Connected Substitutes and Invertibility of Demand*

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July 17, 2012

Abstract

We consider the invertibility (injectivity) of a nonparametric nonseparable demand system. Invertibility of demand is important in several contexts, including identification of demand, estimation of demand, testing of revealed preference, and economic theory exploiting existence of an inverse demand function or (in an exchange economy) uniqueness of Walrasian equilibrium prices. We introduce the notion of "connected substitutes" and show that this structure is sufficient for invertibility. The connected substitutes conditions require weak substitution between all goods and sufficient strict substitution to necessitate treating them in a single demand system. The connected substitutes conditions have transparent economic interpretation, are easily checked, and are satisfied in many standard models. They need only hold under *some* transformation of demand and can accommodate many models in which goods are complements. They allow one to show invertibility without strict gross substitutes, functional form restrictions, smoothness assumptions, or strong domain restrictions. When the restriction to weak substitutes is maintained, our sufficient conditions are also "nearly necessary" for even local invertibility.

Keywords: univalence, injectivity, weak substitutes, complements

*This paper combines and expands on selected material first explored in Gandhi (2008) and Berry and Haile (2009). We have benefitted from helpful comments from Jean-Marc Robin, Don Brown, the referees, and participants in seminars at LSE, UCL, Wisconsin, Yale, the 2011 UCL workshop on "Consumer Behavior and Welfare Measurement," and the 2011 "Econometrics of Demand" conference at MIT. Adam Kapor provided capable research assistance. Financial support from the National Science Foundation is gratefully acknowledged.

1 Introduction

We consider the invertibility (injectivity) of a nonparametric nonseparable demand system. Invertibility of demand is important in several theoretical and applied contexts, including identification of demand, estimation of demand systems, testing of revealed preference, and economic theory exploiting existence of an inverse demand function or (in an exchange economy) uniqueness of Walrasian equilibrium prices. We introduce the notion of "connected substitutes" and show that this structure is sufficient for invertibility.

We consider a general setting in which demand for goods $1, \ldots, J$ is characterized by

$$\sigma(x) = (\sigma_1(x), \ldots, \sigma_J(x)) : \mathcal{X} \subseteq \mathbb{R}^J \to \mathbb{R}^J \tag{1}$$

where $x = (x_1, \ldots, x_J)$ is a vector of demand shifters. All other arguments of the demand system are held fixed. This setup nests many special cases of interest. Points $\sigma(x)$ might represent vectors of market shares, quantities demanded, choice probabilities, or expenditure shares. The demand shifters x might be prices, unobserved characteristics of the goods, or latent preference shocks. Several examples in section 2 illustrate.

The connected substitutes structure involves two conditions. First, goods must be "weak substitutes" in the sense that, all else equal, an increase in x_j (e.g., fall in j's price) weakly lowers demand for all other goods. Second, we require "connected strict substitution"—roughly, sufficient strict substitution between goods to require treating them in one demand system. These conditions have transparent economic interpretation and are easily confirmed in many standard models. They need only hold under *some* transformation of the demand system and can accommodate many settings with complementary goods.

The connected substitutes conditions allow us to show invertibility without the functional form restrictions, smoothness assumptions, or strong domain restrictions relied on previously. We also provide a partial necessity result by considering the special case of differentiable demand. There we show that when the weak substitutes condition is maintained, connected strict substitution is *necessary* for nonsingularity of the Jacobian matrix. Thus, given weak substitutes, connected substitutes is sufficient for global invertibility and "nearly necessary" for even local invertibility. A corollary to our two theorems is a new global inverse function theorem allowing arbitrary open domain.

Important to our approach is explicit treatment of a "good 0" whose "demand" is defined by the identity

$$\sigma_0(x) = 1 - \sum_{j=1}^{J} \sigma_j(x). \tag{2}$$

The interpretation will vary with the application. When demand is expressed in shares (e.g., choice probabilities or market shares), good 0 might be a "real" good—e.g., a numeraire good, an "outside good," or a good relative to which utilities are normalized. The identity (2) will then follow from the fact that shares sum to one. In other applications, good 0 will be a purely artificial notion introduced only as a technical device (see the examples below). This can be useful even when an outside good is also modeled (see Appendix C).

It is clear from (2) that (1) characterizes the full demand system even when good 0 is a real good. Nonetheless, explicitly accounting for the demand for good 0 in this case simplifies imposition of the connected substitutes structure on all goods. When good 0 is an artificial good, including it in the connected substitutes conditions proves useful as well. As will be clear below, it strengthens the weak substitutes requirement in a natural way while weakening the requirement of connected strict substitution.

Also important to our approach is a potential distinction between the set $\mathcal{X}$ and the subset of this domain on which injectivity of σ is in question. A demand system generally will not be injective at points mapping to zero demand for some good j. For example, raising good j's price (lowering x_j) at such a point typically will not change any good's demand. So when considering conditions ensuring injectivity, it is natural to restrict attention to the set

$$\tilde{\mathcal{X}} = \{x \in \mathcal{X} : \sigma_j(x) > 0 \ \forall j > 0\}$$

or even to a strict subset of $\tilde{\mathcal{X}}$—e.g., considering only values present in a given data set, or only positive prices even though σ is defined on all of $\mathbb{R}^J$ (e.g., multinomial logit or probit).

Allowing such possibilities, we consider injectivity of σ on any set

$$\mathcal{X}^* \subseteq \mathcal{X}$$

(typically $\mathcal{X}^* \subseteq \tilde{\mathcal{X}}$). Rather than starting from the restriction of σ to $\mathcal{X}^*$, however, it proves helpful to allow $\mathcal{X}^* \neq \mathcal{X}$. For example, one can impose useful regularity conditions on $\mathcal{X}$ with little or no loss (we will assume it is a Cartesian product). In contrast, assumptions on the shape or topological properties of $\mathcal{X}^*$ implicitly restrict either the function σ or the subset of $\tilde{\mathcal{X}}$ on which injectivity can be demonstrated. Avoiding such restrictions is one significant way in which we break from the prior literature.

There is a large literature on the injectivity of real functions, most of it developing conditions ensuring that a locally invertible function is globally invertible. This literature goes back at least to Hadamard (1906a,b). Although we cannot attempt a full review here, the monograph of Parthasarathy (1983) provides an extensive treatment, and references to more recent work can be found in, e.g., Parthasarathy and Ravindran (2003) and Gowda and Ravindran (2000). Local invertibility is itself an open question in many important demand models, where useful conditions like local strict gross substitutes or local strict diagonal dominance fail because each good substitutes only with "nearby" goods in the product space (several examples below illustrate). Even when local invertibility is given, sufficient conditions for global invertibility in this literature have proven either inadequate for our purpose (ruling out important models of demand) or problematic in the sense that transparent economic assumptions delivering these conditions have been overly restrictive or even difficult to identify.

A central result in this literature is the global "univalence" theorem of Gale and Nikaido (1965). Gale and Nikaido considered a differentiable real function σ with nonsingular Jacobian on $\mathcal{X}^* = \mathcal{X}$.[1] They showed that σ is globally injective if $\mathcal{X}^*$ is a rectangle (a product of intervals) and the Jacobian is everywhere a P-matrix (all principal minors are

[1] Gale and Nikaido (1965) make no distinction between $\mathcal{X}$ and $\mathcal{X}^*$. When these differ, Gale and Nikaido implicitly focus on the restriction of σ to $\mathcal{X}^*$.

strictly positive). While this result is often relied on to ensure invertibility of demand, its requirements are often problematic. Differentiability is essential, but fails in some important models. Examples include those with demand defined on a discrete domain (e.g., a grid of prices), random utility models with discrete distributions, or finite mixtures of vertical models. Given differentiability, the P-matrix condition can be difficult to interpret, verify, or to derive from widely applicable primitive conditions (see the examples below). Finally, the premise of nonsingular Jacobian on rectangular $\mathcal{X}^*$ is often an significant limitation. If σ is differentiable and $\sigma_j(x) = 0$, then typically $\frac{\partial \sigma_j(x)}{\partial x_k} = 0 \ \forall k$, yielding a singular Jacobian at $x \notin \tilde{\mathcal{X}}$. However, $\tilde{\mathcal{X}}$ is often not a rectangle. For example, in a market with vertically differentiated goods (e.g., Mussa and Rosen (1978)), a lower quality good has no demand unless its price is strictly below that of all higher quality goods. So if $-x$ is the price vector, $\tilde{\mathcal{X}}$ generally will not be a rectangle. Other examples include models of spatial differentiation (e.g., Salop (1979)), linear demand models, the "pure characteristics" model of Berry and Pakes (2007), and the Lancasterian model in Appendix C. When $\tilde{\mathcal{X}}$ is not a rectangle, the Gale-Nikaido result can demonstrate invertibility only on a rectangular strict subset of $\tilde{\mathcal{X}}$.

The literature on global invertibility has not been focused on invertibility of demand, and we are unaware of any result that avoids the broad limitations of the Gale-Nikaido result when applied to demand systems. All require some combination of smoothness conditions, restrictions on the domain of interest, and restrictions on the function (or its Jacobian) that are violated by important examples and/or are difficult to motivate with natural economic assumptions.[2] The connected substitutes conditions avoid these limitations. They have clear interpretation and are easily checked based on qualitative features of the demand system. They hold in wide range of models studied in practice and imply injectivity without

[2] A few results, starting with Mas-Colell (1979), use additional smoothness conditions to allow the domain to be any full dimension compact convex polyhedron. However, the natural domain of interest $\tilde{\mathcal{X}}$ can be open, unbounded, and/or nonconvex. Examples include standard models of vertical or horizontal differentiation or the "pure characteristics model" of Berry and Pakes (2007). And while the boundaries of $\tilde{\mathcal{X}}$ (if they exist) are often planes when utilities are linear in x, this is not a general feature. A less cited result in Gale and Nikaido (1965) allows arbitrary convex domain but strengthens the Jacobian condition to require positive quasidefinitess. Positive quasidefiniteness (even weak quasidefiniteness, explored in extensions) is violated on $\tilde{\mathcal{X}}$ in prominent demand models, including that of Berry, Levinsohn, and Pakes (1995).

any smoothness requirement or restriction on the set $\mathcal{X}^*$.

The plan of the paper is as follows. Section 2 provides several examples that motivate our interest, tie our general formulation to more familiar special cases, and provide connections to related work. We complete the setup in section 3 and present the connected substitutes conditions in section 4. We give our main result in section 5. Section 6 presents a second theorem that underlies our partial exploration of necessity, enables us to provide tight links to the classic results of Hadamard and of Gale and Nikaido, and has additional implications of importance to the econometrics of differentiated products markets.

2 Examples

Estimation of Discrete Choice Demand Models. A large empirical literature uses random utility discrete choice models to study demand for differentiated products, building on pioneering work of McFadden (1974, 1981), Bresnahan (1981, 1987) and others. Conditional indirect utilities are normalized relative to that of good 0, often an outside good representing purchase of goods not explicitly under study. Much of the recent literature follows Berry (1994) in modeling price endogeneity through a vector of product-specific unobservables x, with each x_j shifting tastes for good j monotonically. Holding observables fixed, $\sigma(x)$ gives the vector of choice probabilities (or market shares). Because each σ_j is a nonlinear function of the entire vector of unobservables x, invertibility is nontrivial. However, it is essential to standard estimation approaches, including those of Berry, Levinsohn, and Pakes (1995), Berry and Pakes (2007), and Dube, Fox, and Su (2012).[3] Berry (1994) provided sufficient conditions for invertibility that include linearity of utilities, differentiability of $\sigma_j(x)$, and strict gross substitutes.[4] We relax all three conditions, opening the possibility of developing

[3] The Berry, Levinsohn, and Pakes (1995) estimation algorithm also exploits the fact that, in the models they consider, σ is surjective at all parameter values. Given injectivity, this ensures that even at wrong (i.e., trial) parameter values, the observed choice probabilities can be inverted. This property is not necessary for all estimation methods or for other purposes motivating interest in the inverse. However, Gandhi (2010) provides sufficient conditions for a nonparametric model and also discusses a solution algorithm.

[4] Although Berry (1994) assumes strict gross substitutes, his proof only requires that each inside good strictly substitute to the outside good. Hotz and Miller (1993) provide an invertibility theorem for a similar

estimators based on inverse demand functions for new extensions of the standard models, including semiparametric or nonparametric models (e.g., Gandhi and Nevo (2011), Souza-Rodrigues (2011)).

Nonparametric Identification of Demand. Separate from practical estimation issues, there has been growing interest in the question of whether demand models in the spirit of Berry, Levinsohn, and Pakes (1995) are identified without the strong functional form and distributional assumptions typically used in applications. Berry and Haile (2009b, 2010) have recently provided affirmative answers for nonparametric models in which x is a vector of unobservables reflecting latent tastes in a market and/or unobserved characteristics of the goods in a market. Conditioning on all observables, one obtains choice probabilities of the form (1). The invertibility result below provides an essential lemma for Berry and Haile's identification results, many of which extend immediately to any demand system satisfying the connected substitutes conditions.

Inverting for Preference Shocks in Continuous Demand Systems. Beckert and Blundell (2008) recently considered a model in which utility from a bundle of consumption quantities $q = (q_0, \ldots, q_J)$ is given by a strictly increasing C^2 function $u(q, x)$, with $x \in \mathbb{R}^J$ denoting latent demand shocks. The price of good 0 is normalized to 1. Given total expenditure m and prices $p = (p_1, \ldots, p_J)$ for the remaining goods, quantities demanded are given by $q_j = h_j(p, m, x) \quad j = 1, \ldots, J$, with $q_0 = m - \sum_{j>0} p_j q_j$. Beckert and Blundell (2008) consider invertibility of this demand system in the latent vector x, pointing out that this is a necessary step toward identification of demand or testing of stochastic revealed preference restrictions (e.g., Block and Marschak (1960), McFadden and Richter (1971, 1990), Falmagne (1978), McFadden (2004)). They provide several invertibility results. One requires marginal rates of substitution between good 0 and goods $j > 0$ to be multiplicatively separable in x, with an invertible matrix of coefficients. Alternatively, they provide conditions (on

class of models, although they provide a complete proof only for local, not global, invertibility. Berry and Pakes (2007) state an invertibility result for a discrete choice model relaxing some assumptions in Berry (1994), while still assuming the linearity of utility in x_j. Their proof is incomplete, although adding the second of our two connected substitutes conditions would correct this deficiency.

functional form and/or derivative matrices of marginal rates of substitution) implying the Gale-Nikaido Jacobian requirement. We provide an alternative to such restrictions.

One way to translate their model to ours is through expenditure shares. To do this, fix p and m, let $\sigma_j(x) = p_j h(p, m, x)/m$ for $j > 0$. Expenditure shares sum to one, implying the identity (2). Other transformations are also possible (see Example 1 below). And although Beckert and Blundell represent all goods in the economy by $j = 0, 1, \ldots, J$, a common alternative is to consider demand for a more limited set of goods—for example, those in a particular product category. In that case, there will no longer be a good whose demand is determined from the others' through the budget constraint, and it will be natural to have a demand shock x_j for every good j. This situation is also easily accommodated. Holding prices and all other demand shifters fixed, let $\sigma_j(x)$ now give the quantity of good j demanded for $j = 1, \ldots, J$. To complete the mapping to our model, let (2) define the object $\sigma_0(x)$. A hint at the role this artificial good 0 plays below can be seen by observing that a rise in $\sigma_0(x)$ represents a fall in the demand for goods $j > 0$ as a whole.

Existence of Inverse Demand, Uniqueness of Walrasian Equilibrium. Let $-x_j$ be the price of good j. Conditional on all other demand shifters, let $\sigma_j(x)$ give the quantity demanded of good j. A need for invertible demand arises in several contexts. In an exchange economy, invertibility of aggregate Walrasian demand is equivalent to uniqueness of Walrasian equilibrium prices. Our connected substitutes assumptions relax the strict gross substitutes property that is a standard sufficient condition for uniqueness. In a partial equilibrium setting, invertibility of aggregate Marshallian demand is required for competition in quantities to be well defined. The result of Gale and Nikaido (1965) has often been employed to show uniqueness. Cheng (1985) provided economically interpretable sufficient conditions, showing that the Gale and Nikaido (1965) Jacobian condition holds under the dominant diagonal condition of McKenzie (1960) and a restriction to strict gross substitutes. In addition to the limitations of requiring differentiability and, especially, a rectangular domain, the requirement of strict gross substitutes (here and in several other results cited above) rules out many standard models of differentiated products, where substitution is only "local," i.e., between goods that are adjacent in the product space (see, e.g., Figure 1 and Appendix A below). Our invertibility result avoids these limitations. Here we would again use the identity (2) to introduce an artificial good 0 as a technical device.

3 Model

Let $\mathcal{J} = \{0, 1, \ldots, J\}$. Recall that $x \in \mathcal{X} \subseteq \mathbb{R}^J$ is a vector of demand shifters and that all other determinants of demand are held fixed.[5] Given the identity (2), the demand system can be characterized by $\sigma = (\sigma_1, \ldots, \sigma_J) : \mathcal{X} \to \mathbb{R}^J$. Although we refer to σ as "demand" (and to $\sigma_j(x)$ as "demand for good j"), σ may be any transformation of the demand system, e.g., $\sigma(x) = g \circ f(x)$ where $f(x)$ gives quantities demanded and $g : f(\mathcal{X}) \to R^J$. In this case σ is injective only if f is. Our connected substitutes assumptions on σ are postponed to the following section; however, one should think of x_j as a monotonic shifter of demand for good j. In the examples above, x_j is either (minus) the price of good j, the unobserved quality of good j, or a shock to taste for good j. In all of these examples, monotonicity is a standard property. Recall that we seek injectivity of σ on $\mathcal{X}^* \subseteq \mathcal{X}$.

Assumption 1. $\mathcal{X}$ is a Cartesian product.

This assumption can be relaxed (see Berry, Gandhi, and Haile (2011) for details) but appears to be innocuous in most applications. We contrast this with Gale and Nikaido's assumption that $\mathcal{X}^*$ is a rectangle. There is a superficial similarity, since a rectangle is a special case of a Cartesian product. But a fundamental distinction is that we place no restriction on the set $\mathcal{X}^*$ (see the discussion in the introduction). Further, Assumption 1 plays the role of a regularity condition here, whereas rectangularity of $\mathcal{X}^*$ is integral to the proof of Gale and Nikaido's result (see also Moré and Rheinboldt (1973)) and limits its applicability.

[5] When good 0 is an real good relative to which prices or utilities are normalized, this includes all characteristics of this good. For example, we do not rule out the possibility that good 0 has a price x_0, but are holding it fixed (e.g., at 1).

4 Connected Substitutes

Our main requirement for invertibility is a pair of conditions characterizing connected substitutes. The first is that the goods are weak substitutes in x in the sense that when x_j increases (e.g., j's price falls) demand for goods $k \neq j$ does not increase.

Assumption 2 (weak substitutes). $\sigma_j(x)$ is weakly decreasing in x_k for all $j \in \mathcal{J}$, $k \notin \{0, j\}$.

We make three comments on this restriction. First, in a discrete choice model it is implied by the standard assumptions that x_j is excluded from the conditional indirect utilities of goods $k \neq j$ and that the conditional indirect utility of good j is increasing in x_j.

Second, although Assumption 2 appears to rule out complements, it does not. In the case of indivisible goods, demand can be characterized as arising from a discrete choice model in which every bundle is a distinct choice (e.g., Gentzkow (2004)). As already noted, the weak substitutes condition is mild in a discrete choice demand system. In the case of divisible goods, the fact that σ may be any transformation of the demand system enables Assumption 2 to admit some models of complements, including some with arbitrarily strong complementarity. Example 1 below illustrates.

Finally, consider the relation of this assumption to a requirement of strict gross substitutes. If good zero is a real good, then our weak substitutes condition is weaker, corresponding to the usual notion of weak gross substitutes. When good 0 is an artificial good, Assumption 2 strengthens the weak gross substitutes condition in a natural way: taking the case where x is (minus) price, all else equal, a fall in the price of some good $j > 0$ cannot cause the total demand (over all goods) to fall.

To state the second condition characterizing connected substitutes, we first define a directional notion of (strict) substitution.

Definition 1. *Good j substitutes to good k at x* if $\sigma_k(x)$ is strictly decreasing in x_j.

Consider a decline in x_j with all else held fixed. By Assumption 2 this weakly raises $\sigma_k(x)$ for all $k \neq j$. The goods to which j substitutes are those whose demands $\sigma_k(x)$

strictly rise. When x_j is (minus) the price of good j, this is a standard notion. Definition 1 merely extends this notion to other demand shifters that may play the role of x. Although this is a directional notion, in most examples it is symmetric; i.e., j substitutes to k iff k substitutes to j.[6] An exception is substitution to good 0: since any demand shifters for good 0 are held fixed, Definition 1 does not define substitution from good 0 to other goods.[7]

It will be useful to represent substitution among the goods with the directed graph of a matrix $\Sigma(x)$ whose elements are

$$\Sigma_{j+1,k+1} = \begin{cases} 1\{\text{good } j \text{ substitutes to good } k \text{ at } x\} & j > 0 \\ 0 & j = 0. \end{cases}$$

The directed graph of $\Sigma(x)$ has nodes (vertices) representing each good and a directed edge from node k to node ℓ whenever good k substitutes to good ℓ at x.

Assumption 3 (connected strict substitution). For all $x \in \mathcal{X}^*$, the directed graph of $\Sigma(x)$ has, from every node $k \neq 0$, a directed path to node 0.

Figure 1 illustrates the directed graphs of $\Sigma(x)$ at generic $x \in \tilde{\mathcal{X}}$ for some standard models of differentiated products, letting $-x$ be the price vector and assuming (as usual) that each conditional indirect utility is strictly decreasing in price. The connected substitutes conditions hold for $\mathcal{X}^* \subseteq \tilde{\mathcal{X}}$ in all of these models. As panel e illustrates, they hold even when $\mathcal{J}$ is comprised of independent goods and either an outside good or an artificial good 0. Each of these examples has an extension to models of discrete/continuous demand (e.g., Novshek and Sonnenschein (1979), Hanemann (1984), Dubin and McFadden (1984)), models of multiple discrete choice (e.g., Hendel (1999), Dube (2004)), and models of differentiated products demand (e.g., Deneckere and Rothschild (1992), Perloff and Salop (1985)) that

[6] See, e.g., Appendix D. We emphasize that this refers to symmetry of the binary notion of substitution defined above, not to symmetry of any magnitudes.

[7] If good 0 is a real good designated to normalize utilities or prices, one can imagine expanding x to include x_0 and defining substitution from good 0 to other goods prior to the normalization that fixes x_0. If Assumption 3 holds under the original designation of good 0, it will hold for all designations of good 0 as long as substitution (using the expanded vector $x = (x_0, \ldots x_J)$) is symmetric at all $x \in \chi^*$.

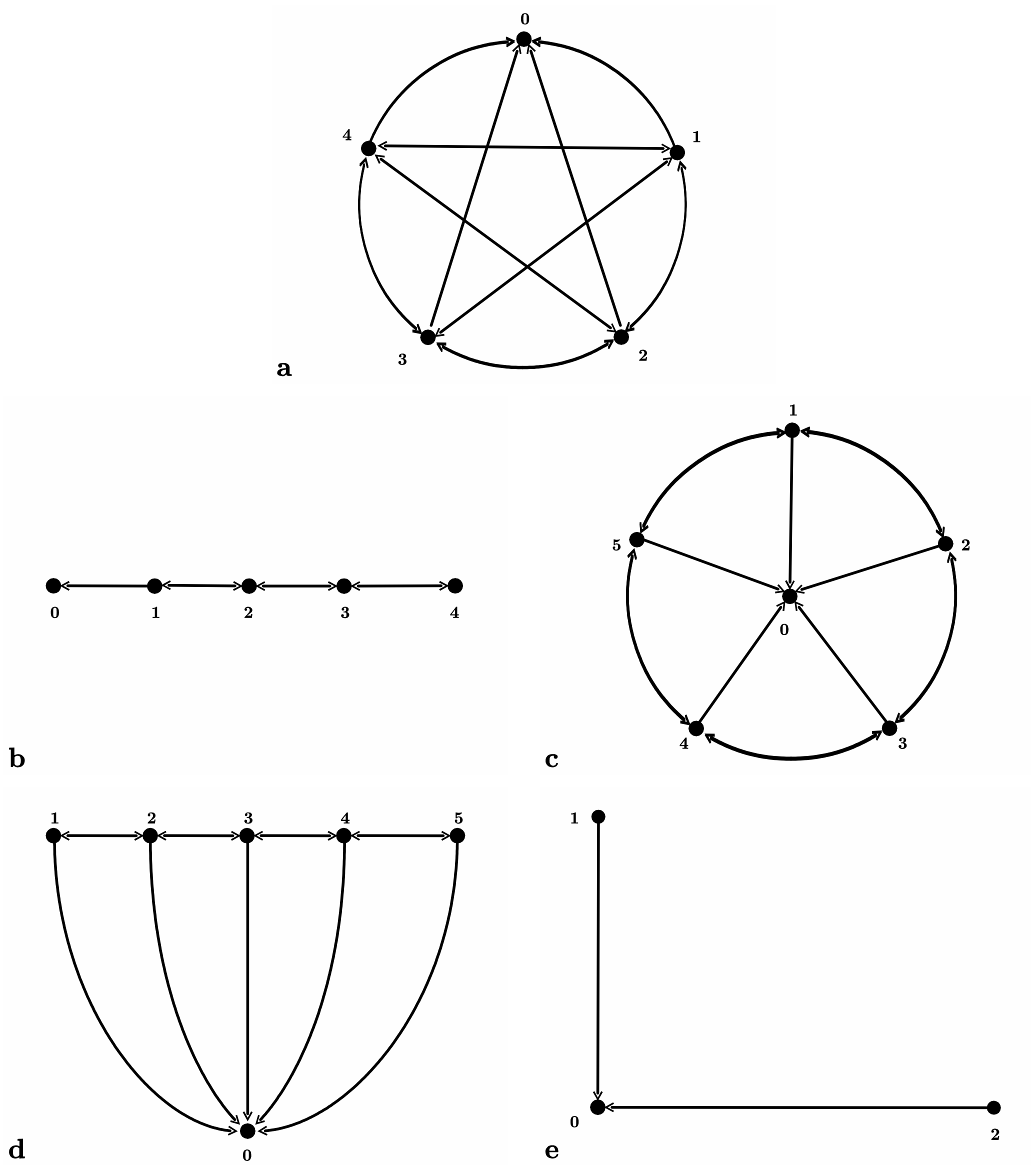

Figure 1: Directed graphs of $\Sigma(x)$ for $x \in \tilde{X}$ (x equals minus price) in some standard models of differentiated products. Panel a: multinomial logit, multinomial probit, mixed logit, etc.; Panel b: models of pure vertical differentiation, (e.g., Mussa and Rosen (1978), Bresnahan (1981b), etc.); Panel c: Salop (1979) with random utility for the outside good; Panel d: Rochet and Stole (2002); Panel e: independent goods with either an outside good or an artificial good 0.

provide a foundation for representative consumer models of monopolistic competition (e.g., Spence (1976), Dixit and Stiglitz (1977)).[8] Note that only the models represented in panel a satisfy strict gross substitutes.

As we have noted already, allowing complementarity is straightforward with indivisible goods. Further, natural restrictions such as additive or subadditive bundle pricing would only help ensure invertibility by restricting the set $\mathcal{X}^*$.[9] The following example shows that complements can be accommodated even in some models of demand for divisible goods.

Example 1 (Demand for Divisible Complements). *Let $q_j(p) : \mathbb{R}_+ \to \mathbb{R}_+$ denote a differentiable function giving the quantity of good j demanded at price vector p (here $x = -p$). Let $q_0(p) = q_0$ be a small positive constant and define $Q(p) \equiv \sum_{j=0}^{J} q_j(p)$. Let $\epsilon_{jk}(p)$ denote the elasticity of demand for good j with respect to p_k and let $\epsilon_{Qk}(p)$ denote that of $Q(p)$ with respect to p_k. If we assume that for all p such that $q_j(p) > 0$, (i) $Q(p)$ is strictly decreasing in p_j $\forall j \geq 1$, and (ii) $\epsilon_{jk}(p) \geq \epsilon_{Qk}(p)$ $\forall k, j \neq k$, then it is easily confirmed that the connected substitutes conditions hold for any $\mathcal{X}^* \subseteq \tilde{\mathcal{X}}$ under the transformation of demand to "market shares" $\sigma_j(p) = \frac{q_j(p)}{Q(p)}$ (see Appendix B). A simple example is the constant elasticity demand system in which $q_j(p) = Ap_j^{-\alpha} \prod_{k \neq j} p_k^{-\beta}$ with $\alpha > \beta > 0$. For for sufficiently small q_0 and any $\mathcal{X}^*$ such that all q_j are bounded above zero, conditions (i) and (ii) above are easily confirmed (see Appendix B).*

The following lemma provides a useful reinterpretation of Assumption 3.

Lemma 1. *Assumption 3 holds iff for all $x \in \mathcal{X}^*$ and any nonempty $\mathcal{K} \subseteq \{1, \ldots, J\}$, there exist $k \in \mathcal{K}$ and $\ell \notin \mathcal{K}$ such that $\sigma_\ell(x)$ is strictly decreasing in x_k.*

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[8] Mosenson and Dror (1972) used a graphical representation to characterize the possible patterns of substitution for Hicksian demand. Suppose x is minus the price vector, expanded to include the price of good zero (see footnote 7). Suppose further that σ is differentiable and represents the Hicksian (compensated) demand of an individual consumer. Let $\Sigma^+(x)$ be the expanded subsitution matrix, with elements $\Sigma^+_{j+1,k+1} = 1\{\text{good } j \text{ substitutes to good } k \text{ at } x\}$. Mosenson and Dror (1972) show that the directed graph of $\Sigma^+(x)$ must be strongly connected. This is a sufficient condition for Assumption 3.

[9] In a recent working paper, Azevedo, White, and Weyl (forthcoming) consider an exchange economy with indivisible goods and a continuum of financially unconstrained consumers with quasilinear utilities. They focus on existence of Walrasian equilibrium prices but also consider uniqueness under a restriction to additive pricing of the elementary goods in each bundle. Their "large support" assumption on preferences makes all bundles strict gross substitutes in the aggregate demand function. See the related discussion in section 2.

Proof. (necessity of Assumption 3) Let $\mathcal{I}_0(x) \subseteq \mathcal{J}$ be comprised of 0 and the indexes of all other goods whose nodes have a directed path to node 0 in the directed graph of $\Sigma(x)$. If Assumption 3 fails, then for some $x \in \mathcal{X}^*$ the set $\mathcal{K} = \mathcal{J} \backslash \mathcal{I}_0(x)$ is nonempty. Further, by construction there is no directed path from any node in $\mathcal{K}$ to any node in $\mathcal{I}_0(x)$. Thus, there do not exist $k \in \mathcal{K}$ and $\ell \notin \mathcal{K}$ such that $\sigma_\ell(x)$ is strictly decreasing in x_k.

(*sufficiency*) Assumption 3 implies that for all $x \in \mathcal{X}^*$ and any nonempty $\mathcal{K} \subseteq \mathcal{J} \backslash 0$, every node $k' \in \mathcal{K}$ has a directed path in $\Sigma(x)$ to node $0 \notin \mathcal{K}$. By definition, on this directed path there exists some $k \in \mathcal{K}$ (possibly $k = k'$) and $\ell \notin \mathcal{K}$ (possibly $\ell = 0$) such that good k substitutes to good ℓ. $\square$

Thus, Assumption 3 requires that there be no strict subset of goods that substitute only among themselves. Note that when good 0 is an artificial good, its presence in $\mathcal{J}$ weakens the requirements of Assumption 3: taking the case where x is (minus) price, when the price of some good $j > 0$ falls it may be only the demand for good zero that strictly declines.

Finally, when introducing the model we suggested that x_j should be thought of as a monotonic shifter of demand for good j. The following remark shows that we have implicitly imposed this monotonicity with the connected substitutes conditions.

Remark 1. *Suppose Assumptions 2 and 3 hold. Then for all $x \in \mathcal{X}^*$ and $j > 0$, $\sigma_j(x)$ is strictly increasing in x_j.*

Proof. Take $x \in \mathcal{X}^*$ and $x' \in \mathcal{X}$ such that $x'_j > x_j$, $x'_k = x_k \; \forall k \neq j$. Assumption 2 implies $\sigma_k(x') \leq \sigma_k(x) \; \forall k \neq j$. Further, by Lemma 1, $\sigma_\ell(x') < \sigma_\ell(x)$ for some $\ell \neq j$. Thus, $\sum_{k \neq j} \sigma_\ell(x') < \sum_{k \neq j} \sigma_\ell(x)$. The claim then follows from (2). $\square$

5 Invertibility of Demand

To establish our main result, we begin with two lemmas. The first shows that under weak substitutes, if x_j weakly increases for (only) a subset of goods j, demand for the remaining goods (taken as a whole) does not increase.[10] Adding the requirement of connected strict substitution, Lemma 3 then shows that, all else equal, a strict increase in x_j for some goods j strictly raises demand for those goods (taken as a whole). This intuitive property is the key to our injectivity result.

Lemma 2. *Given Assumption 1, Assumption 2 implies that for any $\mathcal{I} \subset \mathcal{J}$ and any $x, x' \in \mathcal{X}$ such that for all $j \neq 0$, $x'_j \geq x_j$ if $j \in \mathcal{I}$ and $x'_j \leq x_j$ if $j \notin \mathcal{I}$, $\sum_{k \notin \mathcal{I}} \sigma_k(x') \leq \sum_{k \notin \mathcal{I}} \sigma_k(x)$.*

Proof. Let $\tilde{x}$ be such that, for all $j \neq 0$, $\tilde{x}_j = x_j$ if $j \in \mathcal{I}$ and $\tilde{x}_j = x'_j$ if $j \notin \mathcal{I}$. If $\mathcal{J} \backslash \mathcal{I}$ contains only 0, then $\tilde{x} = x$ and

$$\sum_{k \notin \mathcal{I}} \sigma_k(\tilde{x}) \leq \sum_{j \notin \mathcal{I}} \sigma_k(x) \tag{3}$$

trivially. If instead $\mathcal{J} \backslash \mathcal{I}$ contains any nonzero element, without loss let these be $1, \ldots, n$. Then let $\tilde{x}^{(0)} = x$, $r = 0$, and consider the following iterative argument. Add one to r and, for all $j > 0$, let $\tilde{x}_j^{(r)} = \tilde{x}_j^{(r-1)} + 1\{j = r\}\left(x'_j - x_j\right)$. Assumption 1 ensures that σ is defined at $\tilde{x}^{(r)} = \left(\tilde{x}_1^{(r)}, \ldots, \tilde{x}_J^{(r)}\right)$. By Assumption 2, $\sum_{j \in \mathcal{I}} \sigma_j(\tilde{x}^{(r)}) \geq \sum_{j \in \mathcal{I}} \sigma_j(\tilde{x}^{(r-1)})$. Iterating until $r = n$ and applying (2) we obtain (3). A parallel argument shows that

$$\sum_{k \notin \mathcal{I}} \sigma_k(x') \leq \sum_{k \notin \mathcal{I}} \sigma_k(\tilde{x})$$

and the result follows. $\square$

Lemma 3. *Let Assumptions 1–3 hold. Then for all $x, x' \in \mathcal{X}^*$ such that $\mathcal{I} \equiv \left\{j : x'_j > x_j\right\}$ is nonempty, $\sum_{j \in \mathcal{I}} \sigma_j(x') > \sum_{j \in \mathcal{I}} \sigma_j(x)$.*

Proof. Since $0 \notin \mathcal{I}$, Lemma 1 ensures that for some $k \in \mathcal{I}$ and some $\ell \notin \mathcal{I}$, $\sigma_\ell(x)$ is strictly decreasing in x_k. Take one such pair (k, ℓ). Define a point $\tilde{x}$ by $\tilde{x}_j = x_j +$

[10] The converse also holds: taking $x'_j \geq x_j$, $x'_i = x_i$ $\forall i \neq j$, and $\mathcal{I} = \mathcal{J} \backslash \{k\}$, we obtain $\sigma_k(x)$ weakly decreasing in x_j.

$(x'_k - x_k)\, 1\{j = k\}\ \forall j > 0$. Assumption 1 ensures that σ is defined at $\tilde{x}$. By Assumption 2, $\sigma_j(\tilde{x}) \leq \sigma_j(x)$ for all $j \neq k$. Further, $\sigma_\ell(\tilde{x}) < \sigma_\ell(x)$ by our choice of (k, ℓ). So, since $\ell \notin \mathcal{I}$,

$$\sum_{j \notin \mathcal{I}} \sigma_j(\tilde{x}) < \sum_{j \notin \mathcal{I}} \sigma_j(x).$$

By Lemma 2, $\sum_{j \notin \mathcal{I}} \sigma_j(x') \leq \sum_{j \notin \mathcal{I}} \sigma_j(\tilde{x})$, so we obtain

$$\sum_{j \notin \mathcal{I}} \sigma_j(x') \leq \sum_{j \notin \mathcal{I}} \sigma_j(\tilde{x}) < \sum_{j \notin \mathcal{I}} \sigma_j(x)$$

and the result follows from (2).[11] □

To demonstrate invertibility of demand under the connected substitutes conditions, we will first show that σ is *inverse isotone on* $\mathcal{X}^*$. Below we use $\leq$ to denote the component-wise weak partial order on $\mathbb{R}^n$. Thus for $y, y' \in \mathbb{R}^n$, $y \leq y'$ iff $y_i \leq y'_i$ for all $i = 1, \ldots, n$.

Definition 2. A mapping $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is *inverse isotone* if for any $y, y' \in D$, $F(y') \leq F(y)$ implies $y' \leq y$.

Theorem 1. *Under Assumptions 1–3, σ is inverse isotone on $\mathcal{X}^*$.*

Proof. Take any $x, x' \in \mathcal{X}^*$ such that

$$\sigma(x') \leq \sigma(x) \tag{4}$$

and suppose, contrary to the claim, that the set $\mathcal{I} = \left\{j : x'_j > x_j\right\}$ is non-empty. By Lemma 3 this requires

$$\sum_{j \in \mathcal{I}} \sigma_j(x') > \sum_{j \in \mathcal{I}} \sigma_j(x)$$

which contradicts (4). □

[11] If $\mathcal{X}^*$ is open then, given Assumption 2, Assumption 3 is necessary for the conclusion of this lemma. Suppose Assumption 3 fails. Then by Assumption 2 and Lemma 1 there is some $x \in \chi^*$ and some nonempty $\mathcal{K} \subseteq \mathcal{J} \backslash 0$, such that $\sigma_\ell(x)$ is constant in x_k for all $k \in \mathcal{K}$ and all $\ell \notin \mathcal{K}$. For some $\epsilon > 0$ and each $k \in \mathcal{K}$ let $x'_k = x_k + \epsilon$, while $x'_j = x_j$ for $j \notin \mathcal{K}$. Now $\mathcal{I} = \mathcal{K}$. For sufficiently small ϵ we have $x' \in \mathcal{X}^*$ and $\sum_{j \notin \mathcal{I}} \sigma_j(x') = \sum_{j \notin \mathcal{I}} \sigma_j(x)$, which implies $\sum_{j \in \mathcal{I}} \sigma_j(x') = \sum_{j \in \mathcal{I}} \sigma_j(x)$.

Injectivity follows from Theorem 1, exploiting the following well known observation (e.g., Rheinboldt (1970b)).[12]

Remark 2. *If* $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ *is* inverse isotone, it is injective.

Proof. Suppose $F(y) = F(y')$ for $y, y' \in D$. Since F is inverse isotone this implies both $y \leq y'$ and $y' \leq y$; hence $y' = y$. □

This gives us our main result:

Corollary 1. *Under Assumptions 1–3, σ is injective on $\mathcal{X}^*$.*

6 Discussion

Given the new set of sufficient conditions for invertibility provided by Corollary 1, two questions naturally arise. One is whether these conditions are unnecessarily strong. Another is how these conditions relate to those required by the classic results of Gale and Nikaido (1965) and Hadamard (1906a, 1906b). In this section we provide partial answers to these questions and develop some additional results of independent interest. To facilitate this, we assume σ is differentiable on $\mathcal{X}^*$. In addition, consider the following differentiable version of the connected strict substitution requirement in Assumption 3 (recall Lemma 1).

Assumption 3*. For all $x \in \mathcal{X}^*$ and any nonempty $\mathcal{K} \subseteq \{1, \ldots, J\}$, there exist $k \in \mathcal{K}$ and $\ell \notin \mathcal{K}$ such that $\frac{\partial \sigma_\ell(x)}{\partial x_k} < 0$.

Given differentiability, this condition slightly strengthens Assumption 3 by ruling out a zero derivative $\frac{\partial \sigma_\ell(x)}{\partial x_k}$ where $\sigma_\ell(x)$ is strictly increasing in x_k. Let $\mathbb{J}_\sigma(x)$ denote the Jacobian matrix

$$\begin{bmatrix} \frac{\partial \sigma_1(x)}{\partial x_1} & \cdots & \frac{\partial \sigma_1(x)}{\partial x_J} \\ \vdots & \ddots & \vdots \\ \frac{\partial \sigma_J(x)}{\partial x_1} & \cdots & \frac{\partial \sigma_J(x)}{\partial x_J} \end{bmatrix}.$$

[12] Another application of Theorem 1 appears in a recent paper by Gandhi, Lu, and Shi (2011). They exploit the inverse isotone property shown here in studying identification and estimation of multinomial choice demand models under mismeasurement of market shares.

Theorem 2. *Suppose Assumption 2 holds and that $\sigma(x)$ is differentiable on $\mathcal{X}^*$. Then the following conditions are equivalent:*

(i) $\mathbb{J}_\sigma(x)$ *is nonsingular on* $\mathcal{X}^*$;

(ii) $\mathbb{J}_\sigma(x)$ *is a* P*-matrix on* $\mathcal{X}^*$;

(iii) *Assumption 3*.*

Proof. See Appendix A. □

This result leads to several valuable observations:

Near Necessity. Equivalence between conditions (i) and (iii) suggests that our sufficient conditions for invertibility are not much too strong. Given weak substitutes and differentiability, Assumption 3* (slightly stronger than Assumption 3) is necessary for a nonsingular Jacobian. Thus, given the restriction to demand systems that can be transformed to satisfy weak substitutes, connected substitutes may be viewed as "nearly necessary" for even *local* invertibility.

Relation to Gale-Nikaido. The equivalence between conditions (ii) and (iii) provides a tight link between connected substitutes and the P-matrix condition required by the classic result of Gale and Nikaido (1965).[13] Given differentiability and weak substitutes, Gale and Nikaido's P-matrix requirement is equivalent to a slightly strengthened version of our Assumption 3. One would never use this observation to establish invertibility: if the connected substitutes conditions hold, Corollary 1 establishes invertibility without the additional differentiability and domain restrictions Gale and Nikaido relied on. However, Theorem 2 clarifies the relationship between the two results.[14] One interpretation is that we drop Gale and Nikaido's differentiability requirement, replace the restriction to rectangular $\mathcal{X}^*$ with

[13]Recall that, given differentiability and a nonsingular Jacobian $\mathbb{J}_\sigma(x)$ on a rectangular domain $\mathcal{X}^*$, they required further that $\mathbb{J}_\sigma(x)$ be a P-matrix on $\mathcal{X}^*$.

[14]A secondary result (Theorem 5) in Gale and Nikaido (1965) shows that their injectivity result can be extended to show inverse isotonicity under the additional restriction that $\mathbb{J}_\sigma(x)$ has only nonpositive off-diagonal entries. Our weak substitutes assumption strengthens their restriction on the off-diagonals only by requiring $\sigma_0(x)$ to be nonincreasing in each x_j. This allows us to avoid their problematic requirement of a rectangular domain and implies that their P-matrix requirement would add nothing to the requirement of nonsingular Jacobian.

the weak substitutes condition, and replace the P-matrix requirement with the more easily interpreted but essentially equivalent (slightly weaker) requirement of connected strict substitution.

A Global Inverse Function Theorem. One corollary to our two theorems is the following global inverse function theorem for a mapping with arbitrary open domain.

Corollary 2. *Let $\sigma : \mathbb{R}^J \to \mathbb{R}^J$ be a C^1 function and suppose that, given (2), Assumption* 2 *holds. Then for any open $\mathcal{X}^* \subseteq \mathbb{R}^J$, the restriction of σ to $\mathcal{X}^*$ has a C^1 inverse on $\sigma(\mathcal{X}^*)$ iff $\mathbb{J}_\sigma(x)$ is nonsingular on $\mathcal{X}^*$.*

Proof. Necessity of a nonsingular Jacobian for existence of a differentiable inverse follows from the identity $\sigma^{-1}(\sigma(x)) = x$ and the chain rule. To show sufficiency, observe that by Theorem 2 a nonsingular Jacobian on $\mathcal{X}^*$ implies that Assumption 3* holds. Thus the restriction of σ to $\mathcal{X}^*$ is inverse isotone by Theorem 1, implying that σ has an inverse on $\sigma(\mathcal{X}^*)$. By the standard inverse function theorem this inverse is C^1 in a neighborhood of every point in $\sigma(\mathcal{X}^*)$ and, thus, C^1 on $\sigma(\mathcal{X}^*)$. □

This result shows that if weak substitutes holds, the conclusion of the usual (local) inverse function theorem extends to any open subset of the domain. This follows from the equivalence between conditions (i) and (iii) in Theorem 2.

Corollary 2 may be compared to the classic result of Hadamard (and its extensions), which shows that if both $\mathcal{X}^*$ and $\mathcal{S}^*$ are smooth connected manifolds and $\mathcal{S}^*$ is simply connected, a C^1 map $\sigma : \mathcal{X}^* \to \mathcal{S}^*$ is a diffeomorphism (i.e., a smooth bijection) if and only if it has nonzero Jacobian on $\mathcal{X}^*$ and is "proper."[15] Corollary 2 avoids any connectedness condition on the domain or range and replaces properness with the weak substitutes condition. Whereas properness is not easily verified, we have shown that weak substitutes is a natural property of many demand systems. Finally, our Assumption 3* provides a widely applicable condition equivalent (given weak substitutes) to Hadamard's requirement of local invertibility. Of course, our main result (Corollary 1) requires neither smoothness nor open domain.

[15] See e.g., Palais (1959), Ho (1975), Parthasarathy (1983), Parthasarathy and Ravindran (2003), and Krantz and Parks (2002). A function is proper if the pre-image of any compact set is compact.

Unlike Hadamard's theorem, Corollary 2 avoids the question of surjectivity (and the additional requirements this creates) by seeking a smooth inverse only on $\sigma(\mathcal{X}^*)$. Rheinboldt (1970a) provides necessary and sufficient conditions for surjectivity onto $\mathbb{R}^J$ for inverse isotone functions.

Transforming Demand. As noted already, if $f(x)$ describes a demand system that does not itself satisfy the connected substitutes conditions, it may be possible to find a function g such that $\sigma = g \circ f$ does. Theorem 2 provides some guidance on suitable transformations g. If both f and g have nonsingular Jacobians—a requirement one might not expect to avoid in seeking to show invertibility of a differentiable demand system—then by the equivalence of conditions (i) and (iii) it is sufficient to verify that $g \circ f$ satisfies weak substitutes.

Identification and Estimation in Differentiated Products Markets. The sufficiency of condition (iii) for conditions (i) and (ii) in Theorem 2 is important to the econometric theory underlying standard empirical models of differentiated products markets (e.g., Berry, Levinsohn, and Pakes (1995)). Berry and Haile (2010a) use the latter implication to establish nonparametric identifiability of firms' marginal costs and to testability of alternative models of oligopoly competition. There the P-matrix property ensures invertibility of the derivative matrix of market shares with respect to prices for goods produced by the same firm—a matrix appearing in the first-order conditions characterizing equilibrium behavior. Their results generalize immediately to models with continuous demand satisfying connected substitutes.

Berry, Linton, and Pakes (2004) provide the asymptotic distribution theory for a class of estimators for discrete choice demand models. A key condition, confirmed there for special cases, is that the Jacobian of the demand system (with respect to a vector of demand shocks) is full rank on $\mathcal{X}^*$. Sufficiency of the connected substitutes conditions again establishes economically interpretable sufficient conditions with wide applicability.

7 Conclusion

We have introduced the notion of connected substitutes and shown that this structure is sufficient for invertibility of a nonparametric nonseparable demand system. The connected substitutes conditions are satisfied in a wide range of models used in practice, including many with complementary goods. These conditions have transparent economic interpretation, are easily checked in practice, and allow demonstration of invertibility without functional form restrictions, smoothness assumptions, or strong domain restrictions commonly relied on previously. Further, given a restriction to weak substitutes, our sufficient conditions are also "nearly necessary" for even local invertibility.

Appendices

A Proof of Theorem 2

We first review some definitions (see, e.g., Horn and Johnson (1990)). A square matrix is *reducible* if it can be placed in block upper triangular form by simultaneous permutations of rows and columns. A square matrix that is not reducible is *irreducible*. A square matrix A with elements a_{ij} is (weakly) *diagonally dominant*[16] if for all j

$$|a_{jj}| \geq \sum_{i \neq j} |a_{ij}|.$$

If the inequality is strict for all j, A is said to be *strictly diagonally dominant*. An *irreducibly diagonally dominant* matrix is a square matrix that is irreducible and weakly diagonally dominant, with at least one diagonal being strictly dominant, i.e., with at least one column

[16] Here we refer to column dominance, not row dominance.

j such that

$$|a_{jj}| > \sum_{i \neq j} |a_{ij}|. \tag{A.1}$$

We begin with three lemmas concerning square matrices. The first is well known (see, e.g., Taussky (1949) or Horn and Johnson (1990), p. 363) and the third is a variation on a well known result. The second appears to be new.

Lemma 4. *An irreducibly diagonally dominant matrix is nonsingular.*

Lemma 5. *Let D be a square matrix with nonzero diagonal entries and suppose that every principal submatrix of D is weakly diagonally dominant, with at least one strictly dominant diagonal. Then D is nonsingular.*

Proof. Let $M = D$ and consider the following iterative argument. If M is 1×1, nonsingularity is immediate from the nonzero diagonal. For M of higher dimension, if M is irreducible the result follows from Lemma 4. Otherwise M is reducible, so by simultaneous permutation of rows and columns, it can be placed in block upper triangular form; i.e., for some permutation matrix P,

$$M^* \equiv PMP' = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$$

where A and C are square matrices. Simultaneous permutation of rows and columns changes neither the set of diagonal entries nor the off-diagonal entries appearing in the same column (or row) as a given diagonal entry. Each principal submatrix of A or of C is also a principal submatrix of D. Thus, A and C have only nonzero diagonal entries and are such that every principal submatrix is diagonally dominant with at least one strictly dominant diagonal. M is nonsingular if M^* is, so it is sufficient to show that both A and C are nonsingular. Let $M = A$ and restart the iterative argument. This will show A to be nonsingular, possibly after further iteration. Repeating for $M = C$ completes the proof. $\square$

Lemma 6. *Suppose a real square matrix D is weakly diagonally dominant with strictly positive diagonal elements. Then $|D| \geq 0$.*

Proof. Let D_{ij} denote the elements of D. For $\lambda \in [0,1]$ define a matrix $D(\lambda)$ by

$$D_{ij}(\lambda) = \begin{cases} D_{ij} & i = j \\ \lambda D_{ij} & i \neq j. \end{cases}$$

For $\lambda < 1$, $D(\lambda)$ is strictly diagonally dominant. Since the diagonal elements of $D(\lambda)$ are strictly positive this implies $|D(\lambda)| > 0$ (see, e.g., Theorem 4 in Taussky (1949)). Since $|D(\lambda)|$ is continuous in λ and $|D(1)| = |D|$, the result follows. □

Two observations regarding the demand system σ will be useful.

Remark 3. *Suppose σ is differentiable on $\mathcal{X}^*$ and that Assumptions 2 and 3* hold. Then $\frac{\partial \sigma_j(x)}{\partial x_j} > 0$ for all $j > 0$ and $x \in \mathcal{X}^*$.*

Proof. Differentiate (2) with respect to x_j and applying Assumptions 2 and 3*. □

Lemma 7. *Suppose σ is differentiable on $\mathcal{X}^*$ and that Assumptions 2 and 3* hold. Then for all $x \in \mathcal{X}^*$, every principal submatrix of $\mathbb{J}_\sigma(x)$ is weakly diagonally dominant, with at least one strictly dominant diagonal.*

Proof. Take $x \in \mathcal{X}^*$ and nonempty $\mathcal{K} \subseteq \{1, 2, \ldots, J\}$. Let $D_{\mathcal{K}}(x)$ denote the principal submatrix of $\mathbb{J}_\sigma(x)$ obtained by deleting rows $r \notin \mathcal{K}$ and columns $c \notin \mathcal{K}$. Because $\sum_{k \in \mathcal{J}} \sigma_k(x) = 1$,

$$\sum_{k \in \mathcal{J}} \frac{\partial \sigma_k(x)}{\partial x_j} = 0.$$

By Remark 3 and Assumption 2, $\frac{\partial \sigma_j(x)}{\partial x_j} > 0$ and $\frac{\partial \sigma_k(x)}{\partial x_j} \leq 0 \ \forall j > 0$, $k \neq j$. So for $j \in \mathcal{K}$

$$\left| \frac{\partial \sigma_j(x)}{\partial x_j} \right| = \sum_{k \in \mathcal{K} - \{j\}} \left| \frac{\partial \sigma_k(x)}{\partial x_j} \right| + \sum_{\ell \notin \mathcal{K}} \left| \frac{\partial \sigma_\ell(x)}{\partial x_j} \right|. \tag{A.2}$$

This implies

$$\left| \frac{\partial \sigma_j(x)}{\partial x_j} \right| \geq \sum_{k \in \mathcal{K} - \{j\}} \left| \frac{\partial \sigma_k(x)}{\partial x_j} \right|. \tag{A.3}$$

Furthermore, since $0 \notin \mathcal{K}$, Assumption 3* implies that for some $j \in \mathcal{K}$ the second sum in (A.2) is strictly positive. For that j the inequality (A.3) must be strict. $\square$

With these results in place, we now prove the sufficiency of condition (iii) in Theorem 2 for condition (ii). This will immediately imply sufficiency for condition (i). Take arbitrary $x \in \mathcal{X}^*$ and let $D(x)$ be a principal submatrix of $\mathbb{J}_\sigma(x)$. Since every principal submatrix of $D(x)$ is also a principal submatrix of $\mathbb{J}_\sigma(x)$, Lemma 7 implies that every principal submatrix of $D(x)$ is weakly diagonally dominant with at least one strictly dominant diagonal. Thus, by Lemma 5, $D(x)$ is nonsingular. Since, by Remark 3, $D(x)$ also has strictly positive diagonal entries, it follows from nonsingularity and Lemma 6 that $|D(x)| > 0$.

Finally, we show necessity of condition (iii) in Theorem 2 for condition (i). This will immediately imply necessity for condition (ii). Suppose condition (iii) fails. Then by Assumption 2, for some $x \in \mathcal{X}^*$ there is a nonempty set $\mathcal{K} \subseteq \{1, \ldots, J\}$ such that $\frac{\partial \sigma_j(x)}{\partial x_k} = 0$ for all $k \in \mathcal{K}$ and all $j \in \mathcal{J} \backslash \mathcal{K}$. Fix this value of x and, without loss, permute the labels of goods $1, \ldots, J$ so that $\mathcal{K} = \{1, \ldots, |\mathcal{K}|\}$. If $|\mathcal{K}| < J$, $\mathbb{J}_\sigma(x)$ has block triangular form

$$\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$$

where A is $|\mathcal{K}| \times |\mathcal{K}|$. If instead $|\mathcal{K}| = J$, let $A = \mathbb{J}_\sigma(x)$. Because $0 \notin \mathcal{K}$, $\frac{\partial \sigma_0(x)}{\partial x_k}$ is zero for all $k \in \mathcal{K}$, so (2) requires

$$\sum_{j \in \mathcal{K}} \frac{\partial \sigma_j(x)}{\partial x_k} = 0 \quad \forall k \in \mathcal{K}.$$

Thus, either $|\mathcal{K}| = 1$ and $A = \frac{\partial \sigma_1(x)}{\partial x_1} = 0$, or $\frac{\partial \sigma_1(x)}{\partial x_k} = -\sum_{j=2}^{|\mathcal{K}|} \frac{\partial \sigma_j(x)}{\partial x_k} \quad \forall k \in \mathcal{K}$. In either case, A is singular and the result follows.

B Demand for Divisible Complements

Here we demonstrate two assertions made in the discussion of Example 1 in the text.

Proposition 1. *Let* $q_0(p) = q_0$ *be a small positive constant and define* $Q(p) = \sum_{j=0}^{J} q_j(p)$*. Suppose that for all* p *such that* $q_j(p) > 0$*, (i)* $Q(p)$ *is strictly decreasing in* p_j $\forall j \geq 1$ *and (ii)* $\epsilon_{jk}(p) \geq \epsilon_{Qk}(p)$ $\forall k, j \neq k$*. Then Assumptions 2 and 3 hold for any* $\mathcal{X}^* \subseteq \tilde{\mathcal{X}}$ *under the transformation* $\sigma_j(p) = \frac{q_j(p)}{Q(p)}$.

Proof. We first verify the weak substitutes condition. If $q_j(p) = 0$, $\frac{\partial q_j(p)}{\partial p_k}$ cannot be negative, so the derivative $\frac{\partial \sigma_j(p)}{\partial p_k} = \frac{\partial q_j(p)}{\partial p_k}/Q(p)$ is nonnegative. When $q_j(p) > 0$, $\frac{\partial \sigma_j(p)}{\partial p_k} = \left[Q(p)\frac{\partial q_j(p)}{\partial p_k} - q_j(p)\frac{\partial Q(p)}{\partial p_k}\right]/Q(p)^2$, which is nonnegative if $\frac{\partial q_j(p)}{\partial p_k}\frac{p_k}{q_j(p)} \geq \frac{\partial Q(p)}{\partial p_k}\frac{p_k}{Q(p)}$. We have assumed this in (ii). To show that Assumption 3 holds, observe that since $\sigma_0(p) = \frac{q_0}{Q(p)}$, (i) implies that each good $j \neq 0$ substitutes directly to the artificial good zero on $\tilde{\mathcal{X}}$. □

Proposition 2. *For* p *such that* $q_k(p) \geq \delta > 0$ *for all* $k > 0$*, the hypotheses of Proposition 1 hold for* q_0 *sufficiently small when, for all* $j > 0$*,* $q_j(p) = Ap_j^{-\alpha}\prod_{k \notin \{0,j\}} p_k^{-\beta}$ *with* $\alpha > \beta > 0$.

Proof. Part (i) of the hypotheses is immediate since all real goods have downward sloping demand and are strict gross complements. Since $\epsilon_{Qk}(p) = \sum_{j=1}^{J}\sigma_j(p)\epsilon_{jk}(p)$, part (ii) holds if

$$-\beta \geq -[1 - \sigma_k(p) - \sigma_0(p)]\beta - \sigma_k(p)\alpha$$

i.e.,

$$\beta \leq \frac{q_k(p)}{q_k(p) + q_0}\alpha.$$

Since $q_k(p) \geq \delta > 0$ and $\beta < \alpha$, this holds for sufficiently small q_0. □

C A Lancasterian Example

Consider a simple variation of Lancaster's (1966) "diet example," illustrating a continuous demand system with only local substitution, with a non-rectangular domain of interest, and where the introduction of an artificial good 0 is useful even though an outside good is already modeled. A representative consumer has a budget y and chooses consumption quantities

(q_1, q_2, q_3) of three goods: wine, bread, and cheese, respectively. Her preferences are given by a utility function

$$u(q_1, q_2, q_3) = \ln(z_1) + \ln(z_2) + \ln(z_3) + m$$

where (z_1, z_2, z_3) are consumption of calories, protein, and calcium, and m is money left to spend on other goods. The mapping of goods consumed to characteristics consumed is given by[17]

$$\begin{aligned} z_1 &= q_1 + q_2 + q_3 \\ z_2 &= q_2 + q_3 \\ z_3 &= q_3. \end{aligned}$$

We assume $y > 3$. The set of prices (p_1, p_2, p_3) such that all goods are purchased is defined by

$$0 < p_1 < p_2 - p_1 < p_3 - p_2. \tag{C.1}$$

Since p plays the role of x here, (C.1) defines $\tilde{\mathcal{X}}$, which is not a rectangle. Let $\mathcal{X}^*= \tilde{\mathcal{X}}$.

It is easily verified that demand for each inside good is given by

$$\begin{aligned} \sigma_1(p) &= \frac{1}{p_1} - \frac{1}{p_2 - p_1} \\ \sigma_2(p) &= \frac{1}{p_2 - p_1} - \frac{1}{p_3 - p_2} \\ \sigma_3(p) &= \frac{1}{p_3 - p_2} \end{aligned} \tag{C.2}$$

for $p \in \mathcal{X}^*$. These equations fully characterize demand for all goods. However, we introduce

[17] Unlike Lancaster (1966), we sacrifice accuracy of nutritional information for the sake of simplicity.

the artificial quantity of "good 0", defined by

$$q_0 \equiv 1 - \sum_{j=1}^{3} q_j. \tag{C.3}$$

Observe that this artificial good is not the outside good m. Further, the connected substitutes conditions would not hold if the outside good were treated as good 0.

With (C.2), (C.3) implies

$$\sigma_0(p) = 1 - \frac{1}{p_1}.$$

From these equations, it is now easily confirmed that Assumption 2 holds Further, goods 2 and 3 strictly substitute to each other, goods 1 and 2 strictly substitute to each other, and good 1 strictly substitutes to good 0. Thus, Assumption 3 also holds.

D Symmetric Strict Substitution

Our graphical illustrations of the connected substitutes property involved examples in which (strict) substitution is symmetric, i.e., good j substitutes to good k only if good k also substitutes to good j (excepting substitution from good 0, as discussed in the text). The following result shows that this generically true in discrete choice models given monotonicity in x_j.

Proposition 3. *Consider a discrete choice model in which each consumer i's conditional indirect utility from good j is* $v_{ij} = v_j(x_j; \theta_i)$, $j = 1, \ldots, J$. *Suppose the (possibly infinite-dimensional) parameter* θ_i *is independent of* x *and that* $v_j(\cdot; \theta_i)$ *is strictly increasing for all* θ_i. *Then for all* $j \in \{1, \ldots, J\}, k \in \{1, \ldots, J\} \setminus j$, *and almost all* x, $\sigma_k(x)$ *strictly decreasing in* x_j *implies* $\sigma_j(x)$ *is strictly decreasing in* x_k.

Proof. Strict monotonicity implies that for all θ_i, $v_j(\cdot; \theta_i)$ and $v_k(\cdot; \theta_i)$ are almost everywhere continuous. Take a point of continuity x. By the exclusion of x_j from $v_k(\cdot)$ for $k \neq j$, to have $\sigma_k(x)$ strictly decreasing in x_j requires that for all $\epsilon > 0$

$$\Pr\left(\epsilon > v_{ij} - v_{ik} > -\epsilon, \min\{v_{ij}, v_{ik}\} > \max_{\ell \neq j,k} v_{i\ell}\right) > 0.$$

This implies that $\sigma_j(x)$ is strictly decreasing in x. □

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