Nonparametric Methods for Ascending Auctions with Unobserved Heterogeneity

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Preliminary and Incomplete

1 Introduction

For both theoretical and empirical work on auctions, it is standard to assume that bidders have symmetric and independently distributed private values for the good at auction, and that the set of bidders participating in an auction is independent of the characteristics of the auction itself. These are powerful assumptions. Theoretically, they establish the optimality of any of the standard auction formats with a suitably chosen reserve price (Myerson (1981)). Empirically, they ensure that observations of transaction prices in ascending auctions with a fixed number of bidders nonparametrically identify the entire model, allowing explicit calculation of that profit-maximizing reserve price (see Athey and Haile (2002), Paarsch (1992), Paarsch (1997)); if auctions with different numbers of bidders are observed, the model is overidentified, and therefore testable (Athey and Haile (2002)).

However, both the theoretical prescription and the empirical approach lean heavily on the assumptions of symmetric independent private values and a fixed set of bidders. Even ignoring the question of common versus private values, introducing any small degree of correlation in valuations among bidders (Cremer and McLean (1988), McAfee and Reny (1992)), or endogenizing bidders' participation decisions (Samuelson (1985), McAfee and McMillan (1987), Levin and Smith (1994)), significantly alters the design of the optimal mechanism. And once bidder valuations are allowed to be correlated, the model is not identified from bid data in ascending auctions (Athey and Haile (2002)).

If the objects in different auctions are identical, independence seems like a reasonable assumption: differences in valuations are based on idiosyncratic differences in tastes, which need not be correlated. If the objects are heterogeneous, independence is unlikely to hold in its literal form; standard practice is to control for as much auction-to-auction variation as possible, and hope that whatever residual variation is left is similarly idiosyncratic taste shocks and therefore independent across bidders. (**Cite examples.**) This technique has clear limitations. Controlling for multi-dimensional heterogeneity nonparametrically leads to a severe curse of dimensionality (see, e.g., Silverman (1986)). Controlling for heterogeneity parametrically leaves the concern that any deviation of the parametric model from the true data-generating process will leave residual correlation between bidder valuations; and any sources of heterogeneity which are not observed by the econometrician will similarly lead to correlation.

We propose a different approach. We introduce a model which explicitly allows for unobserved heterogeneity in the objects for sale. While the entire model is not identified by data on winning bids from ascending auctions, we show that its most policy-relevant features are, provided there is exogenous variation in the number of bidders n in each auction. We also introduce a nonparametric test for exogenous variation in n, and argue that the test should have power against the most likely sources of endogeneity. Treating the number of bidders as exogenous begs the question of what process we see as determining n; in the subsequent section, we therefore discuss three standard models of participation in auctions, two of them consistent with exogenous n; and show how, given *any* exogenous variation in reserve prices (or an instrument for reserve price), we can distinguish between them.

We introduce the results under the standard assumption that the second-place bidder bids up to his valuation; we later show how to extend our results to an incomplete model of an English (ascending) auction (as in Haile and Tamer (2003)) if we can observe the highest two bids in each auction.

2 Model

The fundamental identification problem in ascending auctions comes from the fact that neither the winner's nor most of the losers' willingness to pay for the object are reliably observed – the former because the auction ends when all but one bidder has dropped out, the latter because some bidders may bid early (or not at all) and then wait to see how bidding develops, and never bid close to their valuation if others' bids surpass it. Thus, most inference is based on the valuation of the second-highest bidder, since this can be inferred fairly precisely from bidding behavior. (In many instances, it is assumed to be known precisely.)

We assume the econometrician has data from a long (possibly infinite) series of ascending auctions. We consider the simplest case: for each auction, the econometrician knows the number of participants n, and the exact willingness-to-pay of the second-highest bidder.

The latter assumption would be exactly true in a second-price sealed-bid auction, or in a so-called "button auction," where bidders hold down a button to remain active and release it to drop out. It is approximately true (up to the minimum bid increment) in an ascending auction with no "jump bids," and is a common assumption when modeling ascending auctions. In a later section, we show how to extend our analysis to a more general model of bidding behavior. The former assumption – observation of the number of bidders in each auction – is crucial to our analysis. This may be non-trivial in an ascending auction, particularly in a setting such as eBay – even if we know how many people actually submitted bids on a particular item, we may not be able to observe how many people considered bidding and chose not to based on the existing bids. Later on, we discuss why this assumption is reasonable in many applications, including the one we use.

Given enough data, then, the econometrician can exactly identify the probability distribution of the second-highest willingness-to-pay in auctions of a given size. To introduce our results as simply as possible, we assume for now that this distribution is known exactly; later on, we generalize our results to the case where the econometrician instead has pointwise upper and lower bounds on this distribution, as would be the case if we were constructing confidence intervals based on finite data or were employing a more general model of bidding behavior (as in Haile and Tamer (2003)).

For now, we assume the auctions being analyzed had no reserve prices. Finally, we should also note that the data should include any "zeroes" – any auctions in which no bidders arrived and therefore no sale occurred. (If the auctions did have reserve prices, this would include any auctions in which no bid exceeded the reserve.)

As for the model, we assume Symmetric, Conditionally Independent Private Values. Each auction is characterized by a set of characteristics $\theta \in \Theta$.[1] Bidder valuations for the object for sale in an auction with characteristics θ are *i.i.d.* draws from a probability distribution $F(\,\cdot\,|\theta)$. Thus, variation in θ induces correlation in bidders' valuations, while any variation not caused by θ is idiosyncratic and independent across bidders.[2]

[1]These can include both characteristics of the object for sale, and of the auction more generally (reliability of a particular seller, shipping costs being charged, etc.)

[2]De Finetti's theorem, as formalized by Hewitt and Savage (1955), says that if bidders are drawn at random from an *infinite* set potential bidders with an ex-ante joint distribution of valuations which is exchangeable, then this formulation is without loss of generality. Without appealing to infinite bidders, since we place no restrictions on the dimensionality of θ or how it affects bidder preferences (other than that it affects them symmetrically),

Let $V_{j:n}$ be the j^{th} lowest bidder valuation in auctions with n bidders, and $F_{j:n}$ its probability distribution. We've already assumed that auction data identifies the distribution $F_{n-1:n}$ for each n.

3 Exogenous Variation in n

The key assumption of this section is that the number of bidders n in a given auction is independent of both the characteristics of that auction θ, and the valuations of the bidders who participate in that auction. We introduce a test of that assumption, and show that under that assumption, variation in n identifies key items of interest. Later on, we discuss three standard models of participation in auctions, two of which are consistent with this assumption, and show how to test between them.

3.1 Testing Exogenous n

3.1.1 The IPV Case

To explain our test for exogenous variation in n, it is easiest to start with the case of Independent Private Values, where there is no heterogeneity in the objects for sale. For $n \geq 2$ and $k \in \{1, 2, \ldots, n\}$, define functions $\psi_{k:n}$: $[0, 1] \rightarrow [0, 1]$ by

$$\psi_{k:n}(s) \quad = \quad \sum_{j=k}^{n} \binom{n}{j} s^j (1-s)^{n-j} \tag{1}$$

These functions have the property that given n independent draws from a probability distribution $H(\cdot)$, the cumulative distribution of the k^{th} lowest of the draws is $\psi_{k:n}(H(\cdot))$.

Under the assumption of symmetric IPV with exogenous n, valuations of participating bidders are all independent draws from some probability

we still feel this is a very general formulation.

distribution H, which is the same regardless of n. So for any n,

$$F_{n-1:n}(v) \quad = \quad \psi_{n-1:n}\left(H(v)\right) \tag{2}$$

or[3]

$$\psi_{n-1:n}^{-1}\left(F_{n-1:n}(v)\right) \quad = \quad H(v) \tag{3}$$

Since the right-hand side does not depend on n, a test of the symmetric IPV model with exogenous n is whether

$$\psi_{n'-1:n'}^{-1}\left(F_{n'-1:n'}(v)\right) \quad = \quad \psi_{n-1:n}^{-1}\left(F_{n-1:n}(v)\right) \tag{4}$$

for each pair (n', n). (This is the test introduced in Theorem 1 of Athey and Haile (2002).)

3.1.2 The CIPV Case

Returning to our model of conditionally independent private values, the assumption of exogenous n would imply that the valuations of bidders participating in an auction of size n are independent draws from a distribution $H(\cdot|\theta)$, where θ are the characteristics of the auction (unobserved by the econometrician) and H and θ are independent of n.

To see why the test above is not still valid, consider the extreme case of perfectly correlated values, where θ is one-dimensional and each bidder's valuation in an auction with characteristics θ is simply θ. In this case, $F_{n-1:n}(v)$ for $n \geq 2$ is simply the ex-ante distribution of θ, and therefore does not vary with n; $\psi_{n-1:n}^{-1}\left(F_{n-1:n}(v)\right)$, then, is strictly increasing in n, and (4) fails.

However, under CIPV, the way in which the test fails is predictable, and leads us to a similarly-motivated one-sided test. Under our model, $F_{n-1:n}(v)$ must be weakly decreasing in n; but we show below that under CIPV with

[3] From (1), $\psi_{k:n}(0) = 0$ and $\psi_{k:n}(1) = 1$. Athey and Haile (2002) write $\psi_{k:n}(s)$ as $\frac{n!}{(n-k)!(k-1)!}\int_0^s t^{k-1}(1-t)^{n-k}dt$, making it clear that $\psi_{k:n}$ is strictly increasing, and therefore invertible.

exogenous n, it decreases more slowly than it would under IPV, and so $\psi_{n-1:n}^{-1}(F_{n-1:n}(v))$ must be increasing in n. We prove this result, then argue why this test should have power against the most likely source of endogeneity of n.

Theorem 1. *Under symmetric, conditionally independent private values with exogenous n, $\psi_{n-1:n}^{-1}(F_{n-1:n}(v))$ is weakly increasing in n for any v.*

Proof. It suffices to show that $\psi_{n:n+1}^{-1}(F_{n:n+1}(s)) \geq \psi_{n-1:n}^{-1}(F_{n-1:n}(s))$, or, equivalently, that $\left(\psi_{n-1:n} \circ \psi_{n:n+1}^{-1}\right)(F_{n:n+1}(s)) \geq F_{n-1:n}(s)$. The key step of the proof is the observation that $\psi_{n-1:n} \circ \psi_{n:n+1}^{-1} : [0,1] \to [0,1]$ is concave. For any two functions f and g,

$$\frac{d}{ds}\left(f \circ g^{-1}\right)(s) \quad = \quad f'\left(g^{-1}(s)\right) \cdot \left(g^{-1}\right)'(s) \quad = \quad \frac{f'\left(g^{-1}(s)\right)}{g'(g^{-1}(s))} \tag{5}$$

Since $\psi_{n-1:n}(t) = nt^{n-1} - (n-1)t^n$, differentiating yields

$$\frac{d}{ds}\left(\psi_{n-1:n} \circ \psi_{n:n+1}^{-1}\right)(s) \quad = \quad \frac{\psi'_{n-1:n}(t)}{\psi'_{n:n+1}(t)} \quad = \quad \frac{n(n-1)t^{n-2}(1-t)}{(n+1)nt^{n-1}(1-t)} \quad = \quad \frac{n-1}{n+1} \cdot \frac{1}{t} \tag{6}$$

where $t = \psi_{n:n+1}^{-1}(s)$. So $\left(\psi_{n-1:n} \circ \psi_{n:n+1}^{-1}\right)'(s) = \frac{n-1}{n+1} / \psi_{n:n+1}^{-1}(s)$, which is decreasing, making $\psi_{n-1:n} \circ \psi_{n:n+1}^{-1}$ concave.

At a given realization of θ, the distribution of $V_{n-1:n}$ is $\psi_{n-1:n}(H(\,\cdot\,|\theta))$; so the unconditional distribution of $V_{n-1:n}$ is

$$F_{n-1:n}(v) \quad = \quad E_\theta\left\{\psi_{n-1:n}(H(v|\theta))\right\} \tag{7}$$

Since $\psi_{n-1:n} \circ \psi^{-1}_{n:n+1}$ is concave, applying Jensen's inequality yields

$$\begin{aligned}
\left(\psi_{n-1:n} \circ \psi^{-1}_{n:n+1}\right) \left(F_{n:n+1}(v)\right) &= \left(\psi_{n-1:n} \circ \psi^{-1}_{n:n+1}\right) E_\theta \left\{\psi_{n:n+1}(H(v|\theta))\right\} \\
&\geq E_\theta \left\{ \left(\psi_{n-1:n} \circ \psi^{-1}_{n:n+1}\right) \left(\psi_{n:n+1}(H(v|\theta))\right)\right\} \\
&= E_\theta \left\{ \psi_{n-1:n} \left(\left(\psi^{-1}_{n:n+1} \circ \psi_{n:n+1}\right) (H(v|\theta))\right)\right\} \\
&= E_\theta \left\{\psi_{n-1:n}(H(v|\theta))\right\} \\
&= F_{n-1:n}(v)
\end{aligned} \tag{8}$$

so $\psi^{-1}_{n:n+1} \left(F_{n:n+1}(v)\right) \geq \psi^{-1}_{n-1:n} \left(F_{n-1:n}(v)\right)$. □

Since we are assuming $F_{n-1:n}$ is known for each n, this gives us a test of exogenous n under the assumption of symmetric, conditionally independent private values: specifically, that

$$\psi^{-1}_{n-1:n} \left(F_{n-1:n}(v)\right) \quad \geq \quad \psi_{n'-1:n'} \left(F_{n'-1:n'}(v)\right) \tag{9}$$

for any $n > n'$ and any $v \in \mathbb{R}$.

3.1.3 Power

It remains to argue why this test is relevant, that is, why endogenous n would likely lead to a violation of (9). The logic is as follows.

The most intuitively plausible source of endogeneity of n would be if bidders were more prone to enter auctions for more valuable objects – either auctions with characteristics which make them likely to be more valuable (if the bidders knew θ but not their actual valuation at the time the entry decision is made), or auctions for objects that the bidder already knows he values more highly. Such "positive selection" is *exactly* the behavior that the test above will detect. As we mentioned, under IPV, the distribution $F_{n-1:n}$ shifts to the right as n increases, at a predictable rate; the effect of correlation is to slow down this shift. So anything that speeds it up – such as more crowded auctions being associated with more valuable objects – will lead to a violation of (9).

However, two caveats are in order. First, *negative* selection – bidders being more prone to participate in auctions for less-valuable objects – would not lead to a violation of (9). (While counterintuitive, negative selection could occur naturally if more valuable objects also tend to be valued more consistently by different buyers. This could be the case if the most valuable objects tended to be purchased by professional dealers, who were fairly unanimous in their assessments, while lower-value objects appealed to individual collectors, whose tastes were more idiosyncratic.) And second, a small amount of positive selection, accompanied by correlation among values, would be undetectable.

3.2 Identification Under Exogenous n

Having introduced a test for exogenous n in the presence of unobserved heterogeneity, we now show how exogenous variation in n can identify the "most relevant" parts of the model.

3.2.1 Identification of $F_{j:k}$ from Unbounded Data

With the existence of the unobserved demand shock $\theta \in \Theta$, the seller faces a correlated demand structure within an auction, which is not identified under standard assumptions (Athey and Haile (2002)). However, in this section, we will show that the variation induced by exogenous n does identify the *marginal* distribution of each order statistic of bidder valuations, $V_{j:k}$. Short of knowing θ, this is "the best we can hope to do": under reasonable assumptions, the distributions $F_{m-1:m}$ and $F_{m:m}$ pin down ex-ante expected revenue, expected bidder surplus, and everything payoff-relevant that does not require knowledge of θ for a particular auction.

For any fixed number of bidders m, we have assumed $F_{m-1:m}$ is identified from the data, but $F_{m:m}$ is not; we show now how exogenous variation in the number of bidders $n > m$ across auctions identifies $F_{m:m}$ for any m. To see the intuition behind the result, let H be short for $H(v|\theta)$, and recall from (1) that

$$\begin{aligned} F_{n:n}(v) &= E_\theta \left\{ H^n \right\} \\ F_{n-1:n}(v) &= E_\theta \left\{ nH^{n-1} - (n-1)H^n \right\} \end{aligned} \tag{10}$$

We will identify $F_{3:3}$ from $\{F_{n-1:n}\}_{n>3}$. Knowing that

$$\begin{array}{lclccccccccc} F_{3:4}(v) &=& E_\theta \{ & 4H^3 & - & 3H^4 & & & & & & \} \\ F_{4:5}(v) &=& E_\theta \{ & & & 5H^4 & - & 4H^5 & & & & \} \\ F_{5:6}(v) &=& E_\theta \{ & & & & & 6H^5 & - & 5H^6 & & \} \\ F_{6:7}(v) &=& E_\theta \{ & & & & & & & 7H^6 & - & 6H^7 \ \} \end{array} \tag{11}$$

we can construct linear combinations to cancel terms, giving

$$\begin{aligned} &\tfrac{1}{4}F_{3:4}(v) &&= E_\theta \left\{ H^3 - \tfrac{3}{4}H^4 \right\} \\ &\tfrac{1}{4}F_{3:4}(v) + \tfrac{3}{20}F_{4:5}(v) &&= E_\theta \left\{ H^3 - \tfrac{3}{5}H^5 \right\} \\ &\tfrac{1}{4}F_{3:4}(v) + \tfrac{3}{20}F_{4:5}(v) + \tfrac{1}{10}F_{5:6}(v) &&= E_\theta \left\{ H^3 - \tfrac{3}{6}H^6 \right\} \\ &\tfrac{1}{4}F_{3:4}(v) + \tfrac{3}{20}F_{4:5}(v) + \tfrac{1}{10}F_{5:6}(v) + \tfrac{5}{70}F_{6:7}(v) &&= E_\theta \left\{ H^3 - \tfrac{3}{7}H^7 \right\} \end{aligned} \tag{12}$$

With each added term, the right-hand side moves toward $E_\theta H^3 = F_{3:3}(v)$.

We now present the general result. In addition to identifying $F_{m:m}$ for each m, we can then use $\{F_{m:m}\}$ to identify the distribution of any other order statistic $V_{j:k}$.

Theorem 2. *Under symmetric, conditionally independent private values with exogenous n, for any $k \geq 2$ and $j \leq k$, $F_{j:k}(v)$ is uniquely determined by $\{F_{n-1:n}(v)\}_{n\in\{2,3,\ldots\}}$.*

Proof. First, for any m, we will show how $F_{m:m}(v)$ is uniquely determined by $\{F_{n-1:n}(v)\}_{n>m}$. Continue to let H denote $H(v|\theta)$. For any $M > m$, we show in the appendix (Lemma 5) that

$$\frac{1}{m-1}\sum_{n=m+1}^{M}\left(\prod_{i=m}^{n-1}\frac{i-1}{i+1}\right)F_{n-1:n}(v) \;=\; E_\theta H^m - \frac{m}{M}E_\theta H^M \qquad (13)$$

Note that the left-hand side of (13) is simply a linear combination of terms which we have assumed are known. Since $H \leq 1$, $\frac{m}{M}E_\theta H^M$ vanishes as $M \to \infty$, so

$$F_{m:m}(v) \;=\; E_\theta H^m \;=\; \lim_{M\to\infty}\frac{1}{m-1}\sum_{n=m+1}^{M}\left(\prod_{i=m}^{n-1}\frac{i-1}{i+1}\right)F_{n-1:n}(v) \qquad (14)$$

Next, pick arbitrary j, k with $j \leq k$. We know that

$$F_{j:k}(v) \;=\; E_\theta\left\{F_{j:k}(v|\theta)\right\} \;=\; E_\theta\left\{\sum_{i=j}^{k}\binom{k}{i}H^i(1-H)^{k-i}\right\} \qquad (15)$$

If we expand the polynomial $(1-H)^{k-i}$ term and sum over the different i, we can write this as

$$F_{j:k}(v) \;=\; E_\theta\left\{\sum_{m=j}^{k}a_m H^m\right\} \qquad (16)$$

where a_m are scalar coefficients. Then given linearity of expectations,

$$F_{j:k}(v) \;=\; \sum_{m=j}^{k}a_m E_\theta\left\{H^m(v|\theta)\right\} \;=\; \sum_{m=j}^{k}a_m F_{m:m}(v) \qquad (17)$$

completing the proof. □

3.2.2 Bounding $F_{m:m}$ from Bounded Data

Unfortunately, the results above relied on having data from auctions with n bidders for every positive $n \geq 2$. If there is some upper bound M on the number of bidders that appear in auctions in the data, then point identification of $F_{m:m}$ won't be possible, but we can nevertheless provide bounds on $F_{m:m}$, leading to bounds on the relevant measures.

To bound $F_{m:m}$ using only a finite number of terms $F_{m:m+1}$, $F_{m+1:m+2}$, ..., $F_{M-1:M}$,[4] we rearrange (13) to get

$$F_{m:m}(v) = \frac{1}{m-1} \sum_{n=m+1}^{M} \left(\prod_{i=m}^{n-1} \frac{i-1}{i+1} \right) F_{n-1:n}(v) + \frac{m}{M} F_{M:M}(v) \tag{18}$$

Conveniently, conditional independence leads to upper and lower bounds on $F_{M:M}(v)$, as a function of $F_{M-1:M}(v)$:

Lemma 3. *For any n, under conditionally independent private values,*

$$\psi_{n:n} \circ \psi_{n-1:n}^{-1} \left(F_{n-1:n}(v) \right) \leq F_{n:n}(v) \leq F_{n-1:n}(v) \tag{19}$$

Proof. The upper bound is because by definition, $V_{n:n} \geq V_{n-1:n}$. The lower bound is because

$$\tfrac{d}{ds} \left(\psi_{n:n} \circ \psi_{n-1:n}^{-1} \right)(s) = \tfrac{\psi'_{n:n}(t)}{\psi'_{n-1:n}(t)} = \tfrac{n t^{n-1}}{n(n-1)t^{n-2}(1-t)} = \tfrac{t}{(n-1)(1-t)} \tag{20}$$

where $t = \psi_{M-1:m}^{-1}(s)$, which is increasing in t and therefore s, so $\psi_{n:n} \circ \psi_{n-1:n}^{-1}$ is convex. Using Jensen's Inequality, then,

$$\begin{aligned} \psi_{n:n} \circ \psi_{n-1:n}^{-1} \left(F_{n-1:n}(v) \right) &= \left(\psi_{n:n} \circ \psi_{n-1:n}^{-1} \right) \left(E_\theta \psi_{n-1:n} \left(H(v|\theta) \right) \right) \\ &\leq E_\theta \left(\psi_{n:n} \circ \psi_{n-1:n}^{-1} \right) \left(\psi_{n-1:n} \left(H(v|\theta) \right) \right) \\ &= E_\theta \psi_{n:n} \left(H(v|\theta) \right) \\ &= F_{n:n}(v) \end{aligned} \tag{21}$$

completing the proof. □

Plugging the bounds given by (19) into (18) provides upper and lower

[4]Bounds on $F_{m:m}(v)$ for each $m \in \{j, \ldots, k\}$ would also lead to upper and lower bounds on $F_{j:k}(v)$ via (17); but only $F_{m-1:m}$ and $F_{m:m}$ turn out to be payoff-relevant.

bounds on $F_{m:m}(v)$:

$$\frac{1}{m-1}\sum_{n=m+1}^{M}\left(\prod_{i=m}^{n-1}\frac{i-1}{i+1}\right)F_{n-1:n}(v)+\frac{m}{M}\left(\psi_{M:M}\circ\psi_{M-1:M}^{-1}\right)\left(F_{M-1:M}(v)\right)$$

$$\leq \quad F_{m:m}(v) \quad \leq$$

$$\frac{1}{m-1}\sum_{n=m+1}^{M}\left(\prod_{i=m}^{n-1}\frac{i-1}{i+1}\right)F_{n-1:n}(v)+\frac{m}{M}F_{M-1:M}(v) \tag{22}$$

3.2.3 Applying These Estimates

Begin with the assumption that we have pointwise upper and lower bounds on $F_{n-1:n}$: that is, that

$$F_{n-1:n}(v)\in\left[\underline{F}_{n-1:n}(v),\overline{F}_{n-1:n}(v)\right] \tag{23}$$

for every v, and for $n\in\{m,m+1,\ldots,M\}$.

The bounds on $F_{m:m}$ given by (22) are increasing in $F_{n-1:n}$ for each n; so given (22) and (23), $F_{m:m}(v)\in\left[\underline{F}_{m:m}(v),\overline{F}_{m:m}(v)\right]$, with

$$\begin{aligned}\underline{F}_{m:m}(v) &= \frac{1}{m-1}\sum_{n=m+1}^{M}\left(\prod_{i=m}^{n-1}\frac{i-1}{i+1}\right)\underline{F}_{n-1:n}(v)\\ &\quad +\frac{m}{M}\left(\psi_{M:M}\circ\psi_{M-1:M}^{-1}\right)\left(\underline{F}_{M-1:M}(v)\right)\\ \overline{F}_{m:m}(v) &= \frac{1}{m-1}\sum_{n=m+1}^{M}\left(\prod_{i=m}^{n-1}\frac{i-1}{i+1}\right)\overline{F}_{n-1:n}(v)+\frac{m}{M}\overline{F}_{M-1:M}(v)\end{aligned} \tag{24}$$

If $F_{m-1:m}$ and $F_{m:m}$ were known exactly, then letting v_0 denote the seller's own valuation of the unsold object, the expected profit from an ascending auction with n bidders and a reserve price of r is

$$\begin{aligned}\pi_n(r) &= E_{V_{n-1:n},V_{n:n}}\left\{\mathbf{1}_{V_{n:n}\geq r}\left(\max\{V_{n-1:n},r\}-v_0\right)\right\}\\ &= \left(F_{n-1:n}(r)-F_{n:n}(r)\right)(r-v_0)+\int_r^{+\infty}(v-v_0)dF_{n-1:n}(v)\end{aligned} \tag{25}$$

((25) refers exactly to the expected profit from a second-price sealed-bid auction. Since, conditional on the realization of θ, the bidders are in a symmetric independent private values environment, revenue equivalence holds, so given equilibrium play, "any" auction with reserve price r yields expected profit equal to (25). Even without assuming equilibrium play, (25) holds to within a minimum bid increment in any ascending auction where bidders play undominated strategies and do not employ jump bids.)

Quint (2008) shows that (25) is stochastically increasing in both $V_{n-1:n}$ and $V_{n:n}$, and therefore decreasing pointwise in $F_{n-1:n}(v)$ and $F_{n:n}(v)$; combining (23), (24), and (25), then, gives $\pi_n(r) \in [\underline{\pi}_n(r), \overline{\pi}_n(r)]$, with

$$\begin{array}{rcl} \underline{\pi}_n(r) & = & \left(\overline{F}_{n-1:n}(r) - \overline{F}_{n:n}(r)\right)(r - v_0) + \int_r^{+\infty}(v - v_0)d\overline{F}_2(v) \\ \overline{\pi}_n(r) & = & \left(\underline{F}_{n-1:n}(r) - \underline{F}_{n:n}(r)\right)(r - v_0) + \int_r^{+\infty}(v - v_0)d\underline{F}_2(v) \end{array} \tag{26}$$

As in Haile and Tamer (2003), we can use the bounds on $\pi_n(r)$ to calculate bounds on the profit-maximizing reserve price $r^* = \arg\max_r \pi_n(r)$ for auctions of a given size: we can show $r^* \in [\underline{r}, \overline{r}]$, where

$$\begin{array}{rcl} \underline{r} & = & \inf\{r : \overline{\pi}_n(r) \geq \max_{r'} \underline{\pi}_n(r')\} \\ \overline{r} & = & \sup\{r : \overline{\pi}_n(r) \geq \max_{r'} \underline{\pi}_n(r')\} \end{array} \tag{27}$$

(Letting $r^{**} = \arg\max_r \underline{\pi}_n(r)$, we know that $\overline{\pi}_n(r^*) \geq \pi_n(r^*) \geq \pi_n(r^{**}) \geq \underline{\pi}_n(r^{**})$, which leads to (27).)

Similar to (25), we can write the ex-ante expected surplus of a participating bidder in an auction with n bidders as

$$\begin{array}{rcl} u_n(r) & = & \frac{1}{n}E_{V_{n-1:n},V_{n:n}}\left\{\mathbf{1}_{V_{n:n}\geq r}\left(V_{n:n} - \max\{V_{n-1:n}, r\}\right)\right\} \\ & = & \frac{1}{n}E_{V_{n-1:n},V_{n:n}}\left\{\max\{V_{n:n}, r\} - \max\{V_{n-1:n}, r\}\right\} \end{array} \tag{28}$$

(where $V_{0:1} \equiv 0$), which is stochastically increasing in $V_{n:n}$ and decreasing in $V_{n-1:n}$, and calculate bounds on $u_n(r)$ from (23) and (24).

3.3 Extending to an "Incomplete" Model of Bidding

Finally, we can adapt both the test above and the identification results to the incomplete model of an English auction considered in Haile and Tamer (2003). Rather than assuming an exact understanding of equilibrium behavior, Haile and Tamer (2003) assume (roughly) that bidders play undominated strategies: in particular, that nobody bids higher than his valuation, and that nobody allows the auction to end at a price he would willingly outbid.

Given only these two assumptions and observations of the two highest bids in each auction, we can calculate upper and lower bounds on $F_{n-1:n}$. Specifically, these two assumptions imply $V_{n-1:n} \geq B_{n-1:n}$ and $V_{n-1:n} \leq B_{n:n} + \delta$, where δ is the minimum bid increment and $B_{j:n}$ the j^{th} lowest bid in a given auction. This leads to the bounds

$$G_{n:n}(v-\delta) \quad \leq \quad F_{n-1:n}(v) \quad \leq \quad G_{n-1:n}(v) \tag{29}$$

where $G_{j:n}$ is the (observed) distribution of $B_{j:n}$.

Once $F_{n-1:n}$ is bounded pointwise, the test for exogenous n ((9)) becomes a test of whether

$$\psi^{-1}_{n-1:n}\left(\overline{F}_{n-1:n}(v)\right) \quad \geq \quad \psi_{n'-1:n'}\left(\underline{F}_{n'-1:n'}(v)\right) \tag{30}$$

for any $n > n'$, since under the assumptions of our model, $\psi^{-1}_{n-1:n}\left(\overline{F}_{n-1:n}(v)\right) \geq \psi^{-1}_{n-1:n}\left(F_{n-1:n}(v)\right) \geq \psi_{n'-1:n'}\left(F_{n'-1:n'}(v)\right) \geq \psi_{n'-1:n'}\left(\underline{F}_{n'-1:n'}(v)\right)$. We already discussed above how to apply our estimates using upper and lower bounds on $F_{n-1:n}$.

4 Testing Different Entry Models

Treatment of n as independent of θ and bidder values, of course, begs the question of what model of bidder participation we have in mind. In this section, we discuss three standard models of participation, two of which are potentially consistent with our treatment of n as exogenous, and introduce a test to distinguish between them.

4.1 Models of Bidder Participation

4.1.1 Model 1: Random Entry

This is hardly a model; it is simply the assumption that whatever determines bidder participation has nothing to do with the characteristics of the object/auction θ, including the reserve price r. (This is the implicit assumption behind any analysis of auctions as having a fixed number of bidders. Note also that a higher reserve price r still might affect the number of bidders actually submitting bids, since "participating" bidders with valuations below r may not bid at all; we simply assume that it does not affect the number of bidders who bid if the price is favorable.)

4.1.2 Model 2a: Fully Blind Entry Decisions

This is the model of Levin and Smith (1994). Bidders face a (homogeneous) cost c of learning their valuation for the object and participating in the auction. Ex-ante expected surplus from participation is greater than c if no other bidders participate, and less than c if all other potential bidders participate; bidders play a symmetric mixed strategy in choosing whether or not to participate. Bidders know neither θ nor their valuation when they decide whether to enter, but do know r.

4.1.3 Model 2b: Partly Blind Entry Decisions

Identical to Model 2a, except the bidders know θ when they decide whether to enter.

4.1.4 Model 3: Fully Informed Entry Decisions

This is the model of Samuelson (1985). Bidders face a homogeneous cost of participating in the auction, but know their valuations at the time they make the decision.[5] Thus, all bidders employ the same cutoff strategy, participating in the auction if and only if their valuations exceed some cutoff $v^* > r$.

4.2 Testing Between The Models

Next, we introduce a test to distinguish between these models of participation. The key requirement is that there be exogenous variation in r. After introducing the basics of the test, we discuss how to implement it using an instrumental-variables approach if r is endogenous.

The key to the test is the effect of an increase in reserve price, from r to $r' > r$, on the distribution of winning bids, according to each of the three models. (The test does not distinguish between 2a and 2b, but these can be distinguished using the earlier test for exogenous n.)

Under Model 1, increasing the reserve price does not affect participation, and so wherever the winning bid is determined by the second-highest bidder's valuation, the distribution is unchanged. An increase in reserve price from r to r' affects the distribution of winning bids at and below r' – some winning bids below r' become winning bids at r', and others become auctions where no bid exceeds the reserve and the object is not sold – but the distribution above r' is unaffected.

Under Model 2, increasing the reserve price reduces each potential bidder's probability of entry, stochastically decreasing the number of participants, and therefore the distribution of winning bids, which shifts to the left at every point above r'.

[5] In fact, it is important for this model that bidders know both θ and their own valuation. That is, θ must be interpreted as actual observable characteristics of the auction (albeit unobserved by the econometrician), not just as a technical convenience for inducing correlation. If bidders knew their valuations but not θ, they would be in a correlated-private-values world, and the cutoff-strategy equilibrium considered here might not exist.

Under Model 3, increasing the reserve price increases the cutoff below which bidders will not enter. This shifts the distribution of winning bids to the left at points below the new threshold; above the threshold, bidders still enter as before, and the distribution is not affected. (Without variation in θ, Model 3 would predict that an auction with reserve price r would have no winning bids in the interval $(r, v^*(r))$, and that the distribution of winning bids to the right of $v^*(r)$ would be the same as at lower reserve prices. Variation in θ, however, would lead to a different cutoff $v^*(r)$ for each θ, effectively "smoothing" the transition between these two regimes.)

These effects are summarized in the following theorem, which leads directly to a nonparametric test to distinguish between the three models.

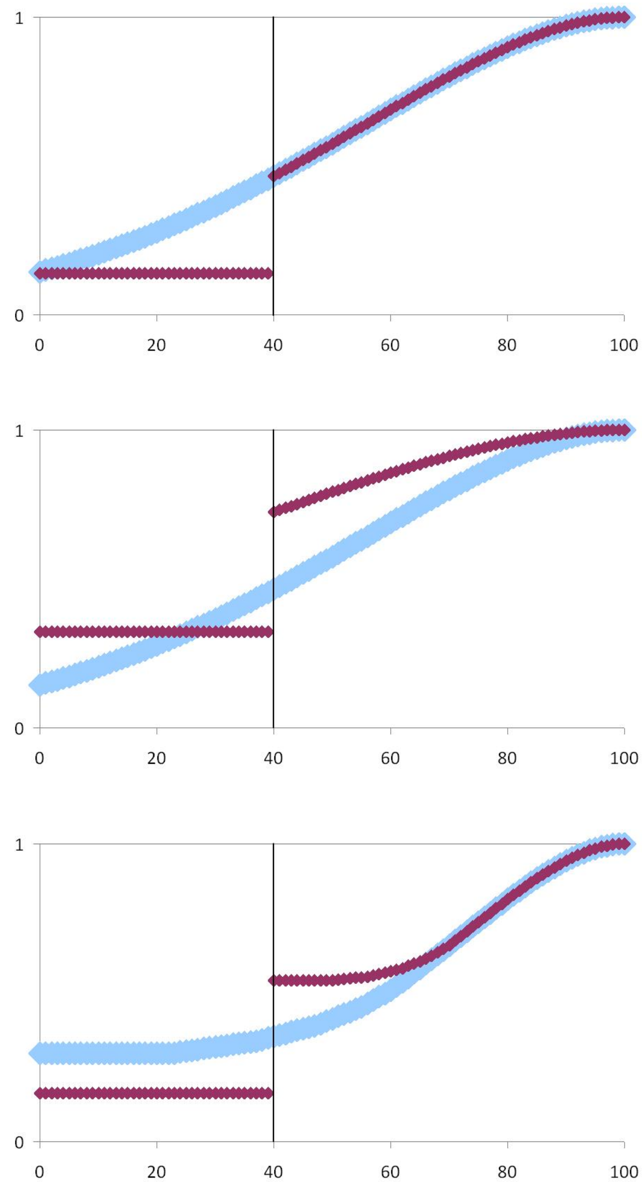
Theorem 4. *Suppose that r varies exogenously in the data between $\underline{r}$ and $\overline{r}$. Let $F_r(v)$ denote the cumulative distribution of transaction prices (winning bids) in auctions with reserve price r,* un*conditional on the number of bidders participating. Then*

- *under Model 1 (random entry), $F_r(v)$ does not vary with r for any $v > \overline{r}$*
- *under Model 2 (blind entry), $F_r(v)$ is increasing in r for any $v > \overline{r}$*
- *under Model 3 (informed entry), $F_r(v)$ is increasing in r for $v \in (\overline{r}, \tilde{v})$, and does not vary with r for $v > \tilde{v}$, for some $\tilde{v} > \overline{r}$*

Each distribution $F_r(\cdot)$ is nonparametrically identified from bid data. Figure 1, on the next page, illustrates the effects shown in Theorem 4; for each model, the blue curve gives the distribution of transaction prices for auctions with no reserve price, and the red curve for auctions with a reserve price of 40. (The exact primitives of these examples are given in the appendix.)

To give an example of the simplest possible test, suppose that the reserve price was chosen randomly for each auction, and always took the value of either 0 or 40. We would pick a price level above but close to 40 (say, 41), and another well above 40 (say, 80). Looking at the CDF of winning bids at

Figure 1: The effect of reserve price under three models of endogenous participation. Blue curves are the CDF of transaction prices with $r = 0$, red with $r = 40$.



these two points, and comparing between the auctions with reserve price of 0 and those with reserve price of 40, we would expect to find:

Model	Pr(winning bid $\leq$ 41)	Pr(winning bid $\leq$ 80)
Random	Same	Same
Blind	Higher	Higher
Informed	Higher	Same

Thus, checking whether the CDF increased in response to an increase in r at just two points is enough to distinguish among the three models.

It's important to note that this test explicitly does *not* condition on n – in fact, it can be run without even observing n. However, as mentioned before, the CDF should account for the "zeroes" – auctions where no bidder exceeded the reserve, and therefore the object was not sold. If variation in r was endogenous but an instrument was available, we could use nonparametric IV to again establish how $F_r(v)$ varied with r at various v.

4.3 Policy Implications of Each Model

Testing between the different models is valuable because each model suggests a different course of action to a profit-maximizing seller.

Under Model 1, the choice of reserve price r does not affect bidder participation. If the seller is unaware of θ, then, his problem is simply to solve

$$\max_r \sum_{n=0}^{\infty} p_n \pi_n(r) \tag{31}$$

where p_n is the probability of exactly n bidders participating. This leads to a reserve price $r > v_0$; we showed above how to calculate upper and lower bounds on this maximizer. (If the seller is able to charge an entry fee, the optimal policy is to set $r = v_0$ and extract all expected bidder surplus via the entry fee.)

Under either Model 2a or 2b, setting $r = v_0$ (and no entry fee even if one is available) is optimal. When bidders already know θ, we are in the IPV world considered in Levin and Smith (1994); they show that setting $r = v_0$ is both profit-maximizing and induces the socially efficient level of entry. When bidders do not yet know θ, they perceive valuations as being correlated; however, the same result holds. (We show this in the appendix, for completeness.)

Under Model 3, a reserve price $r > v_0$ is optimal, but the optimal reserve price is lower than under Model 1. Further, since n is endogenous under Model 3, we do not yet have a strategy for identifying the primitives of the model or the optimal reserve price in this case.

5 Monte Carlo and Empirical Application

(Coming soon.)

Appendix

A.1 Proof of Equation (13)

Lemma 5. *For any $m > 1$ and $M > m$,*

$$\frac{1}{m-1}\sum_{n=m+1}^{M}\left(\prod_{i=m}^{n-1}\frac{i-1}{i+1}\right)F_{n-1:n}(v) \quad = \quad E_\theta H^m - \frac{m}{M}E_\theta H^M \qquad (32)$$

Proof. Fixing arbitrary m, we prove the lemma by induction on M. First, let $M = m + 1$. Then

$$\begin{aligned}
\tfrac{1}{m-1}\textstyle\sum_{n=m+1}^{M}\left(\prod_{i=m}^{n-1}\tfrac{i-1}{i+1}\right)F_{n-1:n}(v) &= \tfrac{1}{m-1}\left(\tfrac{m-1}{m+1}\right)F_{m:m+1}(v)\\
&= \tfrac{1}{m+1}E_\theta\left((m+1)H^m - mH^{m+1}\right)\\
&= E_\theta\left(H^m - \tfrac{m}{M}H^M\right)
\end{aligned}$$

For the inductive step, if (32) holds for $M = K$, then

$$\begin{aligned}
&\tfrac{1}{m-1}\textstyle\sum_{n=m+1}^{K+1}\left(\prod_{i=m}^{n-1}\tfrac{i-1}{i+1}\right)F_{n-1:n}(v)\\
&= \tfrac{1}{m-1}\textstyle\sum_{n=m+1}^{K}\left(\prod_{i=m}^{n-1}\tfrac{i-1}{i+1}\right)F_{n-1:n}(v) + \tfrac{1}{m-1}\left(\prod_{i=m}^{K}\tfrac{i-1}{i+1}\right)F_{K:K+1}(v)\\
&= E_\theta H^m - \tfrac{m}{K}E_\theta H^K + \tfrac{1}{m-1}\tfrac{(m-1)m}{K(K+1)}E_\theta\left((K+1)H^K - KH^{K+1}\right)\\
&= E_\theta H^m - \tfrac{m}{K}E_\theta H^K + \tfrac{m}{K}E_\theta H^K - \tfrac{m}{K+1}E_\theta H^{K+1}\\
&= E_\theta H^m - \tfrac{m}{K+1}E_\theta H^{K+1}
\end{aligned}$$

so it holds for $M = K + 1$. □

A.2 Examples in Section 4

For the example of Model 2, there are six potential bidders, with valuations drawn independently from the uniform distribution on $[0, 100]$. The cost of participation is 10. With no reserve price, this leads to each bidder participating with probability 0.466, leading to zero, one, two, three, four, five, and six bidders participating in a given auction with probabilities 0.023, 0.121, 0.265, 0.308, 0.202, 0.070, and 0.010, respectively. With a reserve price of 40, each bidder enters with probability 0.2867, giving probabilities 0.132, 0.318, 0.319, 0.171, 0.052, 0.008, and 0.001.

For the example of Model 1, bidders again have valuations drawn independently from the uniform distribution on $[0, 100]$; the mix of auction sizes is set to match Model 2 with no reserve price, that is, it is assumed that independent of the reserve price, there will be zero, one, two, three, four, five, or six bidders with probability 0.023, 0.121, 0.265, 0.308, 0.202, 0.070, and 0.010, respectively.

For the example of Model 3, bidders again have valuations drawn independently from the uniform distribution on $[0, 100]$. The cost of participation is 5. It is assumed that there are either 2, 3, 4, 5, or 6 potential bidders in each auction, with equal probabilities, and that the potential bidders know how many of them there are in a given auction. This leads to entry cutoffs of 22.36, 36.84, 47.29, 54.93 and 60.7 when there is no reserve price and two, three, four, five, and six potential bidders, respectively (each bidder enters if and only if his valuation is above the cutoff); and cutoffs of 50, 55.96, 61.5, 66.14, and 69.92 when the reserve price is 40.

A.3 $r^* = v_0$ under Blind Entry

This is essentially the proof from Levin and Smith (1994); since they claimed the result only for $v_0 = 0$ and affiliated values, we reproduce the proof here to show that it extends without difficulty to $v_0 > 0$ and conditionally independent values.

Suppose there are N potential bidders, each of whom face an entry cost $c > 0$ to learn their valuation for the object and participate in the auction. Suppose that N is large enough that all bidders participating in the auction is unprofitable for them. Suppose bidders have exchangeable private values which are otherwise arbitrarily jointly distributed. Assume that the bidders play the unique symmetric equilibrium in which they each enter with the same probability q.

Theorem 6. *Setting reserve price equal to v_0 (regardless of how many entrants there are) maximizes the seller's expected profit.*

Proof. We will first show that if the seller is allowed to charge a fee to each bidder who participates, then setting $r = v_0$ would be optimal; we will then show that the optimal entry fee is 0.

If an entry fee is available, reserve prices $r \neq v_0$ are never optimal (even when the reserve price is allowed to depend on the number of entrants). This is because any set of (potentially entry-dependent) reserve prices $(r_1, r_2, \ldots, r_N)$, along with an entry fee e, leads to some level of entry q which could be replicated by setting all the reserve prices to v_0 and charging a different entry fee e'. This change would leave entry the same, but it would lead to greater ex-post efficiency. But since (by assumption) bidders mix between entering and not entering, they must get 0 expected profit; so the seller captures all the surplus, which means that this change would lead to higher revenue. So now we only worry about setting the optimal e.

Since we now consider only auctions with reserve price $r = v_0$, we can without loss replace each bidder's private value V_i with $\max\{V_i, v_0\}$. This is without loss because a bidder with value $V_i < v_0$ gets the same ex post surplus (0) as a bidder with value $V_i = v_0$, and a seller gets the same profit from a bidder with value $V_i < v_0$ as from a bidder with $V_i = v_0$ (whether this bidder is the highest, second-highest, or lower). So replace each bidder's value V_i with $\max\{V_i, v_0\}$, and let $\overline{v}^{k:n}$ be the expected value of the k^{th} lowest of these when n bidders participate.

Next, put aside the seller's problem and suppose instead that we were a social planner who could choose q (each bidder's probability of entry) to maximize total surplus. Note that when n bidders arrive and participate in an auction with reserve price $r = v_0$, the expected social surplus is simply $E\left(\max\{V^{n:n}, v_0\}\right) = \overline{v}^{n:n}$; so a social planner would maximize

$$S(q) = \sum_{n=0}^{N} \binom{N}{n} q^n (1-q)^{N-n} \overline{v}^{n:n} - Nqc$$

where N is the number of potential bidders and c is the entry cost. Differentiating with respect to q gives

$$\frac{dS}{dq} = \sum_{n=0}^{N} \binom{N}{n} \left[n q^{n-1} (1-q)^{N-n} - (N-n) q^n (1-q)^{N-n-1} \right] \overline{v}^{n:n} - Nc$$

Dividing by N and setting this equal to 0 gives

$$\sum_{n=0}^{N} \left(\begin{array}{c} N \\ n \end{array} \right) \left[\frac{n}{N} q^{n-1} (1-q)^{N-n} - \frac{N-n}{N} q^n (1-q)^{N-n-1} \right] \overline{v}^{n:n} = c$$

Splitting this into two separate sums, and noting that the $n = 0$ term of the first and the $n = N$ term of the second vanish, gives

$$\sum_{n=1}^{N} \left(\begin{array}{c} N \\ n \end{array} \right) \frac{n}{N} q^{n-1} (1-q)^{N-n} \overline{v}^{n:n} - \sum_{n=0}^{N-1} \left(\begin{array}{c} N \\ n \end{array} \right) \frac{N-n}{N} q^n (1-q)^{N-n-1} \overline{v}^{n:n} = c$$

Combining the fractions $\frac{n}{N}$ and $\frac{N-n}{N}$ with the adjacent combinatorial terms gives

$$\sum_{n=1}^{N} \left(\begin{array}{c} N-1 \\ n-1 \end{array} \right) q^{n-1} (1-q)^{N-n} \overline{v}^{n:n} - \sum_{n=0}^{N-1} \left(\begin{array}{c} N-1 \\ n \end{array} \right) q^n (1-q)^{N-n-1} \overline{v}^{n:n} = c$$

Substituting a new index $n' = n - 1$ into the first sum gives

$$\sum_{n'=0}^{N-1} \left(\begin{array}{c} N-1 \\ n' \end{array} \right) q^{n'} (1-q)^{N-n'-1} \overline{v}^{n'+1:n'+1} - \sum_{n=0}^{N-1} \left(\begin{array}{c} N-1 \\ n \end{array} \right) q^n (1-q)^{N-n-1} \overline{v}^{n:n} = c$$

Finally, replacing n' with n and combining the two sums gives

$$\sum_{n=0}^{N-1} \left(\begin{array}{c} N-1 \\ n \end{array} \right) q^n (1-q)^{N-n-1} \left(\overline{v}^{n+1:n+1} - \overline{v}^{n:n} \right) = c$$

Next, start with a sample of $n + 1$ bidders, and randomly select n of them. With probability $\frac{n}{n+1}$, your subsample includes the one with the highest V_i; with probability $\frac{1}{n+1}$, it contains the second-highest but not the highest. Since these probabilities are independent of $\{V_i\}$,

$$\overline{v}^{n:n} = \frac{n}{n+1} \overline{v}^{n+1:n+1} + \frac{1}{n+1} \overline{v}^{n:n+1}$$

Plugging this into the last expression gives

$$\sum_{n=0}^{N-1} \left(\begin{array}{c} N-1 \\ n \end{array} \right) q^n (1-q)^{N-n-1} \frac{1}{n+1} \left(\overline{v}^{n+1:n+1} - \overline{v}^{n:n+1} \right) = c \qquad (33)$$

which defines the q that maximizes social surplus.

Next, consider one bidder's problem. He knows that with probability $\left(\begin{array}{c} N-1 \\ n \end{array} \right) q^n (1-q)^{N-n-1}$, exactly n of his opponents will enter. Should that happen and he chooses to enter, he'll have the highest value with probability $\frac{1}{n+1}$; and should that happen, his expected surplus will be $\overline{v}^{n+1:n+1} - \overline{v}^{n:n+1}$. So the unique symmetric equilibrium to the entry game involves each bidder entering with probability q, where

$$\sum_{n=0}^{N-1} \left(\begin{array}{c} N-1 \\ n \end{array} \right) q^n (1-q)^{N-n-1} \frac{1}{n+1} \left(\overline{v}^{n+1:n+1} - \overline{v}^{n:n+1} \right) \quad = \quad c+e \qquad (34)$$

where e is the entry fee imposed by the seller. Comparing (33) to (34), when $e = 0$, the socially optimal level of entry occurs. Since bidders get expected payoff of 0, social surplus equals seller's expected profit, which is therefore maximized by setting $r = v_0$ and $e = 0$. Since this does not depend on the joint distribution of bidder valuations, it is true whether or not entrants condition entry on the value of θ. $\square$

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