Autoregressive Processes

- The first-order autoregressive process, AR(1) is

$$y_t = \beta y_{t-1} + e_t$$

where e_t is WN(0, σ^2)

- Using the lag operator, we can write

$$(1 - \beta L) y_t = e_t$$

- If $\beta > 0$, y_{t-1} and y_t are positively correlated
- If $\beta < 0$, y_{t-1} and y_t are negatively correlated

Inversion

- By back-substitution

$$\begin{aligned} y_t &= \beta y_{t-1} + e_t \\ &= e_t + \beta\left(\beta y_{t-2} + e_{t-1}\right) \\ &= e_t + \beta e_{t-1} + \beta^2 e_{t-2} + \cdots \\ &= \sum_{i=0}^{\infty} \beta^i e_{t-i} \end{aligned}$$

  a general linear process with geometrically declining coefficients

  – This inversion requires that |β|<1
  – |β|<1 is required for stationarity

Importance of |β|<1

- If β=1 then

$$y_t = e_t + e_{t-1} + e_{t-2} + \cdots$$

does not converge, so the sum is not defined.

Mean and Variance

- By the formula for the unconditional mean and variance of a general linear process

$$E(y_t) = E(\sum_{i=0}^{\infty} \beta^i e_{t-i}) = 0$$

$$\begin{aligned} \mathrm{var}(y_t) &= \mathrm{var}(\sum_{i=0}^{\infty} \beta^i e_{t-i}) \\ &= \left(\sum_{i=0}^{\infty} \beta^{2i} \right) \sigma^2 \\ &= \frac{\sigma^2}{1-\beta^2} \end{aligned}$$

Another Variance Calculation

- Take variance of both sides of

$$y_t = \beta y_{t-1} + e_t$$

- Thus

$$\begin{aligned} \mathrm{var}(y_t) &= \mathrm{var}(\beta y_{t-1} + e_t) \\ &= \mathrm{var}(\beta y_{t-1}) + \mathrm{var}(e_t) \\ &= \beta^2 \,\mathrm{var}(y_{t-1}) + \sigma^2 \end{aligned}$$

- If y is variance stationary, we solve and find

$$\mathrm{var}(y_t) = \mathrm{var}(y_{t-1}) = \frac{\sigma^2}{1-\beta^2}$$

|β|<1

- If $|\beta|=1$ then

$$\text{var}(y_t) = \frac{\sigma^2}{1-\beta^2}$$

is infinite

$|\beta|=1$

- We calculated that

$$\operatorname{var}(y_t) = \beta^2 \operatorname{var}(y_{t-1}) + \sigma^2$$

- When $|\beta|=1$, then

$$\operatorname{var}(y_t) = \operatorname{var}(y_{t-1}) + \sigma^2 > \operatorname{var}(y_{t-1})$$

  so the variance is increasing with t
- $|\beta|=1$ is inconsistent with variance stationarity.
- $|\beta|<1$ is necessary for stationarity.

Random Walk

- An AR(1) with β=1 is known as a random walk or unit root process

$$y_t = y_{t-1} + e_t$$

- By back-substitution

$$y_t = y_0 + \sum_{i=0}^{t} e_{t-i}$$

- The past never disappears. Shocks have permanent effects

Unit Root

- The random walk is called a **unit root** process because the lag operator 1-L has a “root” (intersection with the x-axis) at L=1
- It is called a **random walk** because it tends to wander without mean-reversion.
- If y_t is an AR(1) with a unit root (β=1) then its first difference $\Delta y_t = y_t - y_{t-1}$ is white noise

Conditional Mean and Variance of AR(1)

- Conditional mean:

$$E(y_t \mid \Omega_{t-1}) = E(\beta y_{t-1} + e_t \mid \Omega_{t-1}) = \beta y_{t-1}$$

- Conditional variance:

$$\begin{aligned}\operatorname{var}(y_t \mid \Omega_{t-1}) &= \operatorname{var}(y_t - E(y_t \mid \Omega_{t-1}) \mid \Omega_{t-1}) \\ &= \operatorname{var}(e_t \mid \Omega_{t-1}) \\ &= \sigma^2\end{aligned}$$

Autocovariance of AR(1)

- Take the equation

$$y_t = \beta y_{t-1} + e_t$$

- And then multiply both sides by y_{t-k}

$$y_{t-k} y_t = \beta y_{t-k} y_{t-1} + y_{t-k} e_t$$

- Then take expectations. Since e_t is white noise, it is uncorrelated with

$$E(y_{t-k} y_t) = \beta E(y_{t-k} y_{t-1}) + E(y_{t-k} e_t)$$

or

$$\gamma(k) = \beta \gamma(k-1)$$

Autocorrelation of AR(1)

- Dividing by the variance, this implies

$$\rho(k) = \beta\rho(k-1)$$

- We know

$$\rho(0) = 1$$

- Then

$$\rho(1) = \beta\rho(0) = \beta$$

$$\rho(2) = \beta\rho(1) = \beta^2$$

$$\vdots$$

$$\rho(k) = \beta^k$$

Autocorrelation of AR(1)

- We have derived

$$\rho(k) = \beta^k$$

- The autocorrelation of the stationary AR(1) is a simple geometric decay ($|\beta|<1$)
- If β is small, the autocorrelations decay rapidly to zero with k
- If β is large (close to 1) then the autocorrelations decay moderately
- The AR(1) parameter describes the persistence in the time series

One-Step-Ahead Forecast

- As we showed earlier

$$E(y_t \mid \Omega_{t-1}) = \beta y_{t-1}$$

- Thus

$$E(y_{T+1} \mid \Omega_T) = \beta y_T$$

- The optimal one-step-ahead forecast is a linear function of the final observed value

2-step-ahead forecast

- By back-substitution

$$\begin{aligned} y_t &= \beta y_{t-1} + e_t \\ &= e_t + \beta\left(\beta y_{t-2} + e_{t-1}\right) \\ &= \beta^2 y_{t-2} + e_t + \beta e_{t-1} \end{aligned}$$

- Thus

$$\begin{aligned} E(y_t \mid \Omega_{t-2}) &= E(\beta^2 y_{t-2} + e_t + \beta e_{t-1} \mid \Omega_{t-2}) \\ &= \beta^2 y_{t-2} \end{aligned}$$

- and

$$E(y_{T+2} \mid \Omega_T) = \beta^2 y_T$$

2-step-ahead forecast

- This shows that the optimal 2-step-ahead forecast is also a linear function of the final observed value, but with the coefficient β^2.

$$E(y_{T+2} \mid \Omega_T) = \beta^2 y_T$$

h-step-ahead forecast

- Similarly

$$y_t = \beta^h y_{t-h} + e_t + \beta e_{t-1} + \cdots + \beta^{h-1} e_{t-h+1}$$

- So

$$E(y_t \mid \Omega_{t-h}) = E(\beta^h y_{t-h} + e_t + \beta e_{t-1} + \cdots + \beta^{h-1} e_{t-h+1} \mid \Omega_{t-2})$$

$$= \beta^h y_{t-h}$$

- Optimal forecast:

$$E(y_{T+h} \mid \Omega_T) = \beta^h y_T$$

Inversion of AR(1)

- By inverting the lag operator

$$\begin{aligned} (1-\beta L) y_t &= e_t \\ y_t &= (1-\beta L)^{-1} e_t \\ &= \left(\sum_{i=0}^{\infty} \beta^i L^i \right) e_t \\ &= \sum_{i=0}^{\infty} \beta^i e_{t-i} \end{aligned}$$

- Which is the same as found by back substitution

Condition for Invertibility

- The operator $(1-\beta L)$ is invertible when $|\beta|<1$
- This is the same as for the MA(1) model
- β is the inverse of the root of the polynomial $1-\beta L$
- The root of a function is the value where it crosses the x-axis
- The root of $1-\beta L$ is $1/\beta$, the inverse of the root is β
- Invertibility requires that the inverse of the root be less than one

AR(1) with Intercept

- An AR(1) with intercept is

$$y_t = \alpha + \beta y_{t-1} + e_t$$

Taking expectations

$$E(y_t) = \alpha + \beta E(y_{t-1}) + E(e_t)$$

- Thus

$$\mu = \alpha + \beta\mu$$

- and

$$\mu = \frac{\alpha}{1-\beta}$$

Best Linear Predictor

- A linear predictor of y_t given y_{t-1} is

$$\alpha + \beta y_{t-1}$$

- The forecast error is

$$e_t = y_t - \alpha - \beta y_{t-1}$$

- The linear predictor which minimizes the expected squared forecast error solves

$$\min_{\alpha,\beta} E(y_t - \alpha - \beta y_{t-1})^2$$

Least-Squares

- The estimate of the expected squared linear forecast error is the sum of squared errors
- The least squares estimate

$$y_t = \hat{\alpha} + \hat{\beta} y_{t-1} + \hat{e}_t$$

minimizes the sum of squared errors, so is the estimate of the best linear predictor

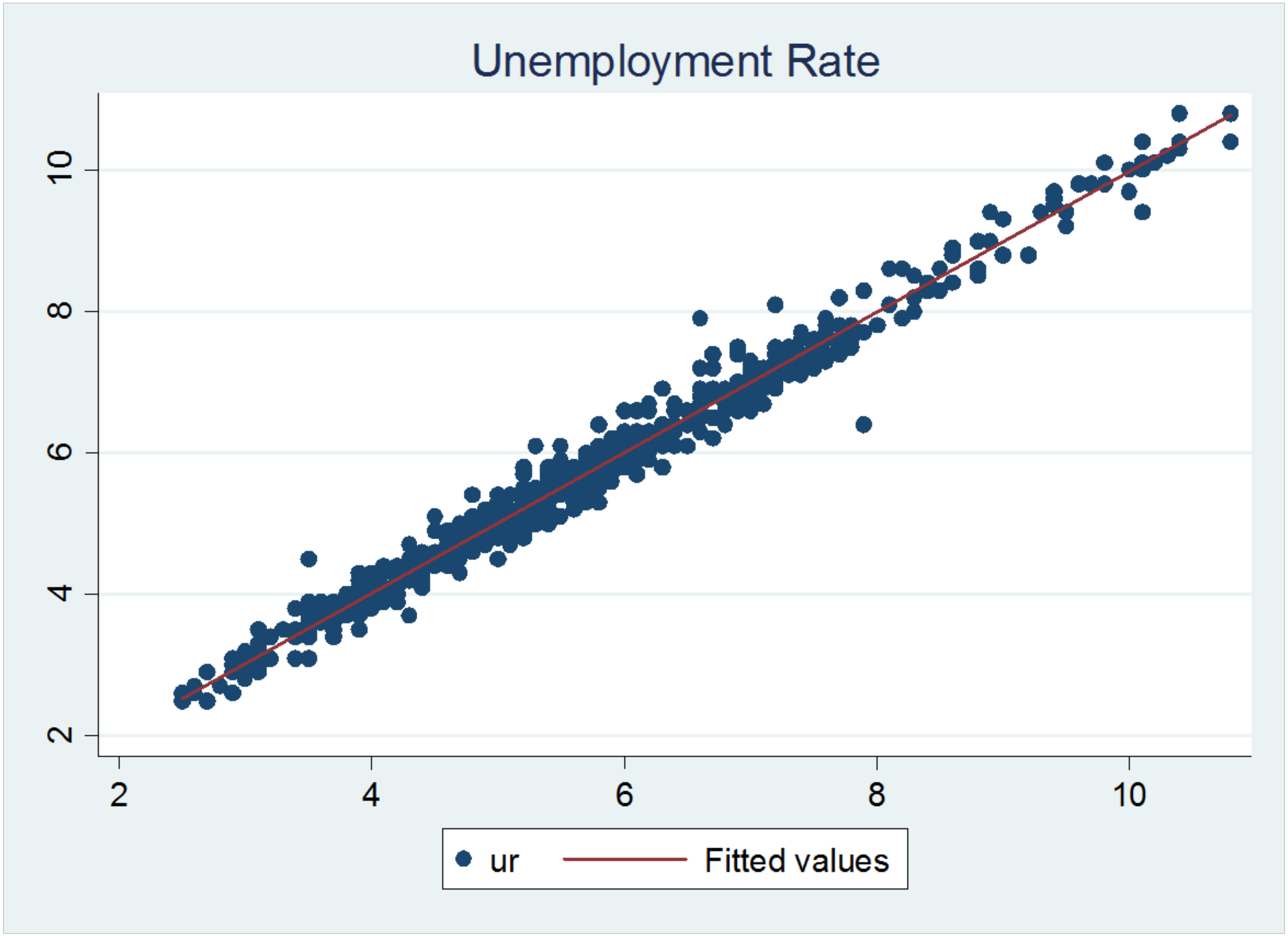
- This is a linear regression, treating y_{t-1} as a regressor.

Unemployment Rate

```
. regress ur L.ur

      Source |       SS       df       MS              Number of obs =     744
-------------+------------------------------           F(  1,   742) =37941.25
       Model |  1775.33245     1  1775.33245           Prob > F      =  0.0000
    Residual |  34.7193756   742  .046791611           R-squared     =  0.9808
-------------+------------------------------           Adj R-squared =  0.9808
       Total |  1810.05182   743  2.43613974           Root MSE      =  .21631

------------------------------------------------------------------------------
          ur |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
          ur |
         L1. |   .9934454   .0051002   194.79   0.000     .9834329    1.003458
             |
       _cons |    .045538   .0299153     1.52   0.128    -.0131907    .1042667
------------------------------------------------------------------------------
```

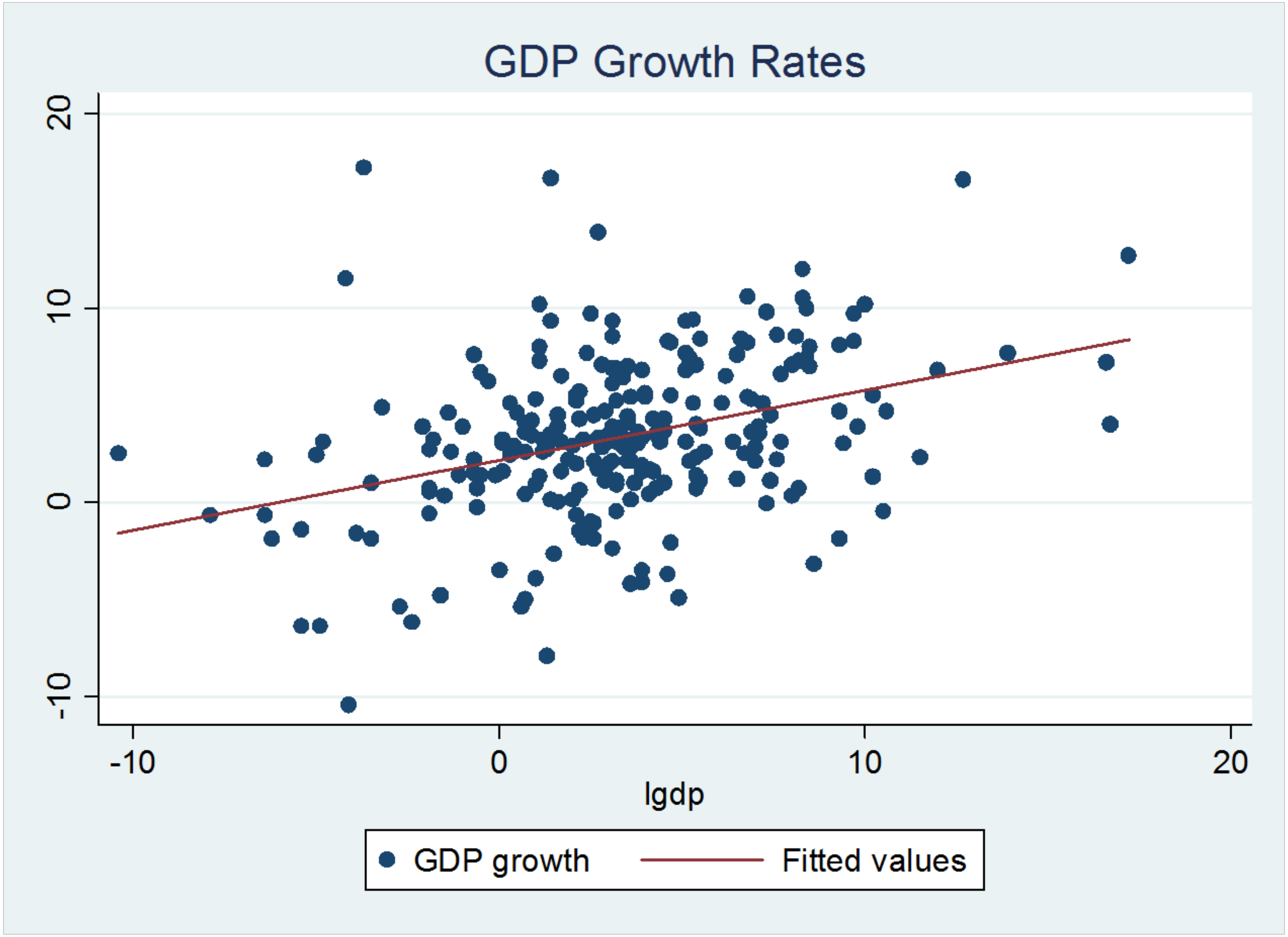

Unemployment Rate
10
8
6
4
2
2
4
6
8
10
ur
Fitted values

GDP Growth Rates

```
. reg gdp L.gdp

      Source |       SS       df       MS              Number of obs =     250
-------------+------------------------------           F(  1,   248) =   37.15
       Model |  548.684959     1  548.684959           Prob > F      =  0.0000
    Residual |  3662.76711   248  14.7692222           R-squared     =  0.1303
-------------+------------------------------           Adj R-squared =  0.1268
       Total |  4211.45207   249  16.9134621           Root MSE      =  3.8431

------------------------------------------------------------------------------
         gdp |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         gdp |
         L1. |   .3605283   .0591503     6.10   0.000     .2440274    .4770292
             |
       _cons |   2.146711   .3123873     6.87   0.000     1.531441    2.761982
------------------------------------------------------------------------------
```


GDP Growth Rates
20
10
0
-10
-10
0
10
20
lgdp
GDP growth
Fitted values

One-Step-Ahead Forecast

- The optimal forecast for T+1 given T is

$$\hat{y}_{T+1|T} = \alpha + \beta y_T$$

- The forecast using the estimates is

$$\hat{y}_{T+1|T} = \hat{\alpha} + \hat{\beta} y_T$$

Example – Unemployment Rate

- The estimates were

ur					
L1.	.9934454	.0051002	194.79	0.000	.9834329
_cons	.045538	.0299153	1.52	0.128	-.0131907

$$y_t = 0.0455 + 0.993 y_{t-1} + \hat{e}_t$$

- The value for Jan 2010 is 9.7%, so

$$\hat{y}_{2010:2} = 0.0455 + 0.993 \times 9.7 = 9.68$$

	y_t	y_{t-1}	fitted value
Jan-09	7.7	7.4	7.40
Feb-09	8.2	7.7	7.70
Mar-09	8.6	8.2	8.19
Apr-09	8.9	8.6	8.59
May-09	9.4	8.9	8.89
Jun-09	9.5	9.4	9.38
Jul-09	9.4	9.5	9.48
Aug-09	9.7	9.4	9.38
Sep-09	9.8	9.7	9.68
Oct-09	10.1	9.8	9.78
Nov-09	10	10.1	10.08
Dec-09	10	10	9.98
Jan-10	9.7	10	9.98
Feb-10	?	9.7	**9.68**

Example – GDP Growth

- The estimates were

gdp	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
gdp						
L1.	.3605283	.0591503	6.10	0.000	.2440274	.4770292
_cons	2.146711	.3123873	6.87	0.000	1.531441	2.761982

$$y_t = 2.14 + 0.361 y_{t-1} + \hat{e}_t$$

- The value for 4th quarter 2009 is 5.7%, so

$$\hat{y}_{2010:1} = 2.14 + 0.361 \times 5.7 = 4.2\%$$

GDP Growth

	y_t	y_{t-1}	fitted
2008q4	-5.4	-2.7	1.2
2009q1	-6.4	-5.4	0.2
2009q2	-0.7	-6.4	-0.2
2009q3	2.2	-0.7	1.9
2009q4	5.7	2.2	2.9
2010q1	?	5.7	**4.2**

One-Step-Ahead Forecast Error

- The forecast error is

$$
\begin{aligned}
y_{T+1} - \hat{y}_{T+1|T} &= \alpha + \beta y_T + e_{T+1} - \left(\alpha + \beta y_T\right) \\
&= e_{T+1}
\end{aligned}
$$

- The forecast variance is

$$\operatorname{var}(y_{T+1} - \hat{y}_{T+1|T}) = \operatorname{var}(e_{T+1}) = \sigma^2$$

Forecast variance estimation

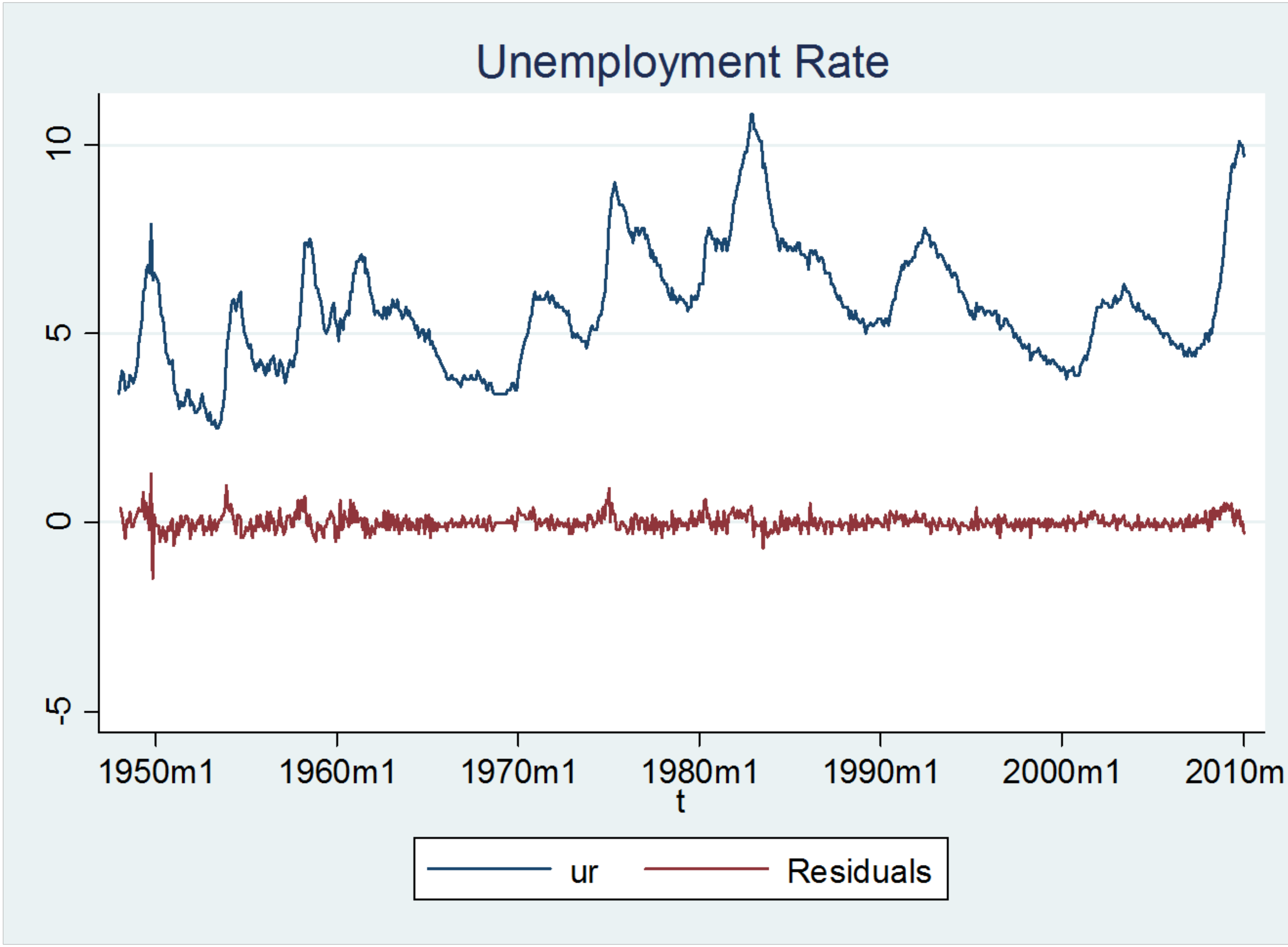
- Average of squared residuals

$$\hat{\sigma}^2 = \frac{1}{T}\sum_{t=1}^{T}\hat{e}_t^2$$

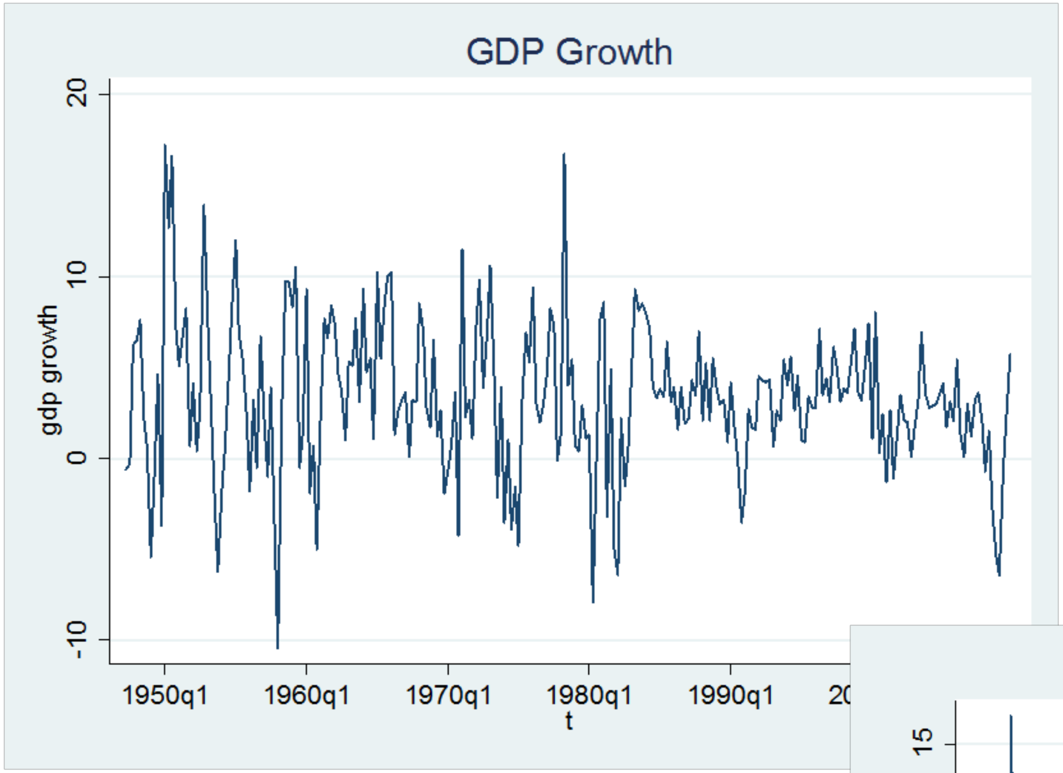
$$\hat{\sigma} = \sqrt{\hat{\sigma}^2}$$

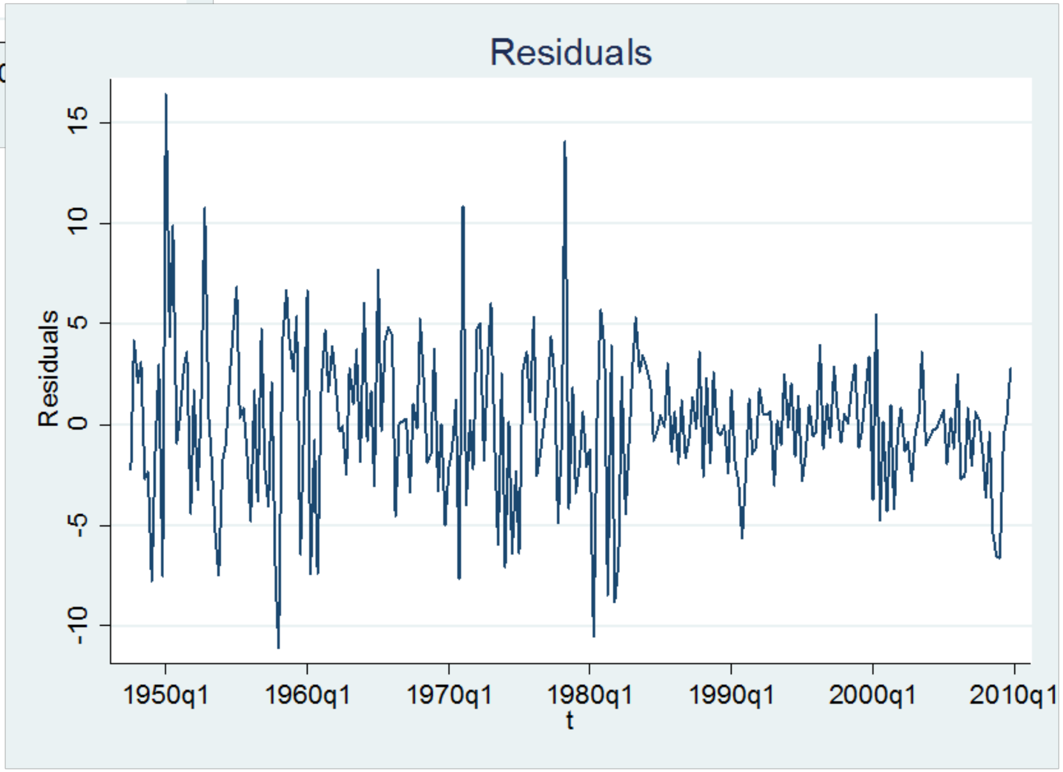
where the least-squares residuals are

$$\hat{e}_t = y_t - \hat{\alpha} - \hat{\beta} y_{t-1}$$



Unemployment Rate
10
5
0
-5
1950m1
1960m1
1970m1
1980m1
1990m1
2000m1
2010m1
t
ur
Residuals


GDP Growth
gdp growth
20
10
0
-10
1950q1
1960q1
1970q1
1980q1
1990q1
t


Residuals
Residuals
15
10
5
0
-5
-10
1950q1
1960q1
1970q1
1980q1
1990q1
2000q1
2010q1
t

One-Step-Ahead Intervals

- Normal Method
  - Assume forecast error is normally distributed
  - Forecast interval is point estimate, plus and minus the estimated standard deviation multiplied by a normal quantile
  - For a 95% interval:

$$\hat{y}_{T+1|T} \pm \hat{\sigma} \cdot z_{.025} = \hat{y}_{T+1|T} \pm \hat{\sigma} \cdot 1.96$$

  - For a 90% interval

$$\hat{y}_{T+1|T} \pm \hat{\sigma} \cdot z_{.05} = \hat{y}_{T+1|T} \pm \hat{\sigma} \cdot 1.645$$

Estimating Forecast Variance

- The estimated variance is 16.9
- The estimated st. dev. is 3.84

```
. reg gdp L.gdp

      Source |       SS       df       MS              Number of obs =     250
-------------+------------------------------           F(  1,   248) =   37.15
       Model |  548.684959     1  548.684959           Prob > F      =  0.0000
    Residual |  3662.76711   248  14.7692222           R-squared     =  0.1303
-------------+------------------------------           Adj R-squared =  0.1268
       Total |  4211.45207   249  16.9134621           Root MSE      =  3.8431

------------------------------------------------------------------------------
         gdp |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         gdp |
         L1. |   .3605283   .0591503     6.10   0.000     .2440274    .4770292
             |
       _cons |   2.146711   .3123873     6.87   0.000     1.531441    2.761982
------------------------------------------------------------------------------
```

Standard Dev Estimation

- In STATA, the estimate of σ or "root mean squared error" is saved after you estimate a regression in "e(rmse)"
- You can access it through the command **e(rmse)**.

```
. dis e(rmse)
3.8430746
```

Forecast Interval Construction

```
. tsappend, add(1)

. predict p if t>tq(2009q4)
(option xb assumed; fitted values)
(251 missing values generated)

. gen p1=p-1.645*e(rmse)
(251 missing values generated)

. gen p2=p+1.645*e(rmse)
(251 missing values generated)
```

- Point estimate = 4.2%
- 90% Interval = [-2.1%, 10.5%]

Two-Step-Ahead Point Forecast

- First Method: Plug-in
- By back-substitution

$$
\begin{aligned}
y_t &= \alpha + \beta y_{t-1} + e_t \\
&= \alpha + \beta(\alpha + \beta y_{t-2} + e_{t-1}) + e_t \\
&= (1+\beta)\alpha + \beta^2 y_{t-2} + e_t + \beta e_{t-1}
\end{aligned}
$$

- Thus

$$y_{T+2} = (1+\beta)\alpha + \beta^2 y_T + e_{T+2} + \beta e_{T+1}$$

$$E(y_{T+2} \mid \Omega_T) = (1+\beta)\alpha + \beta^2 y_T$$

Plug-in Method

- The optimal forecast is

$$\hat{y}_{T+2|T} = (1+\beta)\alpha + \beta^2 y_T$$

- Plug-in our estimates to obtain a feasible forecast

$$\hat{y}_{T+2|T} = (1+\hat{\beta})\hat{\alpha} + \hat{\beta}^2 y_T$$

- This method is feasible but cumbersome for multi-step forecasts and complicated models

Iterated Method

- Take conditional expectations at time T

$$y_{T+2} = \alpha + \beta y_{T+1} + e_{T+2}$$

$$E(y_{T+2} \mid \Omega_T) = \alpha + \beta E(y_{T+1} \mid \Omega_T) + E(e_{T+2} \mid \Omega_T)$$

$$= \alpha + \beta E(y_{T+1} \mid \Omega_T)$$

- The left-hand is the 2-step forecast, the right-hand has the 2-step forecast:

$$\hat{y}_{T+2|T} = \alpha + \beta \hat{y}_{T+1|T}$$

Iterated Method

- We already have the one-step forecast
- The two-step iterates on the one-step

$$\hat{y}_{T+1|T} = \hat{\alpha} + \hat{\beta} y_T$$

$$\hat{y}_{T+2|T} = \hat{\alpha} + \hat{\beta} \hat{y}_{T+1|T}$$

- This method is convenient for linear autoregressions
- It works in **linear** forecasting models (linear autoregressions) but not in nonlinear models
- It is less useful in regression contexts (later sections)

Direct Method

- We showed that

$$
\begin{aligned}
y_t &= (1+\beta)\alpha + \beta^2 y_{t-2} + e_t + \beta e_{t-1} \\
&= \alpha^* + \beta^* y_{t-2} + u_t
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha^* &= (1+\beta)\alpha \\
\beta^* &= \beta^2 \\
u_t &= e_t + \beta e_{t-1}
\end{aligned}
$$

Estimation of Direct Method

- This is a regression

$$y_t = \alpha^* + \beta^* y_{t-2} + u_t$$

- The error is the two-step forecast error
- It can be estimated **directly** by least-squares
- This is actually different than the iterated estimator.

Example – GDP Growth

- α=2.14, β=0.361, y_T=5.7, $y_{T+1|T}$=4.2
- Plug-in:

$$\hat{y}_{T+2|T} = \left(1+\hat{\beta}\right)\cdot\hat{\alpha} + \hat{\beta}^2 y_T$$
$$= \left(1+.36\right)\cdot 2.14 + .36^2\cdot 5.7$$
$$= 3.6\%$$

- Iterated:

$$\hat{y}_{T+2|T} = \hat{\alpha} + \hat{\beta}\hat{y}_{T+1|T}$$
$$= 2.14 + .36\cdot 4.2$$
$$= 3.6\%$$

GDP Growth, Direct 2-step

```
. reg gdp L2.gdp

      Source |       SS       df       MS              Number of obs =     249
-------------+------------------------------           F(  1,   247) =   10.93
       Model |  177.867084     1  177.867084           Prob > F      =  0.0011
    Residual |   4020.2617   247  16.2763632           R-squared     =  0.0424
-------------+------------------------------           Adj R-squared =  0.0385
       Total |  4198.12879   248  16.9279387           Root MSE      =  4.0344

------------------------------------------------------------------------------
         gdp |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         gdp |
         L2. |   .2053005   .0621042     3.31   0.001     .0829792    .3276219
             |
       _cons |   2.675403   .3285317     8.14   0.000     2.028322    3.322484
------------------------------------------------------------------------------
```

- Estimate

$$y_t = 2.67 + 0.21 y_{t-2} + \hat{u}_t$$

$$\hat{\sigma} = 4.03$$

- Notice .21>.13=.36^2 from iterated

Direct 2-step-ahead

- 2-step forecast

$$\begin{aligned}\hat{y}_{T+2|T} &= \hat{\alpha}^{*} + \hat{\beta}^{*} y_T \\ &= 2.67 + 0.21 \cdot 5.7 \\ &= 3.9\%\end{aligned}$$

- This is slightly larger than the 3.6% forecast from the iterated method

2-step forecast variance

- The 2-step equation is

$$y_t = \alpha^* + \beta^* y_{t-2} + u_t$$
$$u_t = e_t + \beta e_{t-1}$$

- The forecast error u has variance

$$\begin{aligned} \mathrm{var}(u_t) &= \mathrm{var}(e_t + \beta e_{t-1}) \\ &= \left(1 + \beta^2\right)\sigma^2 \end{aligned}$$

Plug-in Forecast variance estimation

$$\hat{\sigma}_u^2 = \left(1 + \hat{\beta}^2\right)\hat{\sigma}^2$$

$$\hat{\sigma}_u = \sqrt{\hat{\sigma}_u^2}$$

Iterated Forecast variance estimation

$$\hat{y}_{t|t-1} = \hat{\alpha} + \hat{\beta} y_{t-1}$$

$$\hat{y}_{t|t-2} = \hat{\alpha} + \hat{\beta} \hat{y}_{t|t-1}$$

$$\hat{u}_t = y_t - \hat{y}_{t|t-2}$$

$$\hat{\sigma}_u^2 = \frac{1}{T} \sum_{t=1}^{T} \hat{u}_t^2$$

$$\hat{\sigma}_u = \sqrt{\hat{\sigma}_u^2}$$

Direct Forecast variance estimation

$$\hat{u}_t = y_t - \hat{\alpha}^* - \hat{\beta}^* y_{t-2}$$

$$\hat{\sigma}_u^2 = \frac{1}{T}\sum_{t=1}^{T}\hat{u}_t^2$$

Example: GDP Growth Plug-in Estimate

- β=.36, σ=3.84

$$\hat{\sigma}_u = \sqrt{\left(1+\hat{\beta}^2\right)\hat{\sigma}^2}$$
$$= \sqrt{\left(1+.36^2\right)3.84^2}$$
$$= 4.1$$

Iterated Estimate

```
. use gdp

. reg gdp L.gdp

      Source |       SS       df       MS              Number of obs =     250
-------------+------------------------------           F(  1,   248) =   37.15
       Model |  548.684959     1  548.684959           Prob > F      =  0.0000
    Residual |  3662.76711   248  14.7692222           R-squared     =  0.1303
-------------+------------------------------           Adj R-squared =  0.1268
       Total |  4211.45207   249  16.9134621           Root MSE      =  3.8431

------------------------------------------------------------------------------
         gdp |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         gdp |
         L1. |   .3605283   .0591503     6.10   0.000     .2440274    .4770292
             |
       _cons |   2.146711   .3123873     6.87   0.000     1.531441    2.761982
------------------------------------------------------------------------------

. predict y1
(option xb assumed; fitted values)
(1 missing value generated)

. gen y2=_b[_cons]+_b[L.gdp]*y1
(1 missing value generated)

. gen u=gdp-y2
(1 missing value generated)

. summarize u

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
           u |       250   -.0090853    3.951075  -12.78774   14.76027
```

$$\hat{\sigma}_u = 3.95$$

Using Estimate Coefficients

- Notice on the previous slide how I used the estimated coefficients in a STATA command
- gen y2=_b[_cons]+_b[L.gdp]*y1
- The estimated coefficients are stored in "_b[]"
- You access them by the name of the variable
  - The constant is always called "_cons"
    - You can see this in the output
  - The other variable was L.gdp
- The variable "y1" had been created using **predict**

Direct Estimate

```
. reg gdp L2.gdp
```

Source	SS	df	MS
Model	177.867084	1	177.867084
Residual	4020.2617	247	16.2763632
Total	4198.12879	248	16.9279387

Number of obs	=	249
F(1, 247)	=	10.93
Prob > F	=	0.0011
R-squared	=	0.0424
Adj R-squared	=	0.0385
Root MSE	=	4.0344

gdp	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
gdp						
L2.	.2053005	.0621042	3.31	0.001	.0829792	.3276219
_cons	2.675403	.3285317	8.14	0.000	2.028322	3.322484

$$\hat{\sigma} = 4.0$$

Two-Step-Ahead Intervals

- Normal Method
  - Forecast interval is point estimate, plus and minus the estimated standard deviation multiplied by a normal quantile
  - For a 95% interval:

$$\hat{y}_{T+2|T} \pm \hat{\sigma}_u \cdot z_{.025} = \hat{y}_{T+2|T} \pm \hat{\sigma}_u \cdot 1.96$$

  - For a 90% interval

$$\hat{y}_{T+2|T} \pm \hat{\sigma}_u \cdot z_{.05} = \hat{y}_{T+2|T} \pm \hat{\sigma}_u \cdot 1.645$$

GDP Growth – Plug-in

- Plug-in Intervals
  - $y_{T+2|T}$=3.6%, σ_u=4.1
  - 3.6% ± 1.645*4.1=[-3.1%, 10.3%]
- Iterated
  - $y_{T+2|T}$=3.6%, σ_u=4.0
  - 3.6% ± 1.645*4.0=[-3.0%, 10.2%]
- Direct
  - $y_{T+2|T}$=3.9%, σ_u=4.0
  - 3.9% ± 1.645*4.0=[-2.7%, 10.5%]

h-Step-Ahead back substitution

$$
\begin{aligned}
y_t &= \alpha + \beta y_{t-1} + e_t \\
&= \alpha + \beta(\alpha + \beta y_{t-2} + e_{t-1}) + e_t \\
&= (1+\beta)\alpha + \beta^2(\alpha + \beta y_{t-3} + e_{t-2}) + e_t + \beta e_{t-1} \\
&= (1+\beta+\beta^2)\alpha + \beta^3 y_{t-3} + e_t + \beta e_{t-1} + \beta^2 e_{t-2} \\
&= (1+\beta+\beta^2+\cdots+\beta^h)\alpha + \beta^h y_{t-h} + u_t \\
u_t &= e_t + \beta e_{t-1} + \beta^2 e_{t-2} + \cdots + \beta^{h-1} e_{t-h+1}
\end{aligned}
$$

h-Step-Ahead Point Forecast

- Optimal

$$E\left(y_{T+h} \mid \Omega_T\right) = \left(1 + \beta + \beta^2 + \cdots + \beta^h\right)\alpha + \beta^h y_T$$

- Plug-In

$$\hat{y}_{T+h|T} = \left(1 + \hat{\beta} + \hat{\beta}^2 + \cdots + \hat{\beta}^h\right)\hat{\alpha} + \hat{\beta}^h y_T$$

- Iterated

$$y_{T+h} = \alpha + \beta y_{T+h-1} + e_{T+h}$$

$$E(y_{T+h} \mid \Omega_T) = \alpha + \beta E(y_{T+h-1} \mid \Omega_T)$$

$$\hat{y}_{T+h|T} = \hat{\alpha} + \hat{\beta}\hat{y}_{T+h-1|T}$$

Direct Method

- Best Linear predictor

$$y_t = \alpha^* + \beta^* y_{t-h} + u_t$$

- Least-Squares estimator

$$y_t = \hat{\alpha}^* + \hat{\beta}^* y_{t-h} + \hat{u}_t$$

- h-step forecast

$$\hat{y}_{T+h|T} = \hat{\alpha}^* + \hat{\beta}^* y_T$$

Direct Estimates

- Least Squares

$$y_t = 2.14 + 0.36 y_{t-1} + \hat{e}_t$$

$$y_t = 2.68 + 0.32 y_{t-2} + \hat{u}_t$$

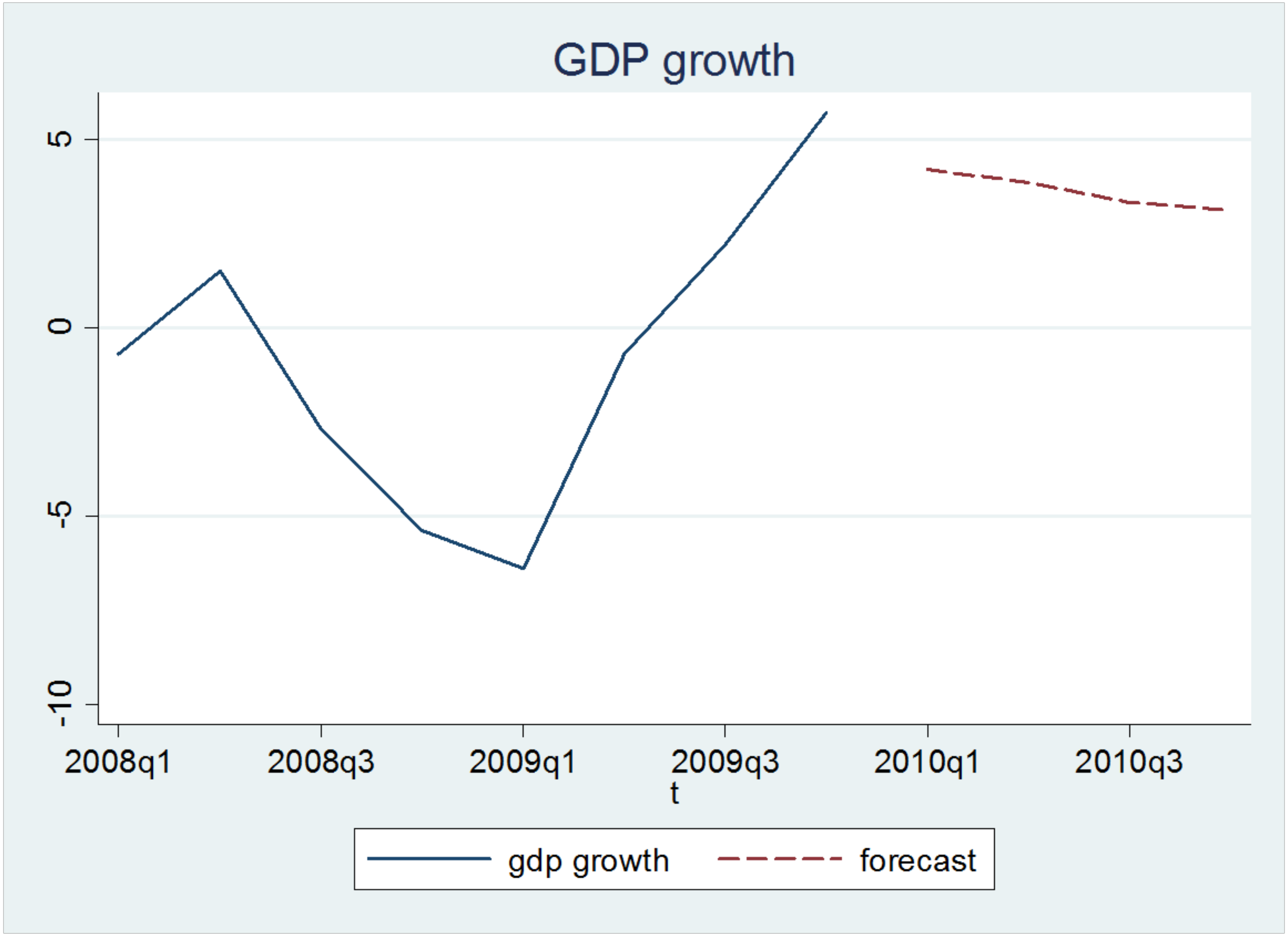
$$y_t = 3.38 - 0.01 y_{t-3} + \hat{u}_t$$

$$y_t = 3.65 - 0.09 y_{t-4} + \hat{u}_t$$

Iterated and Direct Point Estimates

	Iterated	Direct
2010Q1	4.2	4.2
2010Q2	3.7	3.8
2010Q3	3.5	3.3
2010Q4	3.4	3.1

Point Forecast (Direct)



```
use gdp.dta
tsappend, add(4)
reg gdp L.gdp
predict y1
reg gdp L2.gdp
predict y2
reg gdp L3.gdp
predict y3
reg gdp L4.gdp
predict y4
egen p=rowfirst(y1 y2 y3 y4) if t>=tq(2010q1)
label variable p “forecast”
tsline gdp p if t>=tq(2008q1), title(GDP growth) lpattern (solid dash)
```

- There are 4 periods out-of-sample
- The **predict** command computes fitted values for observations which have the needed variables.
- For the regression on the first lag (L.gdp), this works only for the first out-of-sample observation, the remainder are coded as missing.
- For the regression on the second lag (L2.gdp), this works for the fist two out-of-sample observations
- etc

Forecasts

t	y1	y2	y3	y4	p
2009q4	2.94	2.53	3.44	4.16	
2010q1	**4.20**	3.13	3.39	4.26	**4.20**
2010q2		**3.85**	3.36	3.72	**3.85**
2010q3			**3.32**	3.44	**3.32**
2010q4				**3.11**	**3.11**

Multi-Step Forecast Variance

- Can use plug-in, iterated, or direct method
- Easiest method is direct
- Forecast variance is computed from direct regression

$$y_t = \hat{\alpha}^* + \hat{\beta}^* y_{t-h} + \hat{u}_t$$

$$\hat{\sigma}_u^2 = \frac{1}{T} \sum_{t=1}^{T} \hat{u}_t^2$$

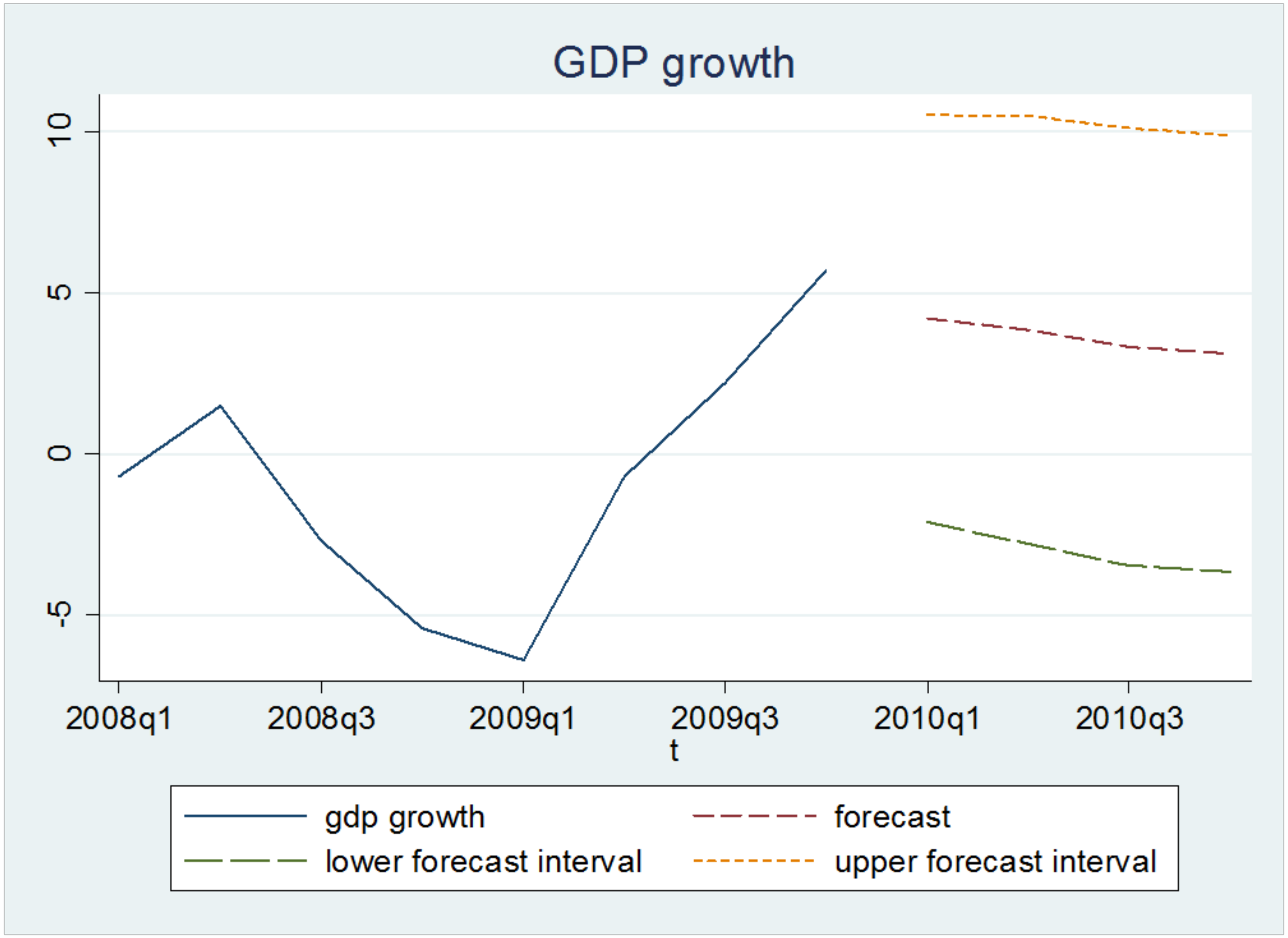
Forecast Variance and Intervals

- Regression standard deviation printed in regression output
  - Stored in “e(rmse)”
- Forecast Intervals (90% normal)

  gen y1L=y1-1.645*e(rmse)

  gen y1U=y1+1.645*e(rmse)

GDP Growth Forecast



.do file

```
tsappend, add(4)
reg gdp L.gdp
predict y1
gen y1L=y1-1.645*e(rmse)
gen y1U=y1+1.645*e(rmse)
reg gdp L2.gdp
predict y2
gen y2L=y2-1.645*e(rmse)
gen y2U=y2+1.645*e(rmse)
reg gdp L3.gdp
predict y3
gen y3L=y3-1.645*e(rmse)
gen y3U=y3+1.645*e(rmse)
reg gdp L4.gdp
predict y4
gen y4L=y4-1.645*e(rmse)
gen y4U=y4+1.645*e(rmse)
```

```
egen p=rowfirst(y1 y2 y3 y4) if t>=tq(2010q1)
egen pL=rowfirst(y1L y2L y3L y4L) if t>=tq(2010q1)
egen pU=rowfirst(y1U y2U y3U y4U) if t>=tq(2010q1)
label variable p "forecast"
label variable pL "lower forecast interval"
label variable pU "upper forecast interval"
tsline gdp p pL pU if t>=tq(2008q1), title(GDP growth)
        lpattern (solid dash longdash shortdash)
```