MODELS OF COMPLEXITY IN ECONOMICS AND FINANCE

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1. INTRODUCTION.

There are two contrasting viewpoints concerning the explanation of observed fluctuations in economic and financial markets. According to the first (New classical) view the main source of fluctuations is to be found in exogenous, random shocks to fundamentals. In the absence of shocks, prices and other variables would converge to a steady state (growth) path, completely determined by fundamentals. According to the second (Keynesian) view a significant part of observed fluctuations is caused by nonlinear economic laws. Even in the absence of any external shocks, nonlinear market laws can generate endogenous business fluctuations. The new classical view is intimately related to the concept of rational expectations, whereas animal spirits or market psychology have been an important Keynesian theme.

In finance the two different viewpoints lead to opposite views concerning the efficiency of financial markets. In the efficient market hypothesis (EMH) the current price already contains all information and past prices can not help in predicting future prices. Parametric stochastic processes have been used in the empirical literature that are consistent with the EMH. Examples include random walk processes, GARCH-processes and the like. In contrast, Keynes already argued that stock prices are not only determined by fundamentals, but in addition market psychology and investors animal spirits influence financial markets significantly. In the Keynesian view, simple technical trading rules, such as extrapolation of a trend, may help predict future price changes. In fact, recently Brock, Lakonishok and LeBaron (1992) have indeed shown that simple technical trading rules, such as moving average and trading range break, when used in predicting the Dow Jones Index, consistently outperform several popular stochastic finance models, such as the random walk and the GARCH-model.

The discovery of chaotic, seemingly random looking dynamical behaviour in simple deterministic models sheds important new light on this debate. A simple deterministic financial market model with a strange attractor may generate erratic, seemingly unpredictable stock price fluctuations very similar to a random walk. Prices moving on a strange attractor may be very difficult to predict and chaos may thus be consistent with a weak form of the EMH. Intermittent chaotic time series, characterized by irregular switching between a stable phase of low volatility and an unstable phase of high volatility, may
explain the well known GARCH effects frequently observed in financial data.

At this point, let us briefly review the "state of the art" concerning chaos in economics and finance. It has been shown that simple nonlinear general equilibrium models, satisfying the currently dominating assumptions in economic theory (i.e. utility or profit maximizing agents, rational, self-fulfilling expectations and market clearing), can generate chaotic equilibrium dynamics. For the popular overlapping generations (OLG) model this has been shown by Benhabib and Day (1982) and Grandmont (1985) and in optimal growth models by Boldrin and Montrucchio (1986), as e.g. surveyed in Boldrin and Woodford (1990) and Nishimura and Sorger (1996). In all these examples the dynamics can be reduced to a one-dimensional nonlinear difference equation. Only very recently there have been some two-dimensional generalisations, e.g. an OLG-economy with production (Medio and Negroni (1993) and de Vilder (1995). In higher dimensional models, chaos may arise with much less nonlinearity than in the one dimensional case. For example in the one dimensional OLG-model a strong income effect is needed to generate chaotic equilibrium cycles, whereas in the two dimensional OLG model with production, chaos may arise even when the two goods, current leisure and future consumption, are gross substitutes.

The (chaotic) time series generated by these models all seem to be clearly different from actual (macro) economic data however. Apparently, the models are still "too simple to be true". In fact, there is little empirical evidence for low dimensional chaos in (macro) economic data. But as Brock and Sayers (1988, p.78) emphasize, "...the methods we utilized may be too weak to detect chaos if it exists". The main reason is that macro economic time series are too short and probably too noisy to detect chaos, even if it were present. For surveys on testing for chaos in economics and finance, see e.g. Brock (1986) and Brock, Hsien and LeBaron (1991).

Daily or weekly financial series are much longer than (macro) economic series and a number of papers have looked for deterministic chaos in financial time series. For example, Scheinkman and LeBaron (1989) present some evidence for a strange attractor with fractal (correlation) dimension of about 6. They are very careful in their claim however: "...the data are not incompatible with a theory where some of the variation in weekly returns could come from nonlinearities as opposed to randomness and are not compatible with a theory that predicts that the returns are generated by IID random variables" (Scheinkman and LeBaron (1989, p.332)). The most optimistic claims for chaos in financial data seem to be due to Barnett and Chen (1988) in their Divisia monetary aggregates series (about 800 observations) for which they find low
correlation dimension of about 1.5. However, Ramsey, Sayers and Rothman (1990) have emphasized that estimates of the correlation dimension may be strongly downward biased in short (say 1000 or less observations) data sets. LeBaron (1994) reviews evidence for out-of-sample short term predictability using nonlinear prediction methods that would work well if (noisy) chaotic dynamics (or other "established" sources of dynamics) were present in returns. There is some evidence for short term predictability provided one conditions on certain events such as near-past volatility. In summary, it seems to be fair to say that there is no convincing evidence for a low dimensional chaotic explanation of economic or financial data. One should add however that the chaos tests are sensitive to noise and series length. On the other hand, by now it is also a mathematical fact that in two and higher dimensional models weak nonlinearities may already lead to bifurcation routes to chaos (e.g. Palis and Takens (1993)). Hence, there seems to be a "chaos model-data paradox" in economics: chaos is hard to find in economic and financial data, but easily generated by economic equilibrium models under increasingly plausible assumptions.

There is evidence of patterns in financial data that seems inconsistent with simple versions of the EMH, especially in high frequency data. See for example the work of the Olsen group as in Guillaume et al. (1994) and Dacorogna et al. (1995). "Financial psychologists" like DeBondt and Thaler (1985) (see also the papers in Thaler (1994, part VI) have also emphasized the role of quasi-rational, overreacting and biased traders in financial markets. Furthermore the large amount of trading volume and the persistence of this trading volume that is observed in real markets suggests that heterogeneous beliefs must play a key role in generating these observed patterns in financial data. Some evidence, for example the evidence on trading rule profits, suggests that some periods may be consistent with the EMH and certain other periods may not be. It thus seems useful to investigate possible routes of departure from the EMH. This chapter contributes to the task of building analytic frameworks that can help shed light on financial economic forces that may cause the market to move towards or move away from the equilibrium predicted by the EMH. We develop a class of heterogeneous belief models that nest the usual rational expectations models (e.g. EMH) and are econometrically tractable. Beliefs are expressed as beliefs about deviations from the EMH-fundamental. The models should help identify periods where the rewards to possessing rational expectations are relatively large or relatively small. This kind of analysis may suggest periods when one would expect larger departures from rational expectations than other periods.
Recently several structural nonlinear financial market models have been introduced. A number of these emphasize heterogeneity in expectation formation, with two or more groups of traders having different expectations about future prices. Two typical investor types are the fundamentalists, expecting prices to return to their fundamental value, and the chartist or technical analysts, using simple technical trading rules and extrapolating trends in past prices (e.g. Day and Huang (1990), DeGrauwe, DeWachter and Embrechts (1993) and Lux (1995)). In the "artificial economic life" literature an evolutionary dynamics in an "ocean" of traders using different expectations has been introduced (e.g. Arthur et al. (1994) or the review of LeBaron (1995)). Most of this work is computational oriented with a large number of different predictors. Brock (1993, 1995) and Brock and LeBaron (1996) have started to build a theoretical framework and analyze simple versions of these adaptive belief systems. Brock and Hommes (1995) investigate a simple demand-supply cobweb type model with rational versus naive expectations and show how an adaptive belief system may lead to market instability and chaos, when rational expectations are more costly than naive expectations.

In this chapter we introduce a tractable form of evolutionary dynamics which we call, *Adaptive Belief Systems*, into the present value asset pricing theory. Asset traders migrate across different beliefs or predictors of the future value of a risky asset according to a "fitness" or "performance" measure based upon how these beliefs perform in the generation of trading profits. Agents switch between fundamentalists beliefs and simple technical trading rules. Predictor choice is *rational* in the sense that, at each date, most agents choose the predictor generating the highest weighted sum of net past profits. One may say that the market is thus driven by *rational animal spirits*. We show how an increase in the "intensity of choice" to switch predictors can lead to emergence of complicated dynamics for asset returns, where an irregular switching occurs between phases where the market is close to the fundamental solution and phases where traders become excited and extrapolate trends by simple technical analysis. These dynamics are suggestive of complicated dynamics for volatility and volume as well.

In our model, not all agents are perfectly rational. We emphasize though that our traders are also not completely "irrational", but in fact boundedly rational. In periods where prices are close to fundamentals, most agents will be fundamentalists, but when prices move away from fundamentals, most agents will abandon fundamentalists beliefs and e.g. use a simple trend predictor to extrapolate an observed price trend. Predictors with highest performance will
dominate the evolutionary dynamics. If the evolution takes place on a strange attractor, simple technical trading rules need not be "systematically wrong" and such expectations may become self-fulfilling and consistent with actual observations (cf. Grandmont (1994), Hommes (1991, 1996) and Sorger (1996)).

This chapter may be seen as a first step to uncover classes of adaptive learning dynamics which are consistent with the main stylized facts of returns, volatility of returns and volume. We state these stylized facts right away so the reader can see what we are after. If \( p_t \) is the price of a financial asset at date \( t \), define returns ("continuously compounded") to be \( r_t = \ln(p_t) - \ln(p_{t-1}) \). Then (i) the autocorrelation function (ACF) of individual security returns is approximately zero at all leads and lags; (ii) the ACF of most measures of high frequency data volatility and volume (detrended) is positive and dies off as the lag increases; (iii) the cross autocorrelation function between volume measures and volatility measures is contemporaneously positive with rapid falloff with leads and lags. We investigate whether our simple evolutionary dynamics can match some of these stylized facts. In particular, whether the evolutionary dynamics can converge to a strange attractor, with price fluctuations "similar" to a random walk (e.g. with slowly decaying ACF) and chaotic returns with close to zero ACF.

The plan of the chapter is as follows. In section two, we set up a mean variance framework in the present discounted value asset pricing model with one risk free and one risky asset. We add adaptive heterogeneous beliefs to the model and develop a general evolutionary theory for this mean variance setting. Section three presents two typical examples of "few" belief type adaptive systems, one example with two and another with four predictors. One type will be fundamentalist, who believe that prices will return to their EMH-fundamental value. Other simple belief types will typically be trend extrapolators and upward or downward biased beliefs. We also present a first "calibration" of the four-belief type model to monthly IBM data.\(^1\) In section four, we sketch how one might use the theory advanced here to help structure empirical work in testing for the presence of "extra endogenous dynamics" in stock returns, above and beyond the "conventional dynamics" stressed in current financial work. In particular, we sketch how one might set up a nested testing situation for the presence of boundedly rational traders, with the "standard", GARCH(1,1) model as the fundamental returns process. Both the calibration and the nested testing exercise are, at best, a sketch of what kind of empirical work might be suggested by the theoretical work presented here. Finally, section 5 concludes and briefly discusses some future lines of research.
2. ADAPTIVE BELIEFS IN THE SIMPLE MEAN VARIANCE FRAMEWORK.

Consider an asset pricing model with one risky asset and one risk free asset. The risk free asset is perfectly elastically supplied at gross return $R$. Let $p_t$ denote the price (ex dividend) per share of the risky asset. Let $\{y_t\}$ denote the stochastic process of dividends of the risky asset. For illustration and because of space limitations, we shall assume that $\{y_t\}$ is Independently and Identically Distributed (IID), but the analysis can be carried out for more general dividend processes. The dynamics of wealth is

\[
W_{t+1} = RW_t + (p_{t+1} + y_{t+1} - Rp_t)z_t,
\]

where bold face type denotes random variables and $z_t$ is the number of shares of the asset purchased at date $t$. Write $E_t$, $V_t$ for the conditional expectation and conditional variance operators. These are based on a publicly available information set, which we here take to be past prices and past dividends. Let $E_{ht}$, $V_{ht}$ denote the "beliefs" of investor type $h$ about these conditional expectation and conditional variance. Note that the choice variable $z_t$ in (2.1) is multiplied by the excess returns per share, $p_{t+1} + y_{t+1} - Rp_t$, so that the conditional variance of wealth is $z_t^2$ times the conditional variance of excess returns per share. Furthermore, the conditional variance of excess returns per share is just the conditional variance of $p_{t+1} + y_{t+1}$, because $p_t$ is part of the information set at date $t$. Beliefs about the conditional variance of excess returns are assumed to be a constant, $\sigma^2$, and the same for all traders. Each investor type is a myopic mean variance maximizer, maximizing expected risk adjusted wealth, so demand for shares $z_{ht}$ by type $h$, solves

\[
\text{Max}(E_{ht} W_{t+1} - (a/2)V_{ht}(W_{t+1})), \text{ i.e.,}
\]

\[
z_{ht} = E_{ht}(p_{t+1} + y_{t+1} - Rp_t)/a\sigma^2,
\]

where "a" denotes the risk aversion, which is assumed to be equal for all traders. Write $z_t$ for the supply of shares per investor and $n_{ht}$ for the fraction of traders of type $h$. Equilibrium of supply and demand at date $t$ implies,

\[
\sum_{ht} n_{ht} (E_{ht} (p_{t+1} + y_{t+1} - Rp_t)/a\sigma^2) = z_t.
\]

In the case of homogenous expectations, i.e. when there is only one type $h$, equilibration of supply and demand yields the pricing equation

\[
Rp_t = E_{ht}(p_{t+1} + y_{t+1}) - a\sigma^2 z_{st}.
\]

Given a sequence of information sets $F_t$ (2.5a) defines a notion of fundamental solution by letting $E_{ht}$, $V_{ht}$ denote conditional mean and variance upon $F_t$. Furthermore, in order to get a benchmark notion of "fundamental solution" put
\( F_t = (p_t, p_{t-1}, \ldots, y_t, y_{t-1}, \ldots) \). In the special case of zero supply of outside shares, i.e. \( z_{st} = 0 \) for all \( t \), (2.5a) becomes

(2.5b) \( R_p = E(p_{t+1} + y_{t+1} | F_t) = E(p_{t+1} | F_t) + \bar{y} \).

Recall that \( (y_t) \) is IID, so \( E(y_{t+1} | F_t) = \bar{y} \) is a constant. The "fundamental" solution \( p^*_t = \bar{p} \) then must satisfy

(2.6) \( R\bar{p} = \bar{p} + \bar{y} \).

Typically (2.5b) has infinitely many solutions but only the constant solution \( \bar{p} = \bar{y} / (R-1) \) satisfies the "no bubbles" condition \( \lim_{t \to \infty} (E_p / R^t) = 0 \). To improve tractability, we will work in the space of deviations from the benchmark fundamental. Let \( x_t = p_t - \bar{p} \) denote deviation from the fundamental solution.

We now introduce heterogeneous beliefs and study the equilibrium dynamical system. In the case of zero supply of outside shares and heterogeneous beliefs, (2.4) becomes

(2.7) \( R_p = \sum h_t E_{ht} (p_{t+1} + y_{t+1}) \).

It seems plausible that more of the disagreement amongst investors will be about the future price rather than the future earnings, because the future earnings are determined by the fortunes of the company whereas the future prices depend upon how the market reacts to those future earnings. The market reaction adds an additional layer of complexity upon the forecasting problem. Note, that Thaler (1994) discusses work by De Bondt and Thaler that documents strong disagreement, overreactions, and biases amongst securities analysts in forecasting earnings which one might argue would be easier to agree upon than future prices. Assume

**Assumption A.2:** All beliefs \( E_{ht} (p_{t+1} + y_{t+1}) \) are of the form

(2.8) \( E_{ht} (p_{t+1} + y_{t+1} | F_t) = E_{ht} (p_{t+1} + y_{t+1} | F_t) + f_{ht} (x_{t-1}, \ldots, x_{t-L}) \),

where \( f_{ht} \) is a deterministic function of past deviations from the fundamental solution and \( (p^*_t) \) denotes the fundamental solution process.

There can be many beliefs about departures from the baseline fundamental solution, \( \bar{p} \), that are not captured by A.2, but A.2 gives us a useful start. Using A.2 write the equilibrium equation (2.7) in deviations form

(2.9a) \( R x_t = \sum h_t h' f (x_{t-1}, \ldots, x_{t-L}) = \sum h_t h' f' \)
At this stage we shall be very general about the beliefs \( \{ \rho_{ht} \} \) which we have expressed in the form of deviations from the fundamental solution \( \{ \rho^*_{t} \} = \{ \bar{\rho} \} \). Note how we are developing the special case of IID \( \{ y_t \} \) and zero supply of outside shares to position ourselves for treatment of more general cases. In these more general cases, we may deal with a large class of \( \{ y_t \} \) processes, even nonstationary ones, by working in the space of deviations from the fundamental. This device allows us to dramatically widen the applicability of stationary dynamical systems analysis to financial modelling.

Recall that all traders are assumed to have common, constant conditional variances \( \sigma^2 = V_{ht} (R_{ht}) \), on excess returns \( R_{ht} = p_{t+1}^* + y_{t+1}^* - R_p \). Let \( \rho_{ht} = E_{ht} (R_{ht}) \) and consider the goal function

\[
\text{Max}_z \left( E_{ht} (z - (a/2)z^2 V_{ht} (R_{ht})) \right) = \text{Max}_z \left( \rho_{ht} z - (a/2)z^2 \right).
\]

Since (2.10) is equivalent to the objective (2.2) up to a constant, the optimum choice of shares of risky asset is the same. Let \( z(\rho_{ht}) \) solve (2.10), and let \( \rho_t = \text{E}_{t} R_{t+1} \) denote rational expectations. Note here that rational expectors take into account the impact of the non rational expectations traders. I.e. \( \text{E}_{t} R_{t+1} \) is the actual conditional expectation of the actual equilibrium stochastic process based upon information at time \( t \). Note that, in deviations form \( \rho_t = \text{E}_{t} R_{t+1} = \text{E}_{t} x_{t+1} - Rx_t \). This brings us to the issue of precise mathematical formulation of the class of deviations \( \{ x_t \} \) that we shall consider. Digress for a moment to consider the case where all expectations are rational. In this case the equilibrium equation (2.9a) can be written as

\[
R_{x_t} = \text{E}(x_{t+1} | F_t), \quad F_t = \{ y_t, y_{t-1}, \ldots; p_{t}, p_{t-1}, \ldots \}.
\]

Hence any stochastic process \( \{ x_t \} \) that is measurable w.r.t. the sequence of sigma-algebras generated by the information sets \( \{ F_t \} \) qualifies as a solution of (2.9a). Any solution of (2.9b) that is not the zero solution is called a "bubble" solution. If \( R > 1 \) the requirement that a solution \( \{ x_t \} \) be almost surely bounded implies that it is the zero solution.

Turn now to the adaptation of beliefs, i.e. the dynamics of the fractions \( n_{ht} \). First, lagging (2.9a) one period, rewrite market equilibrium as

\[
R_{x_t} = \sum n_{h,t-1} f (x_{t-1}, \ldots, x_{t-L}) \equiv \sum n_{h,t-1} f_{ht}
\]

where \( n_{h,t-1} \) denotes the fraction of type \( h \) at the beginning of period \( t \), before the equilibrium price \( x_t \) has been observed. Define

\[
\pi_{ht} = \pi(\rho_t, \rho_{ht}) = \rho_t z(\rho_{ht}) - (a/2)[z(\rho_{ht})]^2 \sigma^2.
\]
The first term in the RHS of (2.11) denotes realized profits for type \( h \), whereas the second term captures risk adjustment. In this chapter, we concentrate on the case without risk adjustment, i.e. the case where the second term in (2.11) is dropped (see Brock and Hommes (1996b) on the risk adjusted case). As the "fitness function" or the "performance measure", we take

\[
U_{ht} = \pi_{ht} + \eta U_{h,t-1}
\]

that is, predictor performance is measured by a weighted sum of (non-risk adjusted) realized profits. For \( \eta = 1 \) memory is infinite and all past profits get equal weight; for \( \eta = 0 \) only last periods profit feeds into the fitness measure. Now write type \( h \) belief's \( \rho_{ht} = \mathbb{E}_h R_{t-1} \) in deviation form. Let the updated fractions \( n_{ht} \) be given by the discrete choice probability (see Anderson, de Palma and Thisse (1993) for an extensive discussion of discrete choice modelling and Brock and Hommes (1995) for using discrete choice models in adaptive belief systems)

\[
n_{ht} = \exp[\beta U_{h,t-1}] / Z_t, \quad Z_t = \sum_h \exp[\beta U_{h,t-1}].
\]

We call the equilibrium dynamics defined by (2.9c) and (2.13) an Adaptive Belief System. The timing of updating of beliefs in (2.13) is important. We can only allow the fitness function that goes into the RHS of (2.13) to depend upon \( \rho \)'s dated \( t-1 \) and further back in the past in order to insure that \( n_{ht} \) only depends upon \( x \)'s dated \( t \) and further back in the past. The parameter \( \beta \) is called the intensity of choice and measures the degree of rationality among traders, i.e. how fast traders switch to "better" predictors. Note that if \( \beta \to \infty \), then (2.13) places all mass on the "best" predictor.

Brock and Hommes (1996b) show for the risk adjusted case that, if memory is infinite, i.e. \( \eta = 1 \), and costs for rational expectations are zero, then all deterministic deviations \( \{x_t\} \) converge to zero as \( t \to \infty \). This result may be seen as an efficient market hypothesis (EMH) for the heterogeneous adaptive beliefs asset pricing model. To put it another way, Thaler's overreacting investors and/or securities analysts would be driven out of the market in an infinite memory world where rational expectations are costlessly available. But since it is argued by Thaler and others that such investors are present and impact actual markets (cf. Thaler (1994) and his references), it behooves us to study what kinds of relaxations of the assumptions in the above theorem can lead to survival of "boundedly rational" traders in equilibrium. This will be the subject of section 3.
3. Few Type Adaptive Belief Systems.

In this section two typical examples of the asset pricing with a few belief types are discussed, one with two and one with four types. Brock and Hommes (1996a) contains a more systematic investigation of the role of simple belief types in deviations from fundamentals and primary and secondary bifurcations in possible routes to complexity. All beliefs will be of the simple form

\[(3.1) \quad f_{ht} = g_h x_{t-1} + b_h,\]

where \(g_h\) is the trend and \(b_h\) is the bias of agent \(h\). These simple predictors could be viewed as the simplest idealization of De Bondt and Thaler's overreacting securities analysts or overreacting investors (cf. Thaler (1994, Part Five)). In the special case \(g_h = b_h = 0\), (3.1) reduces to fundamentalists, believing that prices return to their fundamental value. Fundamentalists do have all past prices and dividends in their information set, but they do not know the fractions \(n_{h,t}\) of the other belief types, and act as if all agents were fundamentalists. The parameter values we select below are for illustrative purposes only. They are not meant to be realistic parameter values for actual financial data.

Rewriting (2.3) in deviations form yields the demand for shares by type \(h\)

\[(3.2) \quad z_{h,t-1} = E_{h,t-1} (x_t - R x_{t-1}) / (a \sigma^2) = (f_{h,t-1} - R x_{t-1}) / (a \sigma^2).\]

We focus on the polar case with zero memory, i.e. \(\eta = 0\), and no risk adjustment, i.e. the second term in (2.11) is dropped, so that the fitness measure reduces to last periods realized profit \(\pi_{h,t-1} = \rho_{t-1} z_{h,t-1}\), which for predictor type \(h\) in (3.1) becomes

\[(3.3) \quad \pi_{h,t-1} = \frac{1}{a \sigma^2} (x_t - R x_{t-1}) (g_h x_{t-1} + b_h - R x_{t-1}).\]

3.1. Two Belief Types: fundamentalists versus trend.

As a typical two belief type example, consider the case where type 1 are fundamentalists, believing that prices return to their fundamental solution \(x_t = 0\) (corresponding to the fundamental \(p_t = \bar{p}\)), whereas type 2 are pure trend chasers, with \(f_{2t} = g x_{t-1}\) (so there is no bias, i.e. \(b = 0\)). The adaptive belief system (2.9a-2.13) becomes

\[(3.4a) \quad R x_t = n_{2,t-1} a x_{t-1}, \]

\[(3.4b) \quad n_{1t} = \exp(\beta(\frac{1}{a \sigma^2} R x_{t-1} (R x_{t-1} - x_t) - C) / Z_t).\]
\[ n_{2t} = \exp(\beta \frac{1}{\alpha \sigma^2} (x_t - Rx_{t-1}) (g x_{t-2} - Rx_{t-1})) / Z_t \]

It will be convenient to work with the difference in fractions

\[ m_t = n_{1t} - n_{2t} = \text{Tanh}(\frac{\beta}{2} [-Dg x_{t-2} (x_t - Rx_{t-1}) - C]) \]

where \( D = 1/(\alpha \sigma^2) \). Here \( C > 0 \) is the cost of obtaining access to the belief system type 1. This cost \( C \) may be positive because "training" costs must be borne to obtain enough "understanding" of how markets work in order to believe that they should price according to the EMH fundamental.

The adaptive belief system (3.4) is a third order difference equation or equivalently a three dimensional system. Asset price dynamics exhibit the following properties (see Brock and Hommes (1996a) for details and proofs):

**PROPOSITION 1. (fundamentalists versus trend extrapolators):**

D1. For \( 0 < g < R \), the fundamental steady state \( E_1 = (0, \text{Tanh}(\frac{-B C}{2})) \) is the unique, globally stable steady state.

D2. For \( g > 2R \), there exist three steady states: the (unstable) fundamental steady state \( E_1 \) and two additional non-fundamental steady states \( E_2 = (x^*, m^*) \) and \( E_3 = (-x^*, m^*) \), where \( m^* = 1 - 2R/g \).

D3. Let \( R < g < 2R \) and assume that costs \( C > 0 \). There exist \( 0 < \beta^* < \beta^{**} \), with

(a) for \( 0 \leq \beta < \beta^* \) the fundamental steady state is globally stable;

(b) at \( \beta = \beta^* \) a pitchfork bifurcation occurs in which two additional non-fundamental steady states are created.

(c) for \( \beta^* < \beta < \beta^{**} \) the fundamental steady state is unstable and both non-fundamental steady states are stable;

(d) at \( \beta = \beta^{**} \) a Hopf-bifurcation of the non-fundamental steady states;

(e) for \( \beta > \beta^{**} \) all three steady states are unstable.

D4. Let \( \beta = +\infty \), \( C > 0 \) and \( R < g < R^2 \). The unstable manifold \( W(E_1) \) of the fundamental steady state is bounded; all orbits converge to the locally unstable (saddle point) fundamental steady state.

Hence, when the trendchasers extrapolate only weakly \( 0 < g < R \), the fundamental steady state \( E_1 \) is globally stable. If costs \( C = 0 \) half of the traders are of type 1 and half of the traders are of type 2 for any \( \beta \). This makes sense because the difference in profits is zero at \( x = 0 \). Now if \( C > 0 \), we see that the mass on type 1 decreases to zero as \( \beta \) (or \( C \)) increases to \( +\infty \). This makes economic sense. There's no point in paying any cost in a steady state for a trading strategy that yields no extra profit in that steady state. As
intensity of choice $\beta$ increases, the mass on the most profitable strategy in net terms, increases. When the trendchasers extrapolate very strongly ($g > 2R$) there are two additional non-fundamental steady states $E_2$ and $E_3$, even when there are no information costs.

The case of strongly extrapolating trendchasers ($R < g < 2R$) and positive information costs for the fundamentalists is the most interesting. As the intensity of choice $\beta$ increases, the fundamental steady state becomes unstable in a *pitchfork* bifurcation, and two additional (stable) non-fundamental steady states are created. As $\beta$ further increases, the two non-fundamental steady states also become unstable in a Hopf bifurcation. Immediately after this secondary bifurcation, the model has two attracting invariant circles around the two (unstable) non-fundamental steady states $E_2$ and $E_3$ with periodic or quasi-periodic dynamics (see the attractors in figure 1a-b).

FIGURE 1 ABOUT HERE

An important question is whether, as the intensity of choice $\beta$ further increases, the invariant circles break into strange attractors as in the cobweb model with rational versus naive expectations in Brock and Hommes (1995). Property D4 above suggests that for $R < g < R^2$, when the intensity of choice to switch predictors is large but finite, all orbits remain bounded and for $\beta$ large the system must be close to having a *homoclinic point*, a notion already introduced by Poincaré. Let us briefly recall the definition of this important notion, which is one of the key features of a chaotic system. Let $p$ be a saddle point steady state (or periodic saddle). A point $q$ is called a homoclinic point if $q \neq p$ is an intersection point between the stable and the unstable manifolds of $p$. From property D4 it follows in fact that for $R < g < R^2$ and $\beta = \pm \infty$ there exist homoclinic points; therefore, also for $\beta$ large but finite one may expect (transversal) homoclinic points. It is well known that homoclinic orbits imply very complicated dynamical behaviour and possibly existence of strange attractors for a large set of parameter values. However, because our system is 3-D, applying recent homoclinic bifurcation theory (Palis and Takens (1993)), as was done in Brock and Hommes (1995) for the 2-D cobweb adaptive belief system with rational versus naive expectations, is much more delicate. The attractor in figure 1b, suggests that the system is already close to having a homoclinic orbit; the stable manifold $W^s(E_1)$ contains the vertical segment $x = 0$, whereas the unstable manifold $W^u(E_1)$ moves to the right and then "folds back" close to the stable manifold.
Figure 1 shows attractors in the \((x_t, m_t)\) plane, with and without noise, for different \(\beta\)-values. Figure 2 shows some corresponding time series. The asset price fluctuations are characterized by a switching between an unstable phase of an upward or downward trend and a stable phase of close to fundamental price fluctuations. In the noise free case, this switching seems to be fairly regular. In the presence of small noise however, the switching becomes highly irregular and unpredictable.

3.2. Four Belief Types: fundamentalists versus trend versus bias.

Next consider an example with four belief types. As before, type 1 are fundamentalists. Belief parameters for the other three types are: \(g_2=0.9, b_2=0.2; g_3=0.9, b_3=-0.2; g_4=1.01\) and \(b_4=0\). Hence, type two is a trend with upward bias, type three a trend with downward bias and type four a pure trend chaser. This example exhibits some typical features observed in many other examples as well. With four predictors, the adaptive equilibrium dynamics (2.9a-2.13) is:

\[
(3.5a) \quad R_{x_t} = \sum_{j=1}^{4} n_{j,t-1} (g_j x_{t-1} + b_j)
\]

\[
(3.5b) \quad n_{j,t} = \exp \left( \frac{-\beta}{\sigma_0^2} (g_j x_{t-2} + b_j - R_{x_{t-1}})(x_t - R_{x_{t-1}})/Z_t \right), \quad j = 1, 2, 3, 4
\]

The system (3.5) is equivalent to a third order difference equation in \(x_t\). The following holds (see Brock and Hommes (1996a)):

**PROPOSITION 2**: With the parameters as above, the fundamental \(x^* = 0, n^*_j = 1/4\), is a steady state of (3.5). This fundamental steady state is stable for \(0 < \beta < 50\) and unstable for \(\beta > 50\). At \(\beta = 50\), a Hopf bifurcation occurs.

Figure 3 shows some attractors projected into the \((x_t, x_{t-1})\) plane for different values of the intensity of choice \(\beta\). For low values of the intensity of adaptation the fundamental steady state is stable. As \(\beta\) increases, the steady state becomes unstable due to a Hopf bifurcation and an "invariant circle" with quasi-periodic dynamics arises. As \(\beta\) further increases, the invariant circle breaks up into a strange attractor with a fractal structure (figures 3a-d). Figure 3e shows a chaotic series of deviations from the fundamental.
Chaos is characterized by an irregular switching between a stable phase with prices close to the fundamental and an unstable phase of an upward or downward trend where most agents are of type 2 respectively type 3. This irregular switching is triggered by a rational choice between the four predictors.

For a very high intensity of choice (say \( \beta > 120 \)), at some point almost all traders become fundamentalists, driving prices back to their fundamental steady state. However, the fundamental steady state is locally unstable. The simulations suggest (see e.g. figure 3c and the enlargement 3d) that, as in the two predictor example, for high values of the intensity of choice, the system is close to having a homoclinic intersection between the stable and unstable manifolds of the fundamental steady state.

Our results imply that in a four type world, even when there are no costs, fundamentalists can not drive out trend chasers and biased beliefs, when the intensity of adaptation is high. Hence, the market can protect a biased or trend trader from his own folly if he is part of a group of traders whose biases are "balanced" in the sense that they average out to zero over the set of types. Centralized market institutions can make it difficult for unbiased traders to prey on a set of biased traders provided they remain "balanced" at zero. Of course, in a pit trading situation, unbiased traders could learn which types are biased and simply take the opposite side of the trade. This is an example where a centralized trading institution like the New York Stock Exchange could "protect" biased traders, whereas in a pit trading institution, they could be eliminated.

3.3. A calibration exercise.

Do the strange attractors in our simple adaptive belief system explain a "significant" part of observed fluctuations in real asset markets? In order to get some insight into this problem, we "calibrate" a (noisy) chaotic time series of the four belief type model, to ten years of monthly IBM-data. Figure 4 shows linearly detrended logs of IBM common stock prices\(^{(2)}\), (i.e. \( \ln(p_t) \) minus a linear trend), returns (i.e. \( \ln(p_t) - \ln(p_{t-1}) \)) and squared returns, from January 1980- December 1989. Figure 5 shows the autocorrelation function (ACF) plots of these IBM-series; all ACF's are displayed with "Bartlett 5% significance bands" (e.g. Box et al. (1994, pp.32-34)). The ACF of linearly detrended logs of IBM-prices is slowly decaying with positive significant lags 1-7; the ACF's of returns and squared returns are both close to zero, with only the first lag significant. These patterns are not uncommon for monthly
data of individual stocks.

Figure 6 shows a chaotic time series of the deviation \( x_t \) from the fundamental, excess returns \( R_t = x_t - R_{x_{t-1}} \) and squared returns \( R^2_t \), all of the same length as the corresponding IBM-series (120 observations). Figure 7 shows the corresponding ACF's. The ACF of deviations \( x_t \) from the fundamental is slowly decaying with positive significant lags 1-4; the ACF's of returns \( R_t \) is significant only at the first lag, whereas the ACF of squared returns has no significant lags. Similar patterns of the ACF's have been observed for other initial states, closeby parameters and also for longer chaotic series. Note that in figure 6, we picked an initial state \(^3\) such that the chaotic series starts with an upward trend, followed by a downward trend, a stable phase of close to fundamental prices and finally two upward trends at the end of the 120 periods. This pattern has been chosen in order to mimic the behaviour of the IBM log price series. For other chaotic series, the pattern may of course be different, since along the strange attractor, exactly the switching between the different phases is highly unpredictable. \(^4\) However, in the chaotic adaptive belief system, the "IBM-pattern" apparently can occur with positive probability, triggered by a 'rational' choice between the four predictors.

Figure 8 shows noisy chaotic deviations from the fundamental, i.e. \( x_t + \varepsilon_t \), with \( \varepsilon_t \sim N(0, \sigma^2) \) and \( \sigma^2 = 0.01 \), noisy returns \( R_t + \varepsilon_t \) and noisy squared returns \( (R_t + \varepsilon_t)^2 \). The variance of the deterministic deviation \( x_t \) is about 0.09, so that the signal to noise ratio in this example is 9. Figure 9 shows that the ACF's of the noisy series are similar to the noise-free case.

The reader may judge for himself or herself how well the noisy chaotic model matches the stylized facts of the data, by comparing figure 4 with 8 and figure 5 with 9. It seems that prices and returns in our simple four belief type financial market exhibit some of the stylized facts. In particular we find decaying ACF's of prices and close to zero ACF's of returns. Noisy chaotic squared returns also seem to be similar to monthly IBM squared returns. The strange attractors show little persistence in volatility however. The adaptive belief system thus does not produce (G)ARCH-effects, at least not in its simplest form as discussed here. In monthly data like the IBM prices (G)ARCH effects are also not too strong, and less strong than in high frequency data.
4. A sketch of an empirical exercise suggested by the evolutionary theory

In this section we sketch a possible strategy, using the theory developed in previous sections, to set up an econometric test for the presence of "extra endogenous dynamics" in stock returns above and beyond the "conventional dynamics" stressed in current financial work. See Altug and Labadie (1994) for work that stresses the interaction of movements in financial returns and movements in macroeconomic aggregates. See Grossman (1989) for work that stresses movements in financial returns due to asymmetric information amongst traders and the role of the price system in communicating information to traders. In these models traders receive signals (measurements with error) on a "latent" fundamental that is not observed by the econometrician. See Brock and LeBaron (1996) for a start on unification of work that stresses information with evolutive dynamics based upon fitness measures. The spirit of this chapter is to see how far one can go in testing for the presence of non-fundamental traders in financial markets in using only returns data.

It was suggested by Brock and Hommes (1995) that it should be possible to extend the deterministic heterogeneous adaptive belief system (where one of the belief systems is rational expectations) into a stochastic setup and nest the Rosen, Murphy, Scheinkman (1994), "RMS," model within this extended setup. More precisely, if \( n_1 \) of the agents are RMS rational expectations types and \( n_2 \) are some backward looking type, then, holding the \( n_1 \) fixed, one could set up a linear quadratic general equilibrium model with heterogeneous beliefs. One could then estimate it along the lines of Anderson et al. (1995). One can then set up a test of the null Hypothesis, \( H_0: n_2 = 0 \) and test \( H_2 \) by the usual "nest it, test it" method. Baak (1995) and Chavas (1995) have carried out a similar research strategy and have adduced evidence for the presence of some backward looking agents in agricultural data sets. We outline a similar project here for financial data that is stimulated by the theoretical findings for trend chasers that were developed in previous sections.

The results on dynamics with trend chasers suggest that, perhaps in the "real" world, certain patterns in returns may "excite" a mass of investors to extrapolate such patterns of deviations from a commonly shared idea of fundamental value. An example might be a positive run of two or three (or more) consecutive periods of positive returns, or the symmetric opposite of this pattern. But a positive measure mass of deviant beliefs from the fundamental may attract an "opposite" mass which would profit by taking a position that deviates away from the fundamental. The appearance of this counter mass should generate extra volatility in returns that would be added to the amount
of volatility one would expect from a fundamental which is estimated from data on earnings. Of course calculation and estimation of a "serious" fundamental is beyond our data resources. Therefore we propose a "first cut" procedure here. Recall the pricing equation from (2.5a) in section 2,

\[ (4.1) \quad R_p^t = E_t (p_{t+1} + y_{t+1}) - ah_t z_t, \]

where \( a \) is the risk aversion parameter, \( h_t \) is conditional variance of \( v_{t+1} = p_{t+1} + y_{t+1} \), \( E_t \) is conditional mean, \( p_t, y_t \) are ex-dividend price and dividends at date \( t \), and \( z_t \) is supply of shares per investor. Here conditional expectation and conditional variance are conditional on past prices, past dividends and past supplies of shares per trader.

Before continuing further, let us briefly discuss how one solves (4.1) for a fundamental solution. If one assumes that \( \langle y_t \rangle \), \( \langle z_t \rangle \) are finite order Markov processes, then under regularity conditions, a fixed point argument may be developed to produce a unique fundamental solution \( p_t^* = P(y_t, z_t) \). The assumed stationarity of the solution that one seeks, avoids the class of "non-fundamental" solutions, e.g. bubble solutions. See Stokey and Lucas (1989) for a general treatment of this kind of theory. Consider the equilibrium excess returns generated by the fundamental solution \( p_t^* \), i.e., consider

\[ (4.2) \quad R_{t+1}/p_t^* = (p_{t+1}^* + y_{t+1} - Rp_t^*)/p_t^*. \]

If no restrictions are placed on \( \langle y_t \rangle \) and \( \langle z_t \rangle \), then the set of \( \langle R_{t+1}/p_t^* \rangle \) that one can generate by (4.2) will be too large to be useful in empirical work. Empirical work on earnings and dividends can be used to restrict \( \langle y_t \rangle \). It is much more difficult to figure out how to use observational data to usefully restrict the \( \langle z_t \rangle \) process, representing net supply of shares per trader to the community of traders that is being modelled. In the case of a closed community of traders, that are trading only contracts written between each other, \( z_t \) is zero for all \( t \). In the case of a stock such as IBM, the process \( \langle z_t \rangle \) represents not only the shares outstanding of IBM per trader, but also a netting out of sources of supply and demand for IBM shares that are "outside" of the community of traders that is being modelled. In view of the simple purpose of this section, to show how use of the theory can help guide empirical work, we shall focus on drastic simplifications of many features of reality, and concentrate on the case with zero supply of outside shares.

Return to equation (4.1). Here is an example where the fundamental solution is easy to calculate. Let the dividend process be given by the random walk with martingale difference sequence errors
\[ y_{t+1} = \mu + y_t + \epsilon_{t+1}, \quad E_{t} \epsilon_{t+1} = 0, \quad t=1,2,\ldots \]

For the special case where the variance of the errors \( \epsilon_t \) is a constant, by equating coefficients, it follows that the fundamental solution of (4.1) is

\[ p_t = P(y_t) = A_0 + A_1 y_t, \quad A_0 = (\mu R/r - a \delta) / r, \quad A_1 = 1/(R-1) = 1/r. \]

We shall drop the asterisk superscript, whenever it is clear that we are talking about the fundamental solution of (4.1). Note that if \( z=0 \), the solution (4.4) holds for any martingale difference sequence (MDS) process \( \epsilon_t \) for dividends. In particular, the conditional variance does not have to be constant. Define excess returns per share by

\[ R_{t+1} = v_{t+1} - R_{t, t+1} = v_{t+1} - E_{t+1} v_{t+1} + E_{t+1} v_{t+1} - R_{t, t+1} = v_{t+1} - E_{t+1} v_{t+1} = (1+A_1) \epsilon_{t+1}. \]

Note how excess returns per share \( R_{t, t+1} \), is just the market's prediction error in equilibrium. Before going further, let us discuss dealing with actual data. Financial researchers typically work with continuously compounded returns

\[ \ln(v_{t+1} / p_t) = \ln((v_{t+1} / p_t - R) + R) \equiv (1/R)(v_{t+1} / p_t - R) + \ln(R) \equiv (v_{t+1} / p_t - 1), \]

for \( R=1 \). For high frequency data we have \( R=1 \) and also the data are adjusted for dividend payouts, so researchers typically replace (4.6) by

\[ r_{t+1} = \ln(p_{t+1} / p_t) \equiv R_{t, t+1} / p_t, \]

and the daily IBM data that we study here is given in the form (4.6').

Return now to theory. Let \( v_{t+1}^f \) denote the "fundamental solution" generated by (4.4-4.5). Now introduce "bounded rationality" by type \( j \) having expectations \( E_{j, t+1} = E_{t, t+1} + f_{j, t} \), and let there be fractions \( n_{j, t} \) of type \( j \) believers at date \( t \). Here \( f_{j, t} \) denotes the deviation in type \( j \)'s beliefs from the fundamental at time \( t \). The pricing equilibrium equation, with zero supply \( z \) of outside shares, is

\[ R_{t, t+1} = E_{t, t+1} v_{t, t+1} + \sum_j n_{j, t} f_{j, t} \equiv E_{t, t+1} v_{t, t+1} + \tilde{f}_{t}. \]

Let us set up a test of \( H_0: \tilde{f}_t = 0 \) for all \( t \). This hypothesis is true when, for example, all traders have fundamental beliefs, i.e., all traders have \( f_{j, t} = 0 \), or the set of nonfundamental traders is "balanced", i.e. \( \tilde{f}_t = 0 \). Let the alternative \( H_1 \) be: There exist some time \( t \) such that \( \tilde{f}_t \) is not zero.

The econometric problem is to use available data to set up an econometric test for the presence of nonfundamental traders. The spirit of this chapter is
to see how far one can go in such testing only using returns data. In order to set up an econometric test, we have to specify an econometrically tractable model to play the role of the "base line" fundamental. A popular class of parametric time series models that is consistent with parametric versions of the EMH is the GARCH(1,1)-M class defined by

\[(4.8)\quad r_{a,t+1} = a_0 + a_1 h_{t+1}^{1/2} + h_{t+1}^{1/2} N_{t+1}, \quad (N_t) \iid \mathcal{N}(0,1), \quad h_t = b_0 + b_1 \eta_t + b_2 h_{t-1}^{1/2}\]

\[(4.9)\quad \eta_t = r_t - a_0 - a_1 h_t^{1/2}\]

The notation \(\{r_{a,t+1}\}\) is meant to suggest that (4.8) is an approximation to the true process. See Bollerslev (1986), Bollerslev, Engle and Nelson (1994) for a large literature on fitting this type of model to financial data. Note that \(E_r r_{a,t+1} = a_0 + a_1 h_t^{1/2}\) in (4.8), and using the equilibrium equation (4.1), also \(E_r r_{a,t+1} = E_r (R_t - p_t)/p_t = ahz_t/p_t^2\). Since the "M-part" \(a_1 + a_1 h_{t+1}^{1/2}\) in (4.8) is statistically small, we shall concentrate on the case \(a_0 = a_1 = 0\), which is consistent with \(z = 0\) for all \(t\).

In order to implement the testing procedure suggested here, we need to find a process \(\{c_{t+1}\}\), for the innovations in the \(\{y_t\}\) process (4.3), so that the equilibrium fundamental returns process \(\{r_{a,t+1}\} = \{R_{a,t+1}^* / p_t^*\}\), generated by (4.1) with \(z = 0\) for all \(t\), is a GARCH(1,1) process of the form (4.8) and (4.9), with \(a_0 = a_1 = 0\). Use (4.5) and (4.8) to write

\[(4.10)\quad R_{a,t+1}^* / p_t^* = (1 + A_t) c_{t+1} / (A_0 + A_t y_t) = h_{t+1}^{1/2} N_{t+1} = r_{a,t+1}\]

i.e.

\[(4.11)\quad c_{t+1} = h_{t+1}^{1/2} N_{t+1} (A_0 + A_t y_t) / (1 + A_t)\]

It is easy to check that \(\{c_{t+1}\}\) given by (4.11) is a Martingale Difference Sequence w.r.t. past prices and dividends. Furthermore \(\{c_{t+1}\}\) generates the "target" process \(\{r_{a,t+1}\}\) when \(\{c_{t+1}\}\) is inserted into (4.3) and \(\{y_{t+1}\}\) is inserted into (4.1), with \(z_t\) set equal to zero for all \(t\).

Next, for testing purposes, we wish to nest the special model which generates (4.8), within the general model (4.7). We shall estimate both the special model and the general model and test whether the extra parameters implied by the general model are "significant". Equations (4.5), (4.7) and (4.8) imply

\[(4.12)\quad R_{a,t+1} = v_{t+1} - R_{p_t} = p_t^* r_{a,t+1} + (\bar{r}_{t+1} / R - \bar{r}_{t}^*).\]

The extra parameters that we shall test for "significance" appear in the term \(\bar{r}_{t+1} / R - \bar{r}_{t}^*\). Let \(n_{tt}\) be the fraction of traders with \(f_{tt} = 0\) and lump all others.
Equations (4.6'), (4.10) and (4.12) suggest to consider the model

\begin{align*}
(4.13a) \quad r_{t+1} &= \left( h_t^{1/2} N_{t+1} p_t^* + n_{2t+1} f_{2,t+1} / R - n_{2t+1} f_{2t+1} / R_t \right) / p_t, \\
(4.13b) \quad h_t &= b_0 + b_1 h_t^{1/2} + b_2 h_{t-1}.
\end{align*}

Our theoretical work above suggests that the fraction of type 2 traders will increase when profits to the fundamental traders have been less than profits to the type two traders in the recent past. Let the type two traders put \( f_{2t} \) proportional to \( r_{t-1} \) so their demand at date \( t \) increases when returns have increased in the near past. Equation (3.4d) suggests setting the term \( n_{2t+1} f_{2t} = (1-m_t) f_{2t+1}^t \), \( m_t = \text{Tanh}(\beta/2 d_{t+1}) \), where \( d_{t+1} \) is the difference in profits in strategy 1 (\( f_{1t}^t = 0 \), since it is the fundamental strategy), and strategy 2. By experimenting with the form of \( f_{2t} \) one can "tune" the term \( (\tilde{f}_{t+1}^t / R - \tilde{f}_{t+1}^t) \) to increase when events of investigative interest such as short runs in returns, either positive runs or negative runs occur. One can do this "tuning" by parametrizing \( f_{2t} \) in terms of past returns and implementing a likelihood ratio test that the "extra" parameters in the term \( (\tilde{f}_{t+1}^t / R - \tilde{f}_{t+1}^t) \) are statistically significant.

In a preliminary version of this empirical exercise on daily IBM-data\(^{(6)}\), the results were not very good, since the estimated standard errors of the coefficients were rather unstable. GARCH-M was also estimated but GARCH-M did not improve much on GARCH. By comparing the sample log likelihood of the GARCH(1,1) model and a version of the GARCH(1,1)-M with biased traders model, the null hypothesis that the extra parameters in the term \( (\tilde{f}_{t+1}^t / R - \tilde{f}_{t+1}^t) \) were all zero was marginally rejected. However, technical problems allowed execution of a likelihood ratio exercise that was only an approximation to the representation in (4.13a).

Thus, we see that there may be some evidence against the "standard" GARCH(1,1) model in favor of the alternative. However, there were also some first order autocorrelations in the returns data. Hence, the rejection of the null could be due to this first order autocorrelation which was not present in the fundamental process \( \{a_{t+1}^t\} \). The rejection could also be due to misspecification of the fundamental and the approximation we made to (4.13a).

The presence of first order autocorrelation raises the issue whether we could design a \( \{z_t^t\} \) process and a \( \{y_t^t\} \) process that had some credibility in the financial literature and solve equation (4.1) to produce an AR(1) returns process with GARCH(1,1) errors for the model returns \( \{r_t^t\} \) that match the AR(1) with GARCH(1,1) errors which were estimated on the actual returns data. Then
one could carry out the process above where one tested the AR(1) with GARCH(1,1) errors as the "fundamental" null hypothesis against an alternative with beliefs which deviated from this benchmark fundamental. We believe this kind of exercise would be very suggestive, but it is beyond the scope of this chapter.

If one studies the form of the alternative for this particular example, we believe it is trying to capture evidence of "excitement" in beliefs caused by short runs in returns of either sign. This suggests that it may be worthwhile to scour the trading literature (technical and fundamental, as well as the work of "finance psychologists" like Werner De Bondt and Richard Thaler) to find pattern sequences (such as runs perhaps) that such literature believes "excites" at least some subset of traders. This search would allow us to frame and test alternatives using a framework much like that set up above.

Of course any parsimoniously parameterized stochastic process is at best an approximation to the true stochastic data generating process of the returns on any asset such as IBM stock. Hence, one would always expect such a model to be rejected in large sample situations such as daily data. However, the direction of the rejection as well as a measure of the economic magnitude of the rejection may be instructive. The very preliminary work presented here suggests rejection in the direction of a type of trader whose presence is stimulated by positive or negative runs in returns measured relative to GARCH(1,1). It is beyond the scope of this article to consider the economic magnitude of the rejection or explore further the space of alternative null hypotheses which are consistent with versions of the efficient markets hypothesis which may be consistent with the data analyzed here.

In any event the character of the evidence adduced by Brock, Lakonishok and LeBaron (1992) against several classes of models, including AR-models and GARCH-models is suggestive that our approach might be on the right track. That is to say the fact that these models underpredict volatility and returns following buy signals is suggestive that endogenous dynamics may be temporarily pushing returns above baseline and keeping volatility below baseline.

5. CONCLUDING REMARKS.
We have set out a general framework for adaptive belief systems in asset pricing theory. Fluctuations in prices and returns are driven by an evolutionary dynamics between traders with different expectations about future prices. In each period, traders revise their beliefs according to a "fitness measure",
such as past realized profits. In particular, we have focussed on the evolutionary dynamics with only a few, i.e. two, three or four different trader types. As the intensity of choice to switch predictors becomes high, complicated asset price fluctuations arise with prices and returns moving on a strange attractor. Chaos is characterized by an irregular switching between upward or downward trends and close to the fundamental price fluctuations. Asset price fluctuations are driven by a rational choice of prediction rules by the traders (rational animal spirits).

We have conducted two empirical exercises to investigate whether our very simple structural nonlinear model can match some stylized facts observed in actual stock market data. Firstly, we calibrated a chaotic time series of the four belief type model to monthly IBM data. It seems that the strange attractors exhibit some of these stylized facts, e.g. with slowly decaying ACF’s of prices and close to zero ACF’s of returns. Unlike most financial high frequency series, the strange attractors in our simple adaptive belief systems show little persistence in volatility however. There may be several ways to extend the adaptive belief models (i) let adaptation of beliefs occur on a slower time scale than the high frequency trading, (ii) increase the memory in the performance measure (i.e. take $\eta > 0$, instead of our focus on $\eta = 0$), (iii) let predictor choice be based on a different performance measure, e.g. risk adjusted profits. (iv) consider “large type limits”, where the number of trader types is much larger than four or even goes to infinity, (v) add weakly correlated noise (e.g. correlated noisy dividends or outside shares) and investigate how this weak correlation is amplified by the dynamics, (vi) consider the case of asynchronous adjustment, where traders update their belief strategies according to some asynchronous adjustment rule rather than all adjusting synchronously as was done here. We leave it for future work to see whether these possible extensions lead to stylized facts closer to actual data, and especially to more persistence in volatility.

Secondly, we sketched a possible strategy for econometric testing, using only returns data, to see whether one might adduce some evidence consistent with the presence of evolutive endogenous dynamics of the type studied here. The second empirical exercise was different in spirit from the first calibration type exercise. It investigated whether the benchmark fundamental stochastic GARCH-(1,1)-model is valid or whether "non-fundamental" traders are present in the market.

We stress that the variance of the fluctuations in the fundamental may be large relative to fluctuations induced by the presence of endogenous dynamics,
if any. A more serious empirical attempt to adduce evidence for, or against, the presence of "extra endogenous" dynamics due to shifting evolutive dynamics would use both volume and price (returns) data. Purposive agent and bootstrap based methods as in Brock, Lakanishok and LeBaron (1992) are going to be required to adduce any convincing evidence for the presence of "extra" dynamics above and beyond the "base line" fundamental. Investigating the impulse responses to difference price and volume shocks, as discussed in the empirical work in Gallant, Rossi and Tauchen (1993), would also be needed to adduce evidence of what kind of forces may be playing a role in determining movements in returns. Our empirical exercise suggests that there may be nonfundamental traders, but much more work is needed to arrive at definite conclusions.

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FOOTNOTES

1. There seem to be only a few attempts to fit or calibrate nonlinear chaotic models to economic or financial data. The heterogeneous beliefs exchange rate models by DeGrauwe, DeWachter and Embrechts (1993) may be seen as a step in that direction. The work by Sterman (1989) shows that in experiments, in a simulated macroeconomic system, agents use suboptimal decision rules, which in about 40% of the cases lead to cycles and chaotic fluctuations.

2. Recall that our adaptive belief system has been formulated in deviations from the fundamental $p^*_t$. So far, we have assumed an IID stochastic process for dividends, with corresponding fundamental price $p^*_t = \bar{p} = \bar{y}/(R-1)$, where $\bar{y} = E_{ht}(y_{t+1})$ is common expectations on dividends. Financial analysts often consider nonstationary stochastic earnings processes, e.g. a geometric random walk with a drift $\mu$, i.e. $\ln(y_{t+1}) = \ln(y_t) + \mu + \epsilon_{t+1}$, $\epsilon_{t+1} \sim N(0, \sigma^2)$, which often fits earnings data fairly well. In that case, the fundamental solution is $p^*_t = A_t y_t$, where $A_t = \exp(\mu + \sigma^2/2)/(R - \exp(\mu + \sigma^2/2))$. Assuming this dividend process, it seems to be natural, to compare deviations $x_t$ in the model to linearly detrended logs of IBM-prices.

3. For the time series in figure 6, the initial state is $x_0 = x_{-1} = x_{-2} = -0.066$ and $n_j = 1/4$, $1 \leq j \leq 4$.

4. See also figure 3e, where a chaotic series of 500 observations, converging to the same strange attractor, is shown.

5. To see this, writing in deviation form $x_t = p_t - p^*_t$ yields $R_{t+1} = v_{t+1} - R p_t = p^*_t + x_{t+1} + y_{t+1} - R x_t - R p^*_t = v_{t+1} - R p_t + y_{t+1} - R x_t = E_{v_{t+1}} - R p_t + v_{t+1} - E_{v_{t+1}} + x_{t+1} - R x_t = (1 + A_t) x_{t+1} + x_t - R x_t = p^*_t a_{t+1} + f_{t+1} / R - f_t$.

6. We would like to thank Kim Sau Chung for carrying out the estimation exercise in section 4.
Figure 1. Trend versus fundamentalists: g=1.2, D=(1/\omega^2)=1.0, C=1.0 and R=1.1. Attractors (a-b) and noisy attractors (c-d) (with dynamic noise \( \epsilon_t \), with uniform distribution over the interval \([-0.05,0.05]\), added to (3.4a)). (a) attracting invariant circle around the positive unstable non-fundamental steady state; there is a second attracting invariant circle (not shown) around the negative unstable non-fundamental steady state; (b) both attractors (only one is shown) have moved close to the stable manifold \( x = 0 \) of the fundamental steady state \( E = (0, \text{Tanh}(-\beta C/2)) \). (c-d) noisy attractors.

Figure 2. Trend versus fundamentalists: time series of (deviations from fundamental) prices: (a) \( \beta = 3.6 \) without noise; (b) \( \beta = 3.5 \), with dynamic noise \( \epsilon_t \), uniformly distributed over the interval \([-0.05,0.05]\), added to (3.4a)); (c) difference in fractions corresponding to noisy price series in (b); (d) \( \beta = 5 \), with dynamic noise \( \epsilon_t \).

Figure 3. (a-d) Strange attractors for different \( \beta \)-values, in four belief type model with \( R = 1.01 \) and belief parameters as in the text. (e) chaotic time series of deviations from fundamental, for \( \beta = 90.5 \).

Figure 4: (a) linearly detrended logs of monthly IBM-prices; (b) monthly IBM returns; (c) monthly IBM squared returns.

Figure 5: Autocorrelation functions with Bartlett 5% significance bands. (a) for linearly detrended logs of monthly IBM-prices; (b) for monthly IBM returns; (c) for monthly IBM squared returns.

Figure 6: Chaotic time series for \( \beta = 90.5 \). (a) deviations \( x_t \) from fundamental; (b) returns \( R_t = x_t - Rx_{t-1} \); (c) squared returns \( R_t^2 \).

Figure 7: Autocorrelation functions with Bartlett 5% significance bands. (a) for deviations \( x_t \) from fundamental; (b) for returns \( R_t = x_t - Rx_{t-1} \); (c) for squared returns \( R_t^2 \).
Figure 8: Noisy chaotic time series for $\beta = 90.5$.
(a) noisy deviations from fundamental, i.e. $x_t + \epsilon_t$, $\epsilon_t \sim N(0,\sigma^2)$, $\sigma = 0.1$;
(b) noisy returns $R_t + \epsilon_t$;
(c) noisy squared returns $(R_t + \epsilon_t)^2$.

Figure 9: Autocorrelation functions with Bartlett 5% significance bands.
(a) for noisy deviations from fundamental;
(b) for noisy returns $R_t + \epsilon_t$;
(c) for noisy squared returns $(R_t + \epsilon_t)^2$. 
Figure 1. Trend versus fundamentalists: $g=1.2$, $D=(1/\alpha^2)=1.0$, $C=1.0$ and $R=1.1$. Attractors (a-b) and noisy attractors (c-d) (with dynamic noise $\varepsilon_1$, with uniform distribution over the interval $[-0.05,0.05]$), added to (3.4a)).

(a) attracting invariant circle around the positive unstable non-fundamental steady state; there is a second attracting invariant circle (not shown) around the negative unstable non-fundamental steady state;

(b) both attractors (only one is shown) have moved close to the stable manifold $x \equiv 0$ of the fundamental steady state $E = (0, \text{Tanh}(-BC/2))$.

(c-d) noisy attractors.
Figure 2. Trend versus fundamentalists: time series of (deviations from fundamental) prices: (a) $\beta = 3.6$ without noise; (b) $\beta = 3.5$, with dynamic noise $\varepsilon_t$, uniformly distributed over the interval $[-0.05, 0.05]$, added to (3.4a)); (c) difference in fractions corresponding to noisy price series in (b); (d) $\beta = 5$, with dynamic noise $\varepsilon_t$. 
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(a) deviations $x_t$ from fundamental;
(b) returns $R_t = x_t - Rx_{t-1}$;
(c) squared returns $R^2_t$. 
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(a) noisy deviations from fundamental, i.e. $x_t + \epsilon_t$, $\epsilon_t \sim N(0, \sigma^2)$, $\sigma = 0.1$;
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(a) for noisy deviations from fundamental;
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