolute value)

- A necessary condition is that

$$\beta_1+\beta_2+\cdots+\beta_p<1$$

Alternative Representation

- We can write it as

$$y_t = \alpha + \gamma_1 y_{t-1} + \gamma_2 \Delta y_{t-1} + \cdots + \gamma_p \Delta y_{t-p+1} + e_t$$

or

$$\Delta y_t = \alpha + (\gamma_1 - 1) y_{t-1} + \gamma_2 \Delta y_{t-1} + \cdots + \gamma_p \Delta y_{t-p+1} + e_t$$

- These are equivalent forecasting models

Estimation of AR(p)

- Least Squares

$$y_t = \hat{\alpha} + \hat{\beta}_1 y_{t-1} + \hat{\beta}_1 y_{t-2} + \cdots + \hat{\beta}_p y_{t-p} + \hat{e}_t$$

Example: Unemployment Rate

```
. reg ur L(1/12).ur

      Source |       SS       df       MS              Number of obs =     733
-------------+------------------------------           F( 12,   720) = 3730.91
       Model |  1742.1994    12  145.183284           Prob > F      =  0.0000
    Residual |  28.0178232   720  .038913643           R-squared     =  0.9842
-------------+------------------------------           Adj R-squared =  0.9839
       Total |  1770.21723   732  2.41832954           Root MSE      =  .19727

------------------------------------------------------------------------------
          ur |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
          ur |
         L1. |   .9906563   .0372599    26.59   0.000     .9175053    1.063807
         L2. |   .2451499   .0521534     4.70   0.000      .142759    .3475408
         L3. |   -.073963   .0527587    -1.40   0.161    -.1775422    .0296162
         L4. |  -.0774851   .0528751    -1.47   0.143    -.1812928    .0263227
         L5. |   .0295265   .0528482     0.56   0.577    -.0742285    .1332815
         L6. |  -.1293371   .0528035    -2.45   0.015    -.2330043   -.0256698
         L7. |  -.0304848   .0529059    -0.58   0.565    -.1343529    .0733834
         L8. |   .0416879   .0527905     0.79   0.430    -.0619539    .1453296
         L9. |  -.0201221   .0528201    -0.38   0.703    -.1238219    .0835776
        L10. |  -.1136112   .0526531    -2.16   0.031    -.2169831   -.0102394
        L11. |   .1582629   .0520668     3.04   0.002      .056042    .2604839
        L12. |  -.0372219    .037227    -1.00   0.318    -.1103084    .0358645
             |
       _cons |   .0991343   .0306168     3.24   0.001     .0390254    .1592431
------------------------------------------------------------------------------
```

Iterated Forecasts

$$\hat{y}_{T+h|T} = \hat{\alpha} + \hat{\beta}_1 \hat{y}_{T+h-1|T} + \hat{\beta}_2 \hat{y}_{T+h-1|T} + \cdots + \hat{\beta}_p \hat{y}_{T-p+1|T}$$

Direct Forecasts

$$y_t = \hat{\alpha} + \hat{\beta}_1 y_{t-h} + \hat{\beta}_1 y_{t-h-1} + \cdots + \hat{\beta}_p y_{t-h-p+1} + \hat{u}_t$$

$$\hat{y}_{T+h|T} = \hat{\alpha} + \hat{\beta}_1 y_T + \hat{\beta}_2 y_{T-1} + \cdots + \hat{\beta}_p y_{T-p}$$

Example: Unemployment Rate

```
. reg ur L(12/23).ur
```

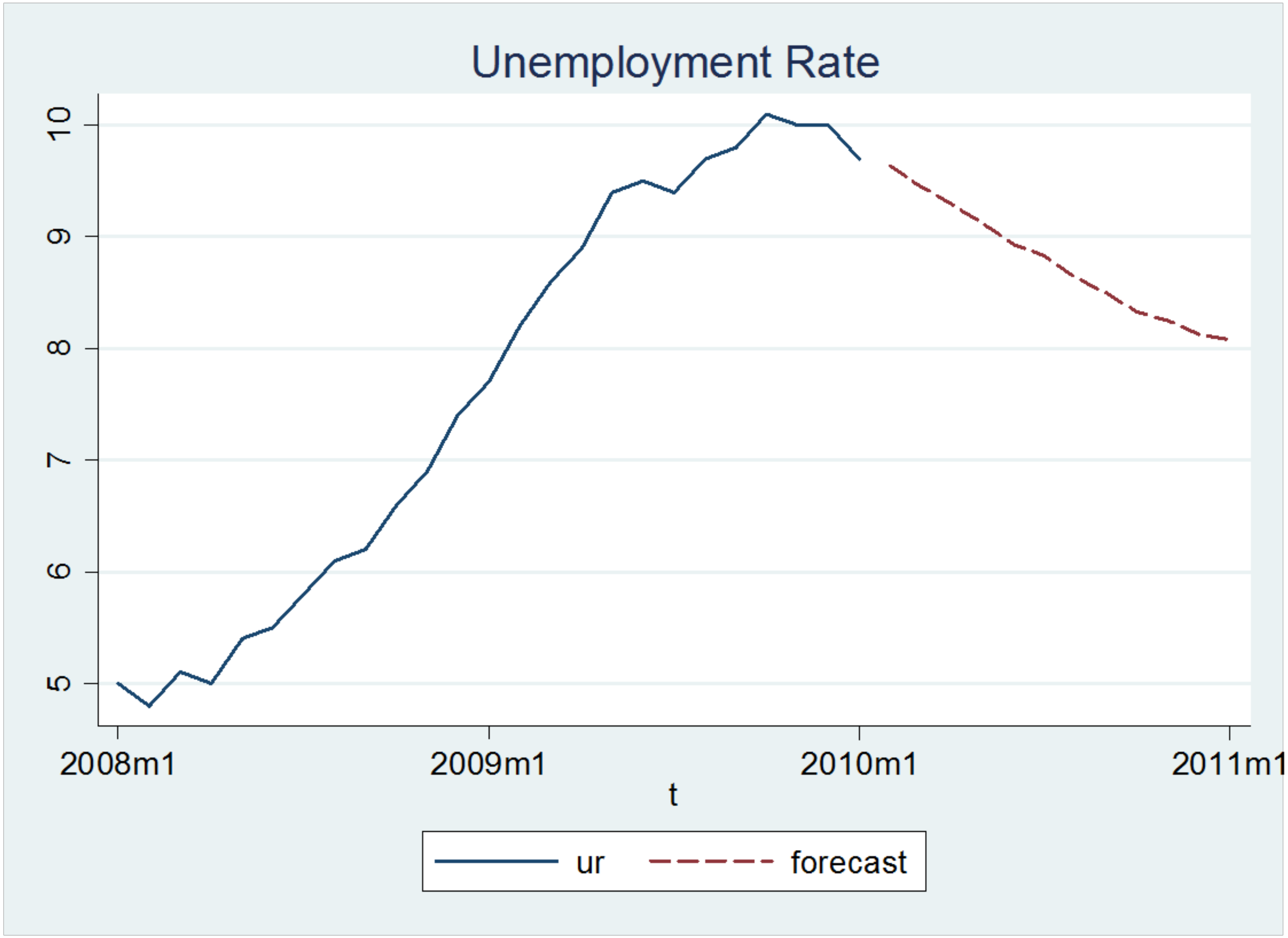
Source	SS	df	MS
Model	916.654935	12	76.3879113
Residual	841.125364	709	1.18635453
Total	1757.7803	721	2.43797545

Number of obs	=	722
F(12, 709)	=	64.39
Prob > F	=	0.0000
R-squared	=	0.5215
Adj R-squared	=	0.5134
Root MSE	=	1.0892

ur	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
ur						
L12.	1.771659	.2075767	8.53	0.000	1.36412	2.179197
L13.	-.0364509	.2897257	-0.13	0.900	-.6052738	.532372
L14.	-.5389651	.2931394	-1.84	0.066	-1.11449	.03656
L15.	-.3469592	.2935564	-1.18	0.238	-.923303	.2293846
L16.	-.1676336	.2933385	-0.57	0.568	-.7435496	.4082824
L17.	-.2296036	.2932366	-0.78	0.434	-.8053195	.3461124
L18.	-.0118408	.2933272	-0.04	0.968	-.5877347	.5640531
L19.	.0747068	.2925796	0.26	0.799	-.4997193	.6491328
L20.	.0162684	.2922787	0.06	0.956	-.557567	.5901037
L21.	.0160849	.2914821	0.06	0.956	-.5561865	.5883562
L22.	.2642944	.2889427	0.91	0.361	-.3029914	.8315801
L23.	-.1004257	.206793	-0.49	0.627	-.5064257	.3055742
_cons	1.674734	.1710406	9.79	0.000	1.338927	2.01054

- 12 periods ahead regression
- Predicted value (Jan 2011)=8.1% (current=9.7%)

12-month forecast with AR(12)



Forecast Intervals at horizon h

- Residuals from the direct forecast estimates

$$y_t = \hat{\alpha} + \hat{\beta}_1 y_{t-h} + \hat{\beta}_1 y_{t-h-1} + \cdots + \hat{\beta}_p y_{t-h-p+1} + \hat{u}_t$$

- Forecast error variance

$$\hat{\sigma}_u^2 = \frac{1}{T} \sum_{t=1}^{T} \hat{u}_t^2$$

- (1-α)% Forecast interval

$$\hat{y}_{T+h|T} \pm \hat{\sigma}_u \cdot z_{\alpha/2}$$

- Identical to AR(1) model

90% forecast intervals