Problem: Assuming that $F(K, L) = (0.5) K^{2/3} L^{1/3}$, $s = 0.2$, $\delta = 0.1$, $n = 0.03$, then:

(a) Show that the production function is constant return to scale.

$$F(\alpha K, \alpha L) = (0.5) (\alpha K)^{2/3} (\alpha L)^{1/3} = \alpha (0.5) K^{2/3} L^{1/3} = \alpha F(K, L)$$

(b) Argue that it has diminishing marginal return in capital and labor.

$$MPK = \frac{\partial Y}{\partial K} = (0.5) \left( \frac{2}{3} \right) K^{-1/3} L^{1/3} \geq 0$$

$$\frac{\partial^2 Y}{\partial K^2} = (0.5) \left( \frac{2}{3} \right) \left( -\frac{1}{3} \right) K^{-4/3} L^{1/3} \leq 0$$

$$MPL = \frac{\partial Y}{\partial L} = (0.5) \left( \frac{1}{3} \right) K^{2/3} L^{-2/3} \geq 0$$

$$\frac{\partial^2 Y}{\partial L^2} = (0.5) \left( \frac{1}{3} \right) \left( -\frac{2}{3} \right) K^{2/3} L^{-5/3} \leq 0$$

(c) Exhibit the growth accounting formula.

$$\frac{\triangle Y}{Y} = \frac{\triangle A}{A} + \left( \frac{2}{3} \right) \frac{\triangle K}{K} + \left( \frac{1}{3} \right) \frac{\triangle L}{L} = \left( \frac{2}{3} \right) \frac{\triangle K}{K} + \left( \frac{1}{3} \right) \frac{\triangle L}{L}$$

since $A$ is assumed to be a constant.

(d) Express the production function in the form of per labor: $y = f(k)$

$$y = \frac{Y}{L} = \frac{F(K, L)}{L} = \frac{(0.5) K^{2/3} L^{1/3}}{L} = \frac{(0.5) K^{2/3}}{L^{2/3}} = (0.5) \left( \frac{K}{L} \right)^{2/3} = (0.5) k^{2/3}$$

(e) Show the equation exhibiting the growth rate of capital per labor.

$$\frac{\triangle k}{k} = s \left( \frac{y}{k} - \delta \right) - n = (0.2) \left( \frac{(0.5) k^{2/3}}{k} - 0.1 \right) - 0.03 = \frac{(0.2)(0.5)}{k^{1/3}} - (0.2)(1) - 0.03 = \frac{0.1}{k^{1/3}} - 0.05$$

(f) Solve for the steady state level of capital per labor.

At steady state, $\frac{\triangle k^*}{k^*} = 0$.

$$0 = \frac{\triangle k^*}{k^*} = \frac{0.1}{k^{1/3}} - 0.05$$

$$\frac{0.1}{k^{1/3}} = 0.05$$

$$k^{1/3} = \frac{0.1}{0.05} = 2$$
(g) Illustrate the graphs for the growth rate of capital and transition path of capital per worker. Show the growth rate in the graphs drawn.

(h) What happens if saving rate increases to 0.3?

\[
\frac{\Delta k}{k} = s \left( \frac{y}{k} - \delta \right) - n = (0.3) \left( \frac{(0.5) k^{2/3}}{k} - 0.1 \right) - 0.03 = \frac{(0.3)(0.5)}{k^{1/3}} - (0.3)(0.1) - 0.03 = \frac{0.15}{k^{1/3}} - 0.06
\]

At a new steady state,

\[
0 = \frac{\Delta k^*}{k^*} = \frac{0.15}{k^{1/3}} - 0.06
\]

\[
\frac{0.15}{k^{1/3}} = 0.06
\]

\[
k^{1/3} = \frac{0.15}{0.06} = 2.5
\]

\[
k^* = 15.625
\]

(z) Repeat item (g) with the information from item (h), graphically explain the effect of a change in saving rate.