

Inverting Demand Systems

Amit Gandhi

University of Wisconsin-Madison

Abstract

In this paper, we present a nonparametric generalization of the inversion theorems for non-separable demand models developed by Berry (1994) and Berry, Levinsohn, and Pakes (1995) (BLP for short). We proceed by developing a set of axioms on consumer choice that ensure invertibility of market demand, which generalize the existence and uniqueness conditions from Berry (1994). Moreover we show that iterating the “element by element” inverse mapping developed in Berry (1994) globally converges to the unique inverse, which generalizes the computational solution to the inverse problem developed by BLP (1995). Our results do not require any parametric structure on the random utility model, and in particular, we allow unobserved product characteristics to enter consumer preferences in a non-separable fashion, thus relaxing the fundamental additivity requirement of Berry (1994) and BLP (1995).

1 Introduction

Starting with Berry (1994), discrete choice models have become the dominant approach to estimating market level demand functions for differentiated products. Under this approach, a random utility model gives rise to a discrete choice demand function over differentiated products at the individual level, which is then aggregated using the distribution of tastes in the population to generate a market demand system. Market level data on prices, quantities, and observable demand shifters can then be used to estimate the underlying distribution of tastes in the model. This approach has allowed researchers to address a wide variety of policy problems (such as welfare gains from the introduction of a new good (Petrin, 2002), or the price effects of a merger (Nevo, 2000), etc).¹

¹For a more comprehensive listing of applications, see Berry and Haile (2009).

However the challenge in using random utility models to estimate market demand functions is that the econometric unobservable in the model does not aggregate up in an additively separable way. Consider a set of J differentiated products with prices $p \in \mathbb{R}_+^J$ and observable demand shifters $x \in X$. If $q \in \mathbb{R}^J$ denotes the quantities demanded of each product, then a traditional econometric model of market demand would be expressed as

$$q_i = D_i(p, x) + \epsilon_i \quad i = 1, \dots, J \quad (1)$$

where $\epsilon = (\epsilon_1, \dots, \epsilon_J) \in \mathbb{R}^J$ denotes a vector of unobservable demand shocks. An endogeneity problem arises as the equilibrium in the market correlates the unobserved demand shocks ϵ and prices p . However the additively separable structure of the demand shock in (1) allows nonlinear instrumental variable techniques, such as parametric GMM estimation or the more recent nonparametric methods (see e.g., Newey and Powell (2003) and Hall and Horowitz (2005)) to be brought to bear on the problem of estimating demand D_i .

Unfortunately, when market demand D_i is modeled as the outcome of the consumption decisions of a population of utility maximizing consumers, the demand shock ϵ does not aggregate up in an additively separable way. That is, the model generates demand functions of the form $q_i = D_i(p, x, \epsilon)$, and the market demand system $D(p, x, \epsilon) = [D_1(p, x, \epsilon), \dots, D_J(p, x, \epsilon)]$ is not consistent with the additively separable representation $D(p, x) + \epsilon$.

Identification and estimation of such a non-separable demand model $D(p, x, \epsilon)$ depends upon the two fundamental issues regarding the structure of demand.

1. Can the model be inverted such that the residual function $\epsilon = \rho(q, p, x)$ uniquely exists for any realization of the observable (q, p, x) ?
2. Does there exist a globally convergent algorithm to compute the inverse $\epsilon = \rho(p, q, x)$ for any realization of the observables (p, q, x) .

In this paper we develop a set of axioms on market demand that provide affirmative answers to both questions. Moreover, the axioms are automatically satisfied by discrete choice models of market demand so long as i) the econometric shock represents a *vertical* demand characteristic (that is, a product characteristics that all consumers positively value) and ii) the set of products in the market cannot be decomposed into two disjoint subsets such that all the substitutability between products in the market occurs within the subsets (that is, the market does not consist of two separate sub-markets).

Our axioms generalize the solution to the above question 1 developed by Berry (1994), and the solution to question 2 developed by Berry et al. (1995) (BLP for short), which today form the basis for the large volume of empirical work on product differentiated demand systems. Both the results in Berry (1994) and Berry et al. (1995) fundamentally rely on the restriction that the demand shock ϵ enters the preferences of agents in the random utility model in an additive way. Thus one way to understand their solution to the non-separability problem is that the assumption of additivity of the econometric error term in market demand is replaced by the assumption of additivity of the econometric error term in consumer preferences. The cost of this approach is that it generates an inherent asymmetry between the role of the observables (p, x) and the unobservable ϵ in consumer preferences that has no basis in economic theory.²

Our solution to the the problem of existence and uniqueness of the inverse is based on a nonparametric formulation of the underlying demand primitives, and thus relaxes altogether functional form restrictions that have formed the basis of the literature to date. As such, it provides a foundation for the nonparametric identification of random utility models of differentiated product demand. Indeed, in concurrent work by Berry and Haile (2009), it is shown that the solution to the inverse problem developed in this paper can be combined with the nonparametric instrumental variable condition from Newey and Powell (2003) to nonparametrically identify (for the first time) a discrete choice differentiated products demand model with price endogeneity.

Of course, since we relax the additivity of the shock ϵ in consumer preferences, the contraction mapping from BLP for computing the inverse is no longer valid. The availability of the contraction mapping for computing the residual function is an important reason for the wide adoption of the BLP technique in the empirical literature.³ Thus in order to empirically implement the nonseparable demand model whose foundations are examined in both this paper and Berry and Haile (2009), an alternative computational approach for computing the inverse is necessary. We show that iterating on the “element by element” mapping introduced by Berry (1994) indeed yields a globally convergent process that can be used computationally in a manner analogous to the BLP contraction

²In particular, it imposes quasilinearity of a consumer’s indifference curves in the demand error ϵ , and thus the marginal rate of substitution among observables is independent the unobservable ϵ .

³Strictly speaking, this literature does not invert the econometric residual directly, but rather a vector of “mean utility levels” δ , which is an additively separable function of the residual ϵ .

mapping.⁴

2 General Statement of the Problem

Consider a market for a differentiated commodity with J products. The products $j = 1, \dots, J$ are each described by a vector of characteristics $z_j \in Z$ and a market price $p_j \in \mathbb{R}$. The market is thus described by a menu of price/characteristic combinations, i.e., $M = [(p_1, z_1), \dots, (p_J, z_J)]$.

A consumer with characteristics $v \in V$ has direct preferences over income Y (i.e., consumption of the numeraire) and characteristics $z \in Z$ of the differentiated commodity given by the utility function $u(Y, z, v)$. We leave as flexible the exact nature of the type space V (it can for example be infinite dimensional space), and require only that it be a measurable space. The conditional indirect utility of a consumer i from product j is thus given by

$$u_{ij} = u(p_j, z_j, v_i).$$

Let P be the probability measure over V that describes the distribution of consumer characteristics in the population.

The market share s_j of product j is simply then the measure of the set of consumers who prefer good j from the set of J products, i.e.,

$$s_j = P\{v_i \in V : u_{ij} \geq u_{ik}, \forall k \in \{1, \dots, J\}\}, \quad (2)$$

This defines a differentiated product demand system, i.e., holding fixed the product characteristics z_j for each product $j \in \{1, \dots, J\}$, s_j is determined by the function $\sigma_j(p_1, \dots, p_J)$ defined by the RHS of (2), which is the demand function for product j . The market demand system $\sigma(p_1, \dots, p_J)$ is defined by the J -tuple of product demand functions.

An econometric problem arises due to the presence of product characteristics that are unobserved to the econometrician. We describe these unobserved product characteristics for any product j by a scalar $\xi_j \in \mathbb{R}$. The interpretation is that ξ_j represents a vertical demand characteristics, or equivalently, it represents an unobserved “quality” shock at the product level. Then for each

⁴Berry (1994) uses the “element by element” inverse for his proof of existence of the inverse. We show that iterating this map generates a convergent algorithm despite the fact that it does not form a contraction in the non-additive setting we consider.

$j \in \{1, \dots, J\}$, $z_j = (x_j, \xi_j)$, where $x_j \in X$ is the product's vector of observed characteristics.

Observe that holding fixed the prices $\{p_j\}_{j=1}^J$ and observed characteristics $\{x_j\}_{j=1}^J$ of all goods in the market, we can express the market demand system in qualities rather than prices, i.e., $\sigma(\xi_1, \dots, \xi_J)$ is defined by the RHS of (2).

We can now state our problem. Given an observed market

$$M = [(p_1, x_1), \dots, (p_J, x_J)],$$

and the observed demand shares $s = (s_1, \dots, s_J) \in \Delta^{J-1}$ (where $\Delta^{J-1} = \{(t_1, \dots, t_J) \in \mathbb{R}^J \mid \forall j, t_j \geq 0, \text{ and } \sum_j t_j = 1\}$ is the $J - 1$ dimensional unit simplex), and the distribution of consumer characteristics P , the problem is whether the demand system is invertible in qualities, i.e., does there exist a unique solution to the system of equations

$$s_j = \sigma_j(\xi_1, \dots, \xi_J) \text{ for } j = 1, \dots, J. \quad (3)$$

We now axiomatize the market demand system $\sigma(\xi_1, \dots, \xi_J)$ so as to both ensure that there exists a unique solution to (3), and that the axioms hold with little further structure on the random utility model (RUM) of demand (2).

3 The Main Result

We now axiomatize conditions on the demand system $\sigma(\xi_1, \dots, \xi_J)$ that are sufficient for existence and uniqueness of an inverse. We divide the conditions into two groups, the first group consists of three regularity conditions. In the appendix, we show that these regularity conditions are satisfied when σ arises from the nonparametric random utility model (2) (with heterogeneity among consumers ranging over a functional space) under only a mild continuity requirement on the distribution of preferences. The second group of conditions consists of two substantive conditions on the nature of preferences. We first introduce the regularity conditions:

R1 $\sigma(\xi_1, \dots, \xi_J) : \mathbb{R}^J \rightarrow \Delta^{J-1}$ and is continuous.

R2 For any $\xi = (\xi_1, \dots, \xi_J) \in \mathbb{R}^J$, and for any strict subset of products $I \subset \{1, \dots, J\}$, if $\xi' = (\xi'_1, \dots, \xi'_J) \in \mathbb{R}^J$ is such that $\xi'_j \geq \xi_j$ for all $j \in I$ and

$\xi'_j \leq \xi_j$ for all $j \notin I$, then

$$\sum_{j \in I} \sigma_j(\xi') \geq \sum_{j \in I} \sigma_j(\xi)$$

and

$$\sum_{j \notin I} \sigma_j(\xi') \leq \sum_{j \notin I} \sigma_j(\xi).$$

Before defining the last regularity condition, let us introduce the convention that setting $\xi_j = -\infty$ means that we have dropped product j from the available set of products in the market.

R3 For any $\xi = (\xi_1, \dots, \xi_J) \in \mathbb{R}^J$ and any strict $I \subset \{1, \dots, J\}$, if $\xi' = (\xi'_1, \dots, \xi'_J)$ is such that $\xi'_j = \xi_j$ for $j \in I$, and $\xi'_j = -\infty$ for $j \notin I$, then for all $j \in I$, $\sigma_j(\xi') \geq \sigma_j(\xi)$.

The appendix provides a general proof that under weak conditions on the underlying random utility model (the most critical being that ξ_j represents a vertical quality characteristic that always raises the utility of consuming a product j), a discrete choice demand system σ defined by (2) satisfies R1-R3.

Observe that in the system of equations (3), the LHS shares $s \in \Delta^{J-1}$ lie in a $J - 1$ dimensional space, but the RHS quality vector $\xi \in \mathbb{R}^J$ takes values in a J dimensional space. Thus we must normalize the qualities of one product, which we label product J , to be fixed at $\xi_J = c$ (which is typically taken to be $c = 0$). The usual interpretation is that product J represents the “outside option” whose characteristics (p_J, x_J, ξ_J) do not vary across markets. Imposing this normalization, the demand system is a function $\sigma(\xi_1, \dots, \xi_{J-1})$ of only the qualities of the $J - 1$ inside goods, and the quality vector $\xi \in \mathbb{R}^{J-1}$ henceforth refers to this $J - 1$ -tuple. We will occasionally use the shorthand notation $\sigma_j(w_j, \xi_{-j}) = \sigma_j(\xi_1, \dots, \xi_{j-1}, w_j, \xi_{j+1}, \dots, \xi_{J-1})$.

We now introduce our two main substantive conditions on preferences. The first condition requires that the market cannot be broken into two separate submarkets. That is, the set of products in the market cannot be broken into nests such that all the substitution between products occurs within nests. As this property that is stronger than weak gross substitutability and weaker than strict gross substitutability (for definitions of weak and strict gross substitutability, see for example page 70 in Mas-Colell et al. (1995)), we use the informal term “strong substitutability” to describe the condition. To ease notation, for a subset $I \subset \{1, \dots, J\}$, let $J - I$ refer to the set difference $\{1, \dots, J\} - I$.

S1 For any $\xi \in \mathbb{R}^{J-1}$ such that $\sigma_j(\xi) > 0$ for all $j \in \{1, \dots, J\}$, and for any non-empty $I \subseteq \{1, \dots, J-1\}$, there exists an $i^* \in I$ with the property that strictly increasing ξ_{i^*} causes σ_j to strictly decrease for some $j \in J-I$, i.e., setting ξ' so that $\xi'_j = \xi_j$ for $j \neq i^*$ and $\xi'_{i^*} > \xi_{i^*}$, then $\sigma_j(\xi') < \sigma_j(\xi)$ for some $j \in J-I$.

Finally, for any $j \in \{1, \dots, J-1\}$, define the function

$$\sigma_j(\xi_j) = \sigma_j(\tilde{\xi}_1, \dots, \tilde{\xi}_{j-1}, \xi_j, \tilde{\xi}_{j+1}, \dots, \tilde{\xi}_{J-1}),$$

where $\tilde{\xi}_k = -\infty$ for $k \neq j$. Thus $\sigma_j(\xi_j)$ is the market share function of product j as a function of ξ_j when only it and the outside good are in the market. We can now state our last substantive condition.

S2 For any $j \in \{1, \dots, J-1\}$, $\sigma_j(\xi_j)$ is monotone increasing in ξ_j and approaches 1 as $\xi_j \rightarrow \infty$ and approaches 0 as $\xi_j \rightarrow -\infty$.

Theorem 1. *Given the normalization $\xi_J = c$, for any vector of market shares $(s_1, \dots, s_J) \in \Delta^{J-1}$ located in the interior of Δ^{J-1} , if the demand system σ satisfies R1-R3 and S1-S2, there exists a unique $(\xi_1, \dots, \xi_{J-1}) \in \mathbb{R}^{J-1}$ that solves (3).*

Proof. We first show existence. For any $j \in \{1, \dots, J-1\}$, define $\bar{\xi}_j$ as a solution to $1 - \sigma_j(\xi_j) = s_J$ (hence $\bar{\xi}_j$ sets the market share of the outside good equal to its observed value s_J under the assumption that only j and the outside good are the only products in the market). Likewise, define $\underline{\xi}_j$ as the solution to $\sigma_j(\xi_j) = s_j$. Let $\bar{\xi} > \max_{j \in \{1, \dots, J-1\}} \bar{\xi}_j$ and $\underline{\xi} < \min_{j \in \{1, \dots, J-1\}} \underline{\xi}_j$, and define the function $f : [\underline{\xi}, \bar{\xi}]^{J-1} \rightarrow [\underline{\xi}, \bar{\xi}]^{J-1}$ as

$$f_j(\xi_1, \dots, \xi_{J-1}) = \begin{cases} \bar{\xi} & \xi_j + s_j - \sigma_j(\xi_1, \dots, \xi_{J-1}) \geq \bar{\xi} \\ \underline{\xi} & \xi_j + s_j - \sigma_j(\xi_1, \dots, \xi_{J-1}) \leq \underline{\xi} \\ \xi_j + s_j - \sigma_j(\xi_1, \dots, \xi_{J-1}) & \text{otherwise.} \end{cases}$$

Observe that by construction, f is a continuous self map over $[\underline{\xi}, \bar{\xi}]^{J-1}$, and hence by the Brouwer fixed point theorem has a fixed point $\xi^* = (\xi_1^*, \dots, \xi_{J-1}^*) \in [\underline{\xi}, \bar{\xi}]^{J-1}$.

The fixed point ξ^* must lie in the interior of $[\underline{\xi}, \bar{\xi}]^{J-1}$. To see why, suppose $\xi_j^* = \bar{\xi}$ for some $j \in \{1, \dots, J-1\}$. Then by R3 and construction of $\bar{\xi}$, $\sigma_j(\xi^*) < s_J$, and since both s and σ lie in Δ^{J-1} , for some $j' \in \{1, \dots, J-1\}$ we have that

$\sigma_{j'}(\xi^*) > s_{j'}$. Observe that if this inequality holds, then by R3 and construction of $\underline{\xi}$, we have that $\xi_{j'}^* \neq \underline{\xi}$. Hence $f_{j'}(\xi^*) < \xi_{j'}^*$, and thus ξ^* could not be a fixed point. A similar argument can be shown for the case where $\xi_j^* = \underline{\xi}$ for some $j \in \{1, \dots, J-1\}$.

Hence ξ^* lies in the interior of $[\underline{\xi}, \bar{\xi}]^{J-1}$, and thus solves $s_j = \sigma_j(\xi^*)$ for each $j \in \{1, \dots, J\}$. This completes the existence argument.

To show uniqueness, let $\xi, \xi' \in \mathbb{R}^{J-1}$ that both satisfy the theorem. WLOG let $\xi'_k > \xi_k$ for some product k and let $I \subset \{1, \dots, J-1\}$ be the non-empty set indexing all products such that $i \in I$ if and only if $\xi'_i > \xi_i$. Now let product k index the product in I that satisfies the “strong substitutability” condition S1, i.e. it strictly substitutes with some product in $J-I$. Consider making the transition from ξ to ξ' in two steps. In the first step we raise only ξ_k to ξ'_k , holding fixed all the other qualities, which yields the vector $\xi^* \in \mathbb{R}^{J-1}$. By R2 we have that for each $i \in \{1, \dots, J\}$ and $i \neq k$, $\sigma_i(\xi^*) \leq \sigma_i(\xi)$. But by “strong substitutability” S1, for some $i^* \in J-I$, $\sigma_{i^*}(\xi^*) < \sigma_{i^*}(\xi)$, and thus $\sum_{i \in J-I} \sigma_i(\xi^*) < \sum_{i \in J-I} \sigma_i(\xi)$. We now make the transition from ξ^* to ξ' , which by R2 gives $\sum_{i \in J-I} \sigma_i(\xi') \leq \sum_{i \in J-I} \sigma_i(\xi^*) < \sum_{i \in J-I} \sigma_i(\xi)$. But this contradicts the assumption that both ξ and ξ' satisfy the theorem, which requires that $\sum_{i \in J-I} \sigma_i(\xi') = \sum_{i \in J-I} \sigma_i(\xi)$. Thus the solution must be unique. \square

Remark 2. The existence part of the proof follows the same basic outline as the existence result in Berry (1994), but generalizes his main substantive condition for existence. In particular, Berry utilized a much stronger version of S2, namely that for any $j \in \{1, \dots, J-1\}$ and any ξ_{-j} , $\sigma_j(\xi_j, \xi_{-j})$ approaches 1 as $\xi_j \rightarrow \infty$ and approaches 0 as $\xi_j \rightarrow -\infty$ (and instead of our map f , he established a fixed point for the “element by element” inverse mapping that we define in the next section). In random utility models where ξ_j may not enter preferences in an additive way, this strong monotonicity condition is not assured to hold even though preferences are monotone in ξ_j . Instead of requiring that ξ_j can always be varied to match any value of product j ’s share s_j for any vector of competing qualities ξ_{-j} in the market, our axiom S1 requires only that ξ_j can be varied to match s_j when the outside option is the only other competing product. We present an example below to illustrate this logic.

Our proof of uniqueness used the strong substitutes axiom S1, which relaxes a key restriction used in the Berry (1994) and Berry et al. (2007) proof of uniqueness. These papers use the condition that for any vector of qualities

(including the outside good) $(\xi_1, \dots, \xi_J) \in \mathbb{R}^J$, and for any scalar $c \in \mathbb{R}$,

$$\sigma(\xi_1, \xi_2, \dots, \xi_J) = \sigma(\xi_1 + c, \xi_2 + c, \dots, \xi_J + c). \quad (4)$$

This condition asserts that the absolute levels of quality do not matter in determining demand, but rather it is only their relative differences that possibly matter. In order for the underlying random utility model to yield market shares that satisfy both of these conditions (i.e., satisfy it for any market M and any distribution of tastes P), we must restrict preferences so that $u(p_j, x_j, \xi_j, v_i)$ is additive separable in ξ_j , i.e.,

$$u(p_j, x_j, \xi_j, v_i) = g(p_j, x_j, \lambda_{1i}) + \lambda_{2i}\xi_j, \quad (5)$$

where the consumer characteristics $v_i = (\lambda_{1i}, \lambda_{2i})$ are such that $\lambda_{2i} > 0$. This is a strong functional form restrictions that implies a quasilinearity representation of preferences in the differentiated product's unobserved characteristic ξ_j . Among many other things, it implies that a consumer's willingness to pay for an change in product characteristics x_j is independent of the level of the unobservable ξ_j . More generally, it imposes an inherent asymmetry between the observable product characteristics and the unobservable characteristic that is not grounded in economic theory. Our strong substitutes condition shows that such a functional form assumption is not necessary for uniqueness. Instead, strong substitutes is based on ξ_j being a vertical characteristic and the assumption that the market is not really two markets, i.e., it cannot be broken into nests such that there is no substitution across nests.

Remark 3. Observe that strong substitutability generalizes strict gross substitutability. Since any random utility model $u_{ij} = u(p_j, x_j, \xi_j, v_i) + \epsilon_{ij}$, where ϵ_{ij} is an idiosyncratic “love for variety” taste shock with full support over the reals, will yield a demand system σ that satisfies strict gross substitutability so long as u is monotone increasing in ξ (see Anderson et al. (1992)), such models automatically satisfy strong substitutability. The simplest demand systems that do not depend on such an idiosyncratic taste shock are models where competition takes place on a line, representing either horizontal differentiation (such as a Hotelling line) or the vertical differentiation (such as the models of Mussa and Rosen (1978) and Shaked and Sutton (1982)). Such models do not satisfy strict gross substitutability (because each product has at most two neighbors on the line). However as can be easily shown, our “strong substitutes” condition S1 is

satisfied for such models (any nest of products has a nearest neighbor outside of the nest).

Example 4. We now present an example of a model that satisfies both of our substantive axioms S1-S2, but neither of the existence and uniqueness conditions used in Berry (1994) and Berry et al. (2007). The example is a generalization of the “pure characteristics” model presented in Berry et al. (2007). They consider a random utility model of the form

$$u_{ij} = -\alpha_i p_j + g(x_j, \beta_i) + \xi_j, \quad (6)$$

where $\alpha_i \in \mathbb{R}_+$ and β_i is an arbitrary vector. The outside option has utility $u_{iJ} = \xi_J$, which is normalized so that $\xi_J = 0$. Let α follow the continuous CDF F with support \mathbb{R}_+ , and let β follow the empirical CDF G^n from a simulated sample $T = \{\beta_1, \dots, \beta_n\}$ of size n from an underlying distribution G . Letting the consumer type variables α and β be independent, we have that the distribution of consumer types in the population is a product measure $P = F \times G^n$.

Observe that (indirect) utility is required to exhibit additivity separability in the unobserved characteristic ξ_j . Unlike linearity of (indirect) utility in prices p_j (which can be motivated by direct utility being quasilinear in income), there is no principle for ξ_j entering additively as an economic restriction on the model. Our axioms allow us to generalize the above formulation and dispense with additivity. We instead consider

$$u_{ij} = \frac{\alpha_i p_j}{\exp(\xi_j)} + g(x_j, \xi_j, \beta_i), \quad (7)$$

where g is increasing in ξ_j . In addition to allowing ξ_j to enter preferences in a non-additive way, we also allow a product’s ξ_j to directly interact with the price coefficient in the random utility model. Such an interaction would reflect the fact that “unobserved quality” has an income effect in consumer preferences. Thus, for example, if the model describes the market for automobiles, we can allow for a car’s prestige level, which cannot be observed and is thus captured by ξ_j , to impact the marginal utility of income, which suggests a new role of the demand shock ξ_j for future empirical applications. More generally, the interaction $(\frac{p_j}{\exp(\xi_j)})$ has the natural economic interpretation of being a measure of “quality adjusted” prices. Finally, since we want the model to have the implication that if all “inside goods” are offered at zero price, no consumer would prefer the outside good, we restrict g so that it is positive valued.

The interpretation that Berry et al. (2007) give to (6) as a “horizontal mixture” over vertical demand systems continues to hold true in the more general formulation (7). That is, conditioning on a fixed consumer type $\beta_k \in T$, the resulting random utility model u_{ij}^k has heterogeneity ranging only over the price coefficient α . It is straightforward to see that this model conditioning on β_k generates a vertical demand system in which the products have qualities $g(x_j, \xi_j, \beta_k)$ and “quality adjusted” prices $\frac{p_j}{\exp(\xi_j)}$ for $j = 1, \dots, J-1$. We denote this vertical demand system conditioned on β_k by $\sigma^k(\xi_1, \dots, \xi_{J-1})$. The unconditional market demand $\sigma(\xi_1, \dots, \xi_J)$ is then a mixture over the vertical demand systems σ^k with mixing weights given by G^n (for details, see Berry et al. (2007)).

It is clear that S2 holds, because as ξ_j goes to ∞ , then the quality adjusted price becomes arbitrarily close to zero, and hence conditional on any consumer type $\beta_k \in T$, an arbitrarily large fraction of the α types prefer product j to the outside good (by positive valuedness of g , any good available at 0 price is preferred to the outside alternative for any consumer type). In addition, as ξ_j approaches $-\infty$, then the quality weighted price becomes arbitrarily large, and hence conditional on any type $\beta_k \in T$, an arbitrarily large fraction of the α types prefers the outside good to product j . On the other hand, if g is bounded from above or below, the stronger monotonicity condition used for existence in Berry (1994) is not assured to hold.

Now let us check strong substitutability S1. Thus take a $\xi = (\xi_1, \dots, \xi_{J-1}) \in \mathbb{R}^{J-1}$ such that $\sigma_j(\xi) > 0$ for all $j \in \{1, \dots, J\}$, and consider $I \subset \{1, \dots, J-1\}$. Now all we have to do is find *some* $\beta_k \in T$ for which $\sigma^k(\xi_1, \dots, \xi_J)$ is such that there is some $i^* \in I$ that strictly substitutes with some $j \in J - I$ (because the mixture G^n over types T will preserve this strict substitution by R2). Since by assumption in S1, $\sum_{i \in I} \sigma_i > 0$, then it must be the case that for some β_k we have $\sum_{j \in I} \sigma_j^k(\xi_1, \dots, \xi_{J-1}) > 0$. Moreover given the model (7), we must have $\sum_{j \in I} \sigma_j^k(\xi_1, \dots, \xi_{J+1}) < 1$ since there are always some arbitrarily large α types that consume the outside option conditional on β_k . Finally, the following is a straightforward fact about vertical demand systems: If we have a strict subset of products I whose market share is less than 1, then there is some $i^* \in I$ that strictly substitutes with $j \in J - I$ (this can be seen visually, i.e., any set of products on a line that doesn’t cover the market competes with a neighbor not in the set). Pick such an i^* for the vertical demand system σ^k , and we have that the strong substitutes condition is satisfied.

4 Computing the Inverse Function

The previous section showed that given the observables (s, p, x) in any market, a residual function $\xi = \rho(s, p, x)$ uniquely exists under the axioms we presented. In this section we show that the residual function can be computed by a natural iterative process. We focus on the Gauss-Siedel iteration of the nonlinear system, which is closely related to the “element by element” inverse used in the existence proof in Berry (1994).

Before we proceed, we strengthen our two main substantive condition S1-S2, and show that these stronger versions of the axioms provide a foundation for using the “element by element” inverse as a computational approach to the inverse problem. The stronger forms of both S1 and S2 that we use are satisfied in typical empirical specifications, and both conditions remain compatible with the non-separable role of ξ_j in preferences. As noted in Remark 3, whenever an idiosyncratic taste shock ϵ_{ij} with full support over the reals is added to the random utility model (such as the typical logit shock), which is typical in practice, then “strong substitutes” is always satisfied. Indeed, the demand model then becomes one that satisfies strict gross substitutability. In this section, we strengthen the substitutability condition in S1 to the case of strict gross substitutes.

S1' For $j \in \{1, \dots, J-1\}$ and $k \in \{1, \dots, J\}$ with $k \neq j$, $\sigma_k(\xi)$ is strictly decreasing in ξ_j .

We also strengthen S2 to S2', which is the equivalent monotonicity condition used in Berry 1994 to define the “element-by-element” inverse.

S2' For any $\xi \in \mathbb{R}^{J-1}$ and any $j \in \{1, \dots, J-1\}$, $\sigma_j(\xi)$ is strictly increasing in ξ_j and approaches 1 as $\xi_j \rightarrow \infty$ and approaches 0 as $\xi_j \rightarrow -\infty$.

Under S2', we can follow Berry 1994 and define the “element by element” inverse in the following way. For any $\xi \in \mathbb{R}^{J-1}$ and for any $j \in \{1, \dots, J-1\}$, the function $r_j : \mathbb{R}^{J-1} \rightarrow \mathbb{R}$ is such that $r_j(\xi)$ is the unique value that solves $\sigma_j(r_j(\xi), \xi_{-j}) = s_j$.⁵ Let $r(\xi) = [r_1(\xi), \dots, r_{J-1}(\xi)]$.

The Gauss-Siedel iteration of the the system r defines an iterative process T over \mathbb{R}^{J-1} such that for any point $\xi^t \in \mathbb{R}^{J-1}$, T updates ξ^t to ξ^{t+1} in the following way. For any $j \in \{1, \dots, J-1\}$, $\xi_j^{t+1} = T_j(\xi^t) = r_j(\xi_1^{t+1}, \dots, \xi_{j-1}^{t+1}, \xi_j^t, \dots, \xi_{J-1}^t)$.

⁵Observe that strictly speaking, r_j only depends on ξ_{-j} , but we let it be a function of the entire quality vector ξ as it helps keep the notation consistent.

Observe that the unique inverse $\xi^* \in \mathbb{R}^{J-1}$ recovered in Theorem 1 is a unique fixed point for the process T .

We now wish to show that for any starting value $\xi^0 \in \mathbb{R}^{J-1}$, the process of iterating T converges to ξ^* , i.e., $T^n(\xi^0) \rightarrow \xi^*$ as $n \rightarrow \infty$. To show this, we construct a Lyapunov function for the process T . To construct a Lyapunov function, we draw an analogy to competitive equilibrium, and view $z_j(\xi) = s_j - \sigma_j(\xi)$ as the excess demand function for product $j \in \{1, \dots, J\}$. Since both $s, \sigma \in \Delta^{J-1}$, then it follows that

$$\sum_{j=1}^J z_j(\xi) = 0, \quad (8)$$

which is a property analogous to Walras' law. Adapting an insight exploited by Negishi (1962) for continuous time convergence to Walrasian equilibrium (who credits Maurice Allais for the original idea), we can use the sum of the absolute values of the excess demand functions, i.e.,

$$L(\xi_1, \dots, \xi_{J-1}) = \sum_{j=1}^J |z_j(\xi)| \quad (9)$$

as a Lyapunov function.

For any $\xi \in \mathbb{R}^{J-1}$, let $I_+(\xi) \subset \{1, \dots, J\}$ index those products with positive excess demand, i.e., $j \in I_+(\xi)$ implies $z_j(\xi) > 0$. Likewise we define $I_-(\xi)$ as indexing those products with negative excess demand. Then observe by (8) that

$$\frac{1}{2}L(\xi) = \sum_{j \in I_+(\xi)} z_j(\xi) = - \sum_{j \in I_-(\xi)} z_j(\xi). \quad (10)$$

The relationship in (10) is the key simplification we need to show that L is a Lyapunov function, which we now define.

Definition 5. A function $F : \mathbb{R}^{J-1} \rightarrow \mathbb{R}$ is positive definite with respect to $\xi^* \in \mathbb{R}^{J-1}$ if $F(\xi) > 0$ for all $\xi \neq \xi^*$, and $F(\xi^*) = 0$.

Given the continuous function defining the Gauss-Siedel process $T : \mathbb{R}^{J-1} \rightarrow \mathbb{R}^{J-1}$, and given a function $H : \mathbb{R}^{J-1} \rightarrow \mathbb{R}$, let $\dot{H}(\xi) = H(T(\xi)) - H(\xi)$.

Definition 6. A continuous function $H : \mathbb{R}^{J-1} \rightarrow \mathbb{R}$ is a Lyapunov function for T if $H(\xi)$ and $-\dot{H}(\xi)$ are both positive definite functions with respect to the unique equilibrium point $\xi^* \in \mathbb{R}^{J-1}$ of T .

Theorem 7. *The function L defined by (9) is a Lyapunov function for T .*

Proof. It is clear that L is continuous and positive definite, and clear that $\dot{L}(\xi^*) = 0$. The only thing remaining to be shown is that $\dot{L}(\xi) < 0$ for all $\xi \neq \xi^*$. Thus for $\xi \neq \xi^*$, define $\xi' = T(\xi)$. We will consider the sign of $\dot{L}(\xi) = L(\xi'_1, \dots, \xi'_{J-1}) - L(\xi_1, \dots, \xi_{J-1})$.

Observe that we can express this as

$$\begin{aligned} L(\xi'_1, \dots, \xi'_{J-2}, \xi'_{J-1}) &- L(\xi'_1, \dots, \xi'_{J-2}, \xi_{J-1}) \\ &+ L(\xi'_1, \dots, \xi'_{J-2}, \xi_{J-1}) \\ &- L(\xi'_1, \dots, \xi_{J-2}, \xi_{J-1}) \\ &\vdots \\ &+ L(\xi'_1, \xi_2, \dots, \xi_{J-1}) \\ &- L(\xi_1, \dots, \xi_{J-1}). \end{aligned}$$

To ease the exposition moving forward, we introduce some notation. For any $i \in \{1, \dots, J-1\}$, and $z_i \in \mathbb{R}$, let us define a vector of qualities derived from ξ and ξ' (which are given as above) as $(w_i, \xi'_{i-}, \xi_{i+}) = (\xi'_1, \dots, \xi'_{i-1}, w_i, \xi_{i+1}, \dots, \xi_{J-1})$. We now show that for any $i \in \{1, \dots, J-1\}$, each term in the above of the form $L(\xi'_i, \xi'_{i-}, \xi_{i+}) - L(\xi_i, \xi'_{i-}, \xi_{i+})$ is ≤ 0 , with a strict inequality for at least one i , thus completing the proof.

Observe that by definition of ξ'_i , $\xi'_i = r_i(\xi_i, \xi'_{i-}, \xi_{i+})$. Assume WLOG that $z_i(\xi_i, \xi'_{i-}, \xi_{i+}) < 0$, and thus by definition of r_i , $\xi'_i < \xi_i$. Then

$$\begin{aligned} \frac{1}{2}(L(\xi'_i, \xi'_{i-}, \xi_{i+}) - L(\xi_i, \xi'_{i-}, \xi_{i+})) &= \tag{11} \\ \sum_{j \in I_+(\xi'_i, \xi'_{i-}, \xi_{i+})} z_j(\xi'_i, \xi'_{i-}, \xi_{i+}) &- \\ \sum_{j \in I_+(\xi_i, \xi'_{i-}, \xi_{i+})} z_j(\xi_i, \xi'_{i-}, \xi_{i+}) & \end{aligned}$$

Also observe that by construction, $i \notin I_+(\xi_i, \xi'_{i-}, \xi_{i+})$ (since product i by assumption has negative excess demand at $(\xi_i, \xi'_{i-}, \xi_{i+})$), and

$$I_+(\xi'_i, \xi'_{i-}, \xi_{i+}) \subseteq I_+(\xi_i, \xi'_{i-}, \xi_{i+}).$$

That is, when ξ_i is lowered to ξ'_i , the demands for all other products increases by $S1'$, and thus the set of products with positive excess demand does not increase as a result of lowering ξ_i . Also since $z_i(\xi_i, \xi'_{i-}, \xi_{i+}) < 0$, $I_+(\xi_i, \xi'_{i-}, \xi_{i+})$ is non-empty. For any $k \in I_+(\xi'_i, \xi'_{i-}, \xi_{i+})$, observe that $z_k(\xi'_i, \xi'_{i-}, \xi_{i+}) - z_k(\xi_i, \xi'_{i-}, \xi_{i+}) <$

0 since i strictly substitutes with k . Moreover if there is a $k \in I_+(\xi_i, \xi'_{i-}, \xi_{i+})$ such that $k \notin I_+(\xi'_i, \xi'_{i-}, \xi_{i+})$, then all terms involving product k in (11) will be negative. Hence all the terms in the sum (11) are negative, and thus $L(\xi'_i, \xi'_{i-}, \xi_{i+}) - L(\xi_i, \xi'_{i-}, \xi_{i+}) < 0$.

A similar argument would hold if we had assumed $z_i(\xi_i, \xi'_{i-}, \xi_{i+}) > 0$ (using the I_- sets rather than the I_+ sets in the above argument). Only if $z_i(\xi_i, \xi'_{i-}, \xi_{i+}) = 0$ would $L(\xi'_i, \xi'_{i-}, \xi_{i+}) - L(\xi_i, \xi'_{i-}, \xi_{i+}) = 0$ since in this case, $\xi_i = \xi'_i$. However since by assumption $\xi \neq \xi^*$, then for some $i \in \{1, \dots, J-1\}$ we must have that $z_i(\xi_i, \xi'_{i-}, \xi_{i+}) \neq 0$. \square

Having established a Lyapunov function for the process T , it becomes a straightforward application of Lyapunov's theorem (see for example Lemma 6.2 in Stokey et al. (1989)) to attain the following result

Theorem 8. *For any starting value $\xi^0 \in \mathbb{R}^{J-1}$, the process $T^n(\xi^0)$ converges to ξ^* as $n \rightarrow \infty$.⁶*

Thus we have shown that iterating on the “element by element” inverse introduced by Berry (1994) generates a globally convergent algorithm for locating the inverse of the differentiated product demand system presented in this paper. This computational approach works despite the possibly non-additive role of ξ in preferences, and thus provides a general algorithm for non-separable demand models that fall under our existence and uniqueness theorem but fall outside of the scope of the contraction mapping in Berry et al. (1995). Preliminary computational experiments reveal that even on high dimensional problems (i.e., the number of products equaling 300), iterating the “element by element” inverse performs with at least the same efficacy (and sometimes greater) than the contraction mapping for similar sized problems.

5 Appendix

In this appendix, we show that a fully nonparametric random utility model subject only to the restriction that unobserved product quality is a vertical characteristic, along with a mild continuity condition on the distribution of tastes, generates market demands that satisfy our regularity conditions R1-R3. As the author is unaware of a development of the analytic properties of market

⁶Strictly speaking, to apply Lyapunov's theorem, in addition to constructing a Lyapunov function, we also require that the motion $\{T^n(\xi^0)\}_{n>0}$ is bounded, the proof of which we omit.

demand functions arising from a fully nonparametric random utility models with heterogeneity ranging over a functional space (the space of utility functions), we provide this development both as a foundation for the model considered in this paper and future work related to nonparametric random utility models.

Consider a continuum of consumers T . Each consumer $t \in T$ has a complete, continuous, transitive, and strictly monotonic preference relation \succsim_t over products $(p, x, \xi) \in \mathbf{Z} \times \mathbb{R}$ that characterize the agent's indirect utility function, where \mathbf{Z} is the support of product characteristics (p, x) . Recall that p is the product's price, x is the product's observed (to the econometrician) characteristics, and ξ is the unobserved (to the econometrician) characteristic. Thus each consumer t 's preference relation can be represented by a continuous utility function $V_t : \mathbf{Z} \times \mathbb{R} \rightarrow \mathbb{R}$ that is strictly increasing in the second argument (and hence ξ represents a vertical product characteristic).

Let $\mathbf{V} \subset \mathbb{R}^{\mathbf{Z} \times \mathbb{R}}$ be the set of all such utility functions. We endow \mathbf{V} with the relative product topology, otherwise known as the topology of pointwise convergence. Let the measurable sets in \mathbf{V} be the Borel subsets (the σ -algebra of subsets generated by the open sets) in this topology. Our population T gives rise to a probability measure \mathbf{P}_V over the space \mathbf{V} . The probability measure \mathbf{P}_V describes the distribution of consumer preferences over product characteristics $\mathbf{Z} \times \mathbb{R}$.

Now consider any finite set of n products

$$G = \{(Z_1, \xi_1), \dots, (Z_n, \xi_n)\} \subset \mathbf{Z} \times \mathbb{R}.$$

The subset of the population T that prefers the i^{th} product from the set G is denoted

$$S_i = \{V \in \mathbf{V} : V(Z_i, \xi_i) \geq V(Z_j, \xi_j) \text{ for all } j \neq i\}.$$

The share of the population T that prefers the i^{th} product from the set G is thus ⁷

$$\sigma_i(G) = \sigma_i(Z_1, \dots, Z_n; \xi_1, \dots, \xi_n) = \mathbf{P}_V(S_i). \quad (12)$$

We refer to σ_i as the *market share* of the i^{th} product from G (i.e., if the market offered a choice of products from the set G , then σ_i is the share of the population T that chooses the i^{th} product). Notice that for any given n -tuple of product characteristics (Z_1, \dots, Z_n) , σ_i is a well defined function over $(\xi_1, \dots, \xi_n) \in \mathbb{R}^n$. In our use of the market share functions, we shall only

⁷ S_i is measurable based on the topology on \mathbf{V} .

consider variation in the n -tuple of qualities (ξ_1, \dots, ξ_n) over the n dimensional \mathbb{R}^n for a fixed vector of Z 's.

We make a mild regularity assumptions on the distribution of preferences \mathbf{P}_V , which we refer to as *continuity*. Continuity requires that the probability measure \mathbf{P}_V be sufficiently continuous, or atomless, so as to not permit a positive mass of consumers to be indifferent between two distinct products.

Definition 9. The probability measure \mathbf{P}_V is continuous if for any two distinct products (Z_i, ξ_i) and (Z_j, ξ_j) , the number of consumers indifferent between product i and j has a probability measure of zero. More precisely,

$$\mathbf{P}_V(\{V \in \mathbf{V} : V(Z_i, \xi_i) = V(Z_j, \xi_j)\}) = 0.$$

Lemma 1. *If \mathbf{P}_V is continuous, then for any finite set of products G ,*

$$\mathbf{P}_V(S_i \cap S_j) = 0 \text{ for every } i \neq j.$$

Proof. Observe that for $i \neq j$, $S_i \cap S_j$ is a subset of

$$\{V \in \mathbf{V} : V(Z_i, \xi_i) = V(Z_j, \xi_j)\},$$

which by continuity has measure 0 under \mathbf{P}_V . □

Theorem 10. *If \mathbf{P}_V is continuous, then for any finite set of distinct products G , the sum of the market shares equals 1, i.e.,*

$$\sum_{i=1}^n \sigma_i(Z_1, \dots, Z_n; \xi_1, \dots, \xi_n) = 1.$$

Proof. Recall that

$$\sum_{i=1}^n \sigma_i(Z_1, \dots, Z_n; \xi_1, \dots, \xi_n) = \sum_{i=1}^n \mathbf{P}_V(S_i).$$

Moreover, since G is finite, each $V \in \mathbf{V}$ attains a maximum over G , and thus

$$\bigcup_{i=1}^n S_i = \mathbf{V}.$$

However by Lemma 1, we have that for all $i \neq j$,

$$\mathbf{P}_V(S_i \cap S_j) = 0.$$

Thus

$$\sum_{i=1}^n \mathbf{P}_V(S_i) = \mathbf{P}_V\left(\bigcup_{i=1}^n S_i\right) = 1.$$

□

Theorem 11. *If \mathbf{P}_V is continuous, then for any n -tuple of distinct product characteristics (Z_1, \dots, Z_n) (i.e., for all $j \neq k$, $Z_j \neq Z_k$),*

$$\sigma_i(Z_1, \dots, Z_n; \xi_1, \dots, \xi_n)$$

is a continuous function in $(\xi_1, \dots, \xi_n) \in \mathbb{R}^n$.

Proof. Let us fix any n -tuple of distinct product characteristics (Z_1, \dots, Z_n) . Now consider any n -tuple of qualities (ξ_1, \dots, ξ_n) . Define a function $F : \mathbf{V} \rightarrow \{0, 1\}$ as

$$F(V) = \prod_{j \neq i} \mathbf{1}[V(Z_i, \xi_i) \geq V(Z_j, \xi_j)],$$

where $\mathbf{1}(\cdot)$ is the indicator function. Then clearly

$$\sigma_i(Z_1, \dots, Z_n; \xi_1, \dots, \xi_n) = \int F(V) \mathbf{P}_V(dV).$$

Now consider any sequence of n -tuples of qualities $\{(\xi_1^t, \dots, \xi_n^t)\}_{t \in \mathbb{N}_+}$ that converges to (ξ_1, \dots, ξ_n) . For each $t \in \mathbb{N}_+$, define $F^t : \mathbf{V} \rightarrow \{0, 1\}$ as

$$F^t(V) = \prod_{j \neq i} \mathbf{1}[V(Z_i, \xi_i^t) \geq V(Z_j, \xi_j^t)],$$

and thus,

$$\sigma_i(Z_1, \dots, Z_n; \xi_1^t, \dots, \xi_n^t) = \int F^t(V) \mathbf{P}_V(dV).$$

We need to establish that $\sigma_i(Z_1, \dots, Z_n; \xi_1^t, \dots, \xi_n^t) \xrightarrow{t} \sigma_i(Z_1, \dots, Z_n; \xi_1, \dots, \xi_n)$.

For every $V \in \mathbf{V} - S_i$ we have $F(V) = 0$. Thus for every $V \in \mathbf{V} - S_i$,

$$V(Z_i, \xi_i) < V(Z_j, \xi_j) \text{ for some } j \neq i.$$

By continuity of every utility function $V \in \mathbf{V}$, we have that for every $V \in \mathbf{V} - S_i$,

$$F^t(V) \xrightarrow{t} F(V).$$

On the other hand, for every $V \in S_i$, $F(V) = 1$. For almost every $V \in S_i$,⁸

$$V(Z_i, \xi_i) > V(Z_j, \xi_j) \text{ for every } j \neq i.^9$$

Once again, by continuity of every utility function $V \in \mathbf{V}$, we have that for almost every $V \in S_i$,

$$F^t(V) \xrightarrow{t} F(V).$$

Thus for almost every $V \in \mathbf{V}$, $F^t(V)$ converges pointwise to $F(V)$. By Lebesgue's dominated convergence theorem,

$$\sigma_i(Z_1, \dots, Z_n; \xi_1^t, \dots, \xi_n^t) \xrightarrow{t} \sigma_i(Z_1, \dots, Z_n; \xi_1, \dots, \xi_n).$$

□

This completes the proof of the regularity condition R1. We now show that the monotonicity condition R2 holds. Let us recall the condition R2 in terms of the current setup. For any n -tuple of distinct product characteristics (Z_1, \dots, Z_n) , any n -tuple of qualities (ξ_1, \dots, ξ_n) , and any strict subset $\mathbf{I} \subset \{1, \dots, n\}$, consider a change to the qualities in the choice set G that weakly increases the qualities indexed by \mathbf{I} and weakly decreases the qualities of the remaining products. This change leads to a new choice set $G^* = \{(Z_i, \xi_i^*)\}_{i \in \{1, \dots, n\}}$ with

$$\xi_i^* \geq \xi_i \text{ for } i \in \mathbf{I}$$

and

$$\xi_i^* \leq \xi_i \text{ for } i \notin \mathbf{I}.$$

We say that the distribution of consumer preferences \mathbf{P}_V generates market demands that satisfy R2, which we refer to as *monotonicity in qualities*, if the sum of the shares of the products indexed by \mathbf{I} weakly increase as a result of any such change in qualities, and the sum of the shares of the products indexed by

⁸The statement "for almost every $V \in S_i$ " means "for all $V \in S_i$ except possibly in a subset $S \subset S_i$ with $\mathbf{P}_V(S) = 0$ ".

⁹ $S_i = \{V \in \mathbf{V} : V(Z_i, \xi_i) > V(Z_j, \xi_j) \text{ for every } j \neq i\} \cup_{j \neq i} (S_i \cap S_j)$.

$\{1, \dots, n\} - \mathbf{I}$ weakly decrease as a result of the change in returns. That is

$$\sum_{i \in \mathbf{I}} \sigma_i^* \geq \sum_{i \in \mathbf{I}} \sigma_i$$

and

$$\sum_{i \notin \mathbf{I}} \sigma_i^* \leq \sum_{i \notin \mathbf{I}} \sigma_i.$$

Theorem 12. *If \mathbf{P}_V is continuous, then it satisfies monotonicity in qualities.*

Proof. Let (Z_1, \dots, Z_n) , (ξ_1, \dots, ξ_n) , and $(\xi_1^*, \dots, \xi_n^*)$ be the n -tuples described in the definition of monotonicity. Similarly to the proof of Theorem 11, define

$$F_i(V) = \prod_{j \neq i} \mathbf{1}[V(Z_i, \xi_i) \geq V(Z_j, \xi_j)] \quad \text{and} \quad F_i^*(V) = \prod_{j \neq i} \mathbf{1}[V(Z_i, \xi_i^*) \geq V(Z_j, \xi_j^*)]$$

By the linearity of the integral operation

$$\sum_{i \in \mathbf{I}} \sigma_i(Z_1, \dots, Z_n; \xi_1, \dots, \xi_n) = \int \sum_{i \in \mathbf{I}} F_i(V) \mathbf{P}_V(dV)$$

and

$$\sum_{i \in \mathbf{I}} \sigma_i(Z_1, \dots, Z_n; \xi_1^*, \dots, \xi_n^*) = \int \sum_{i \in \mathbf{I}} F_i^*(V) \mathbf{P}_V(dV).$$

By Lemma 1, for almost every $V \in \mathbf{V}$, $\sum_{i \in \mathbf{I}} F_i(V)$ equals 0 or 1.¹⁰ However by the monotonicity of each $V \in \mathbf{V}$, $\sum_{i \in \mathbf{I}} F_i(V) = 1$ implies $\sum_{i \in \mathbf{I}} F_i^*(V) \geq 1$. Thus for almost every $V \in \mathbf{V}$,

$$\sum_{i \in \mathbf{I}} F_i^*(V) \geq \sum_{i \in \mathbf{I}} F_i(V)$$

and since the integral is an increasing linear operation,

$$\sum_{i \in \mathbf{I}} \sigma_i(Z_1, \dots, Z_n; \xi_1^*, \dots, \xi_n^*) \geq \sum_{i \in \mathbf{I}} \sigma_i(Z_1, \dots, Z_n; \xi_1, \dots, \xi_n).$$

□

The regularity condition R3 is straightforward to verify - it is an artifact of the discrete choice nature of individual demand that if a product drops out of the choice set, the shares of all the remaining goods weakly increases.

¹⁰The set of V for which $\sum_{i \in \mathbf{I}} F_i(V) > 0$ equals $\cup_{i \neq j; i, j \in \mathbf{I}} (S_i \cap S_j)$

References

- Anderson, S.P., A. De Palma, and J.F. Thisse**, *Discrete choice theory of product differentiation*, MIT press, 1992.
- Berry, S., A. Pakes, and L. Center**, “The pure characteristics demand model,” *International Economic Review*, 2007, 48 (4), 1193–1225.
- and **PA Haile**, “Identification of Discrete Choice Demand From Market Level Data,” Technical Report, Working Paper, Yale University 2009.
- , **J. Levinsohn, and A. Pakes**, “Automobile prices in market equilibrium,” *Econometrica: Journal of the Econometric Society*, 1995, pp. 841–890.
- Berry, S.T.**, “Estimating discrete-choice models of product differentiation,” *The RAND Journal of Economics*, 1994, pp. 242–262.
- Hall, P. and J.L. Horowitz**, “Nonparametric methods for inference in the presence of instrumental variables,” *Annals of Statistics*, 2005, 33 (6), 2904.
- Mas-Colell, A., M.D. Whinston, and J.R. Green**, *Microeconomic theory*, Oxford university press New York, 1995.
- Mussa, M. and S. Rosen**, “Monopoly and product quality,” *Journal of Economic Theory*, 1978, 18 (2), 301–317.
- Negishi, T.**, “The stability of a competitive economy: A survey article,” *Econometrica: Journal of the Econometric Society*, 1962, pp. 635–669.
- Nevo, A.**, “Mergers with differentiated products: The case of the ready-to-eat cereal industry,” *The RAND Journal of Economics*, 2000, pp. 395–421.
- Newey, W.K. and J.L. Powell**, “Instrumental variable estimation of nonparametric models,” *Econometrica*, 2003, pp. 1565–1578.
- Petrin, A.**, “Quantifying the benefits of new products: The case of the minivan,” *Journal of Political Economy*, 2002, 110 (4), 705–729.
- Shaked, A. and J. Sutton**, “Relaxing price competition through product differentiation,” *The Review of Economic Studies*, 1982, pp. 3–13.
- Stokey, N.L., R.E. Lucas, and E.C. Prescott**, *Recursive methods in economic dynamics*, Harvard University Press, 1989.