ASSET PRICE BEHAVIOR IN COMPLEX ENvironments

by

William A. Brock

Department of Economics
The University of Wisconsin, Madison
August, 1995
Revised, December, 1995
This Revision, April 9, 1996

1. Introduction

Eight years ago, the Santa Fe Institute organized a conference called The Economy as An Evolving Complex System" and produced a book with the same title. The purpose of that conference was to bring together "free spirits" who were willing to examine the economy using techniques from "complex systems theory."

"Complex systems theory" is hard to define, but I shall use the term to refer to recent methods of time series analysis which are inspired by dynamical systems theory as well computer assisted analysis of complex adaptive systems consisting of many interacting components. Complex systems analysis as used here includes neural nets, genetic algorithms, as well as artificial life analysis such as Arthur etal. (1993). I shall sometimes speak loosely and use "complex systems theory" to denote the research practices of complex system theorists including those at Santa Fe.

This paper gives a brief review of the use of complex systems techniques in the study of asset prices dynamics and related subjects since the SFI conference. The paper is organized by listing the facts we wish to explain, followed by explanation of works that address those facts. For lack of space, I shall concentrate on works by myself, my coauthors, and my students. The reader should be warned that finance is a huge field that contains many researchers and that the use of "complex systems techniques" in this field has grown rapidly. Hence there is a vast amount of work that we shall not be able to discuss in this article. I shall give frequent reference to recent surveys to give the reader a pot of entry into the literature. Many of the works discussed here give extensive discussion of the work of others.

1.1 TIME SERIES ANALYSIS

One set of research practices which are associated with complex systems scientists, in which SFI affiliated researchers played a major role in developing, supporting, and popularizing, is the study of time series data using tools inspired by dynamical systems theory and related methods that are tailored to the study of "deep" nonlinearity.

A survey of this type of work up to 1991 is contained in the book by Brock, Hsieh, and LeBaron (1991). Two recent surveys with an emphasis on methods of testing for "deep" nonlinearity are contained in Abhyankar etal. (1995), and LeBaron (1994a). Excellent general surveys on financial time series findings, which include findings on nonlinear dynamics, are Goodhart and O'Hara (1995), and Guillaume etal. (1994)

SFI affiliated researchers also played a major role in developing statistical methods to detect nonlinearities in data. Many of these methods use versions of "bootstrapping" type ideas from statistical theory. Examples of this kind of work include Brock, Lakonishok, and LeBaron (1992), LeBaron (1994a,b), and Theller etal. (1992).

Let us be clear what we are talking about here. At an abstract level,
all serious analysis of time series data is inspired by "dynamical systems theory." We are concerned here, with methods that study, for example, nonlinear stochastic dynamical systems where the nonlinearity is so "deep" that it cannot be dealt with by minor modifications of linear techniques such as (i) change of variables after "detrending," (ii) adaptation of linear techniques to conditional variances (e.g. ARCH-type models, c.f. Bollerslev, Engle, and Nelson (1994)), and (iii) stochastic analogues of "Taylor" type expansions where all remaining nonlinearity is tucked into the "remainder" terms and treated as "errors" in econometric models.

INITIAL REACTIONS TO EMPHASIS ON "DEEP" NONLINEARITIES

Before I give a discussion of research findings, let me give a few remarks on the initial reaction to the new methods that were inspired by dynamical systems theory. Eight years ago, I think it is fair to say that there was some controversy over whether application of dynamical systems theory and related nonlinear techniques would add much of value to received methodology in time series econometrics in economics and finance. I believe the main reason for this opposition is the following.

Many economists believe that the strength of "deep" nonlinearities in financial data is small enough relative to the size of non forecastable nonstationarities that reliable detection could not be had with the datasets available. After all, if such predictively useful nonlinearities were present, they would be found and exploited by expert trading houses. Such activity would cause asset prices to move to eliminate any predictive structure in such nonlinearities. Furthermore, nonlinear prediction algorithms were widely available before methods were developed which were based upon dynamical systems theory, so there was no reason to believe that these newer methods would find structure that had not been already found by earlier, perhaps cruder, methods.

I believe that there is rough evidence that opposition to using "fancier" technology including methods based upon dynamical systems theory, neural nets, genetic algorithms, and the like, has fallen faster in areas where better data has become available at a more rapid rate.

For example, "high technology" groups such as Olsen and Associates (cf. Corcella (1995) for a cover story on Olsen) which are quite common in High Frequency financial applications seem less common in Low Frequency financial applications as well as in aggregative macroeconomics applications such as business cycle analysis and analysis of economic growth.

However, while academics initially debated the usefulness of the new methods, at least some practitioners appeared to rapidly adopt and use the new methods. Not only did some researchers associated with the Santa Fe Institute form their own company (The Prediction Company) but also many other like minded companies and groups have emerged in recent years (see, for example, Ridley (1993)).

While the debate on the value added of methods such as neural nets, genetic algorithms, and dynamical systems theoretic time series analysis has not yet died down in academia, it appears that the rate of adoption of the new methods by academics has increased rapidly in the last few years. The recent surveys by Abhyankar, Copeland, and Wong, (1995), LeBaron (1994a), Creedy and Martin (1994), Goodhart and O'Hara (1995), and Guillaume etal. (1994) lists many recent studies in academia that use "complex system techniques" as defined here.

Why did this rather silly (viewed from hindsight from the vantage point of the widespread adoption by sectors that "have to meet the bottom line") opposition (especially in academia) to the use of "high nonlinear" technology occur and what has caused the change in attitude? Manski, in his well known
paper (1993), stresses that when there is not enough data to resolve issues one way or another, divergent beliefs can be held very firmly. When large and widely held investments in established techniques are threatened, opposition can run deep.

But, in finance, the availability of high quality high frequency datasets such as that collected and made widely available by the Olsen Group (cf. Corcella (1995) and Ridley (1993) for popular articles on the Olsen Group) have made it possible to detect usually predictable "nonlinearities" in financial data. In fields such as aggregative macroeconometrics, there is still a widely held attachment to linear methods. But aggregative macroeconometrics as a field does not have access to data of such quality and plentitude as does high frequency finance.

Of course, much of the opposition will melt away against those techniques which prove useful in more conventional practices which have nothing to do with "deep" nonlinearities such as chaos. An example is the use of the BDS test (which originally had its roots in chaos theory) as a specification test for GARCH-type models which have nothing at all to do with complex systems theoretic techniques. See Bollerslev, Engle, and Nelson (1994) for a discussion of the use of the BDS test as a specification test.

Someone once cynically said that scientific advances take place "funeral by funeral." Viz. when the powerful old guard dies off younger and more open minded scientists can fill the vacated niche space. While there may be some truth to this, I believe instead that economic science advances dataset by dataset rather than funeral by funeral because the evidence can be adduced with more precision when more and better data become available.

This is especially true in situations where the goal function of scientific activity is more precise. For example, corporations and government agencies may be subject to a stronger practical performance discipline to understand how the "system actually works" (e.g. better "prediction and control") than academics. Even in that much maligned institution, government, there is strong pressure to understand how the system really works.

In agencies such as private corporations and government departments academic paradigm attachment may not matter as much as delivery of useful results. This may be so because there is an incentive structure operating in these environments that resembles more the profit motive in conventional competition analysis then the incentive structure in academic environments.

I also believe that the SFI meeting of 1988 and subsequent activities by SFI-affiliated researchers played a key role in popularizing new "high technology" ideas in economics and finance. Turn now to a list of stylized facts we wish to explain.

A LIST OF STYLIZED FACTS WE WISH TO EXPLAIN

Before any theorizing is done we should ask the following questions: (i) What are the facts and regularities that we wish to explain? (ii) What are the goals of the explanation? While a standard commercial goal is prediction for the purpose of generating trading profits, we shall take the position here that we are only interested in understanding the stochastic properties of stock returns and trading volume and the forces that account for those properties. This basic scientific understanding is essential to design of intelligent regulatory policy. It also has practical commercial value for the management of risk.

Here is a list of stylized facts. Most of these concern high frequency data, ranging from the tic by tic frequency to the weekly frequency. The facts are taken from the book by Brock, Hsieh, and LeBaron (1991), and articles by Guillaume et al. (1994), Goodhart and O'Hara (1995), LeBaron (1994a, b), and Brock and de Lima (1995). We especially stress the article by
Guillaume et al. (1994) for an excellent list of high frequency facts.

FACT 1: Complexity. Using estimated correlation dimension and lack of predictability as a measure of complexity, financial asset returns are highly complex.

I.e. the estimated correlation dimension is high (and is also unstable across subperiods, in some cases) and there is little evidence of low dimensional deterministic chaos. Evidence against low dimensional chaos includes not only high estimated (and unstable) correlation dimension, but also very little evidence of out of sample predictability using nonlinear prediction methods that have high power against low dimensional deterministic chaos. Here the term "low dimensional deterministic chaos" is used for chaos that is not only "low dimensional" but also "usably" low dimensional in the sense that nonlinear short term prediction can be conducted out of sample.

We doubt if anyone in finance believes that asset returns are wholly a deterministic chaos. The issue is whether there might be a chaos of low enough dimension and "regular" enough lurking in the sea of nonstationarity and noise that marginally effective short term prediction might be possible. Unfortunately the evidence for even this rather limited notion of chaos is weak.

FACT 2: Nonlinearity. There is good evidence consistent with nonlinearity, but effort must be made to avoid confusion of evidence of nonlinearity with evidence of nonstationarity. Much of the evidence of nonlinearity is also consistent with evidence of neglected nonstationarity.

Although evidence for low dimensional deterministic chaos is weak, there is, however, evidence of stochastic nonlinearity. Let "IID" denote "Independently and Identically Distributed." There is extremely strong evidence against IID-linearity and there is weaker evidence against MDS-linearity. Here a stationary stochastic process \( \{ X_t \} \) is said to be IID-linear (MDS-linear) if \( X_t = \sum a_j \epsilon_{t-j} + \sum a_j x_j \), where \( \{ \epsilon_t \} \) is an IID stochastic process (Martingale Difference Sequence, i.e. \( E(\epsilon_t | \epsilon_{t-1}, \epsilon_{t-2}, \ldots) = 0 \), all \( t \)).

While the evidence may be due to nonstationarity rather than nonlinearity, many studies reviewed in, for example, Abhyankar et al. (1995), Hsieh (1991), and Guillaume et al. (1994) attempt to control for nonstationarities by adjusting for known seasonalties from higher to lower frequencies such as daily seasonalties in the bid/ask spread, volatility and volume over the trading day to the January Effect.

Guillaume et al. (1994) introduce a rescaling of time, which they call "theta" time, which shortens periods with little trading activity and magnifies periods with a lot of trading activity. They show that this device is very useful for controlling for nonstationarities. It is also sensible from an economic point of view and plays an important role in the Olsen group's vision of how the markets work--a vision which emphasizes trader heterogeneity at all time scales.

Studies such as Abhyankar et al. (1995), and Hsieh (1991) adjust for other types of nonstationarities by studying the pattern of rejections of nonlinearity by frequencies. At a very rough level of expository approximation one may describe their results as showing that the pattern of rejections of linearity have a "self similar" structure. I.e. the pattern of rejection is similar at all frequencies. This forces some discipline on heterogeneous belief or heterogeneous characteristics theories of traders.

Evidence of nonsymmetry such as skewness allows rejections of many models
of the form $X_t = \sum_{j=1}^{\infty} a_j \varepsilon_{t-j}$ for example $\{\varepsilon_t\}$ Gaussian. Gallant, Rossi, and Tauchen (1993, p. 873), adduce evidence against a Vector AutoRegressive model driven by AutoRegressive Conditionally Heteroskedastic errors, abbreviated as VAR-ARCH in stock price and volume data by showing that the symmetry imposed by such a model conflicts with "the intrinsic nature of the price-volume relationship." In general as Gallant, Rossi, and Tauchen (1993) show for asset returns and volume data, and as Potter (1991) shows for macroeconomic data, nonlinear dynamic impulse response analysis coupled with a bootstrapping type of statistical significance analysis is a good way to adduce evidence against linear models.

FACT 3: Predictability. Evidence of out of sample predictability is weak.

The studies of Diebold and Nason (1990), Meese and Rose (1990) put a damper on the effort to find reliable out of sample predictability in asset returns. They showed that evidence for predictability out of sample beyond that of simple random walk models was nil.

Later studies such as Brock, Lakonishok, and LeBaron (1992), LeBaron (1992) for stock market indices, Antoniewicz (1993) for individual stocks; and LeBaron (1994b) and Guillaume et al. (1994) for foreign exchange showed that evidence for short term out of sample predictability is good provided conditioning was done on the "right" information sets.

If opportunities for such predictability were rare enough over the whole sample, they would be hard to detect in bulk average measures of predictability over the whole sample. Tests such as the BDS test (Brock, Dechert, and Scheinkman 1987)) are useful for exhibiting evidence for such "pockets of predictability" but more refined tools such as bootstrapping economically motivated quantities such as profits from different trading strategies under null models whose development was guided by structural economic modelling (e.g. Brock, Lakonishok, and LeBaron (1992)) are required to locate such pockets and measure the "statistical and economic" significance of such conditional predictability. Blind application of nonlinear high technology is not likely to be successful.

As we said before, there is evidence of short-term predictability provided one conditions on the "right information sets." For example, LeBaron (1992a) has shown, for stock indices and for foreign exchange, that predictability increases when near-past volatility and volume decreases. He has also shown, LeBaron (1994b), that certain technical trading rules have predictive power for foreign exchange. But this predictive power drops when central bank intervention periods are deleted.

Lagged volume has predictive power for near future returns. Recall that volume and volatility measures are contemporaneously correlated for both indices and individual stocks (see Antoniewicz (1992, 1993) for documentation for a large collection of individual stocks). For indices LeBaron (1992a) (LeBaron (1992b)) showed (for several indices and for IBM stock returns) that decreases in lagged volatility (volume) are associated with increases in first order autocorrelation of returns. To put it another way, price reversals tend to follow abnormally high volume.

Conrad et al. (1992) find the following for weekly returns on individual securities: "returns autocovariances are negative only for last period's heavily traded securities; autocovariances in returns are positive if trading declined last week." Campbell, Grossman, and Wang (1993) give a theoretical story for this type of finding and document it for indices and some data on individual firms.

Campbell, Grossman, and Wang (1993) consider the regression
\( r_t = \alpha_0 + (\alpha_1 + \beta_1 V_{t-1}) r_{t-1} + \epsilon_t \)

where \( r, \ V \) denote returns and detrended turnover and find \( \alpha_1 > 0, \beta_1 M < 0 \) for the NYSE/AMEX value weighted index. While they also study some individual firms, Antoniewicz (1993) runs regressions of the form (R) for a large sample of individual firms, for more lags, and for individual firm specific volume as well as market volume.

The theoretical story consists of two classes of mean variance investors, one type has constant risk aversion, the other type has random risk aversion. The constant risk aversion types serve as "liquidity providers" and they must be rewarded for this task. There is no heterogeneity of beliefs. Since the random risk aversion process is persistent, a burst of volume signals not only an innovation to the random risk aversion process but also an increase in expected returns which must revert to the "normal" level. This leads to a volume spike predicting a decrease in first order autocorrelation of future returns. As we said before, Campbell, Grossman, and Wang (1993) find evidence consistent with the theory using data on indices and individual firms.

Antoniewicz (1993) has shown that "spikes in aggregate volume do not influence the serial correlation for large capitalized securities. For smaller securities, episodes of abnormally high firm specific volume can induce a change from negative serial correlation to positive serial correlation in returns. The effects from market volume and firm specific volume on the serial correlation in returns depend on firm size and the sign of the previous day's return." Her findings may be consistent with the presence of different types of "liquidity providers" for large securities than for small securities. The differential presence of derivative securities such as put and call options across different size classes may play a role in the apparently different volume/return dynamics for large capitalized securities.

SIMPLE HETEROGENEOUS BELIEF STORIES TO "EXPLAIN" VOLUME AND AUTOCORRELATION

A very simple story based upon sequences of two period temporary equilibria under heterogeneous beliefs may help point the way towards building models that may help explain some of the Antoniewicz findings for smaller securities. Let one set of traders have belief biases away from the fundamental where the bias process is persistent. The traders which have no bias serve as the "liquidity" providers. Volume increases when there is an exogenous shock to the bias process. Since this exogenous shock is persistent, this generates positive first order serial correlation in returns even though fundamental returns have no serial correlation.

A simple story based upon three period temporary equilibrium adds the new ingredient that expectations must be formed by the "fundamentalists" on how many biased traders will be around next period when they wish to trade. One can tie down the sequence of equilibria by having the "fundamentalists" believe that values will revert to fundamental values three periods from now. This kind of model can induce endogenous changes in the quantity of risk in the system especially when it is coupled with the modelling strategy laid out in Brock (1993a) where the interaction between intensity of choice and the desire to choose strategies used by people most like yourself can create large movements in the strategy mix induced by small differences in the payoff measure to each strategy.

FACT 4: Autocorrelation Functions of Returns, Volatility of Returns, Volume Measures, and Cross Correlations of these measures have similar shapes across different securities and different indices.
Autocorrelation functions of returns are roughly zero at all leads and lags. This is a version of the Efficient Markets Hypothesis. However, at high frequencies, autocorrelations of returns are slightly negative for individual securities and for foreign exchange at small lags. This may be due to bid/ask bounce (Ross (1992)). For stock indices, autocorrelations of returns are slightly positive at small lags. This is, at least, partly due to non-synchronous trading effects (Ross (1992)). I.e. stocks which have most recently traded have adjusted to the latest "news" whereas sluggishly traded stocks have not adjusted. Their slow adjustment leads to a move in the same direction later on. This "non-synchronous trading" effect leads to positive autocorrelation in returns. Yet if one tried to trade on it, the typically wider bid/ask spread and higher liquidity costs of the sluggish securities may wipe out the apparent profits.

Autocorrelation functions of volatility measures are positive at all lags with slow (approximately hyperbolic) decay for stock indices and foreign exchange. The decay is more rapid for individual securities.

This is the famous stylized fact that lies behind the huge ARCH literature surveyed by Bollerslev, Engle, and Nelson (1994). Microeconomic explanations for this stylized fact, which we call the "ARCH" fact, are discussed by Brock and LeBaron (1995).

Autocorrelation functions of volume measures look similar to those of volatility measures. Cross autocorrelations of volatility with volume are contemporaneously large and fall off rapidly at all leads and lags. There is some asymmetry in the fall off. Antoniewicz (1992, 1993) documents many of these facts as well as others for a large collection of individual NYSE/AMEX and NASDAQ stocks. She also surveys many other studies documenting similar facts for indices and individual assets.

SOME THEORETICAL STORIES FOR THE ARCH-FACT

Models of endogenous belief heterogeneity (where belief types are driven by profitability of trading on those beliefs) by Brock and LeBaron (1995) for short lived assets and multiple time scales, and de Fontnouvelle (1995a) for long lived assets and one time scale are shown to produce time series roughly consistent with Fact 4 which includes the ARCH-Fact. The general phenomenon of volatility bursting in complex interconnected systems has been recently treated by Hogg et al. (1995).

FACT 5: Seasonalities and Other Predictable Nonstationarities; Non Predictable Nonstationarities.

Seasonalities include the widening of the bid/ask spread at daily open and daily close, increase in volatility and trading volume at daily open and daily close as well as spillover effects across major international markets as they open and close over the trading day. See Goodhart and O’Hara (1995). The major theories are asymmetric information and differential arrival rates of information to portfolio rebalancing and peak load pricing of trading services. Guillaume et al. (1994) show that use of an appropriate rescaling of chronological time into "economic" time is very useful for removing many seasonalities. They also show how estimates of objects like autocorrelation functions are badly contaminated and how statistical inference is damaged if these seasonalities are not removed.

FACT 6: Lead/lag relationships (Linear and Nonlinear)

Lo and MacKinlay (1990) have documented that large firms in the same
industry tend to lead smaller ones in returns. Conrad et al. (1992) references work documenting similar leading effects for conditional variances. Kleidion and Whaley (1992) reviews evidence that futures markets on stock indices tend to lead cash markets on stock indices. Abhyankar (1994) and Hiemstra and Jones (1994a) have recently used nonlinearity tests based upon U-statistics theory and correlation integrals to document lead-lag relationships over and above those documented by linear methods.

Let us digress here to explain the correlation integral-based tests of Hiemstra and Jones (1994a) and Abhyankar (1994). We do this by expositing a simple special case of the general method.

Let \( \{X_t\}, \{Y_t\} \) be a pair of stationary stochastic processes, let \( \Pr(E) \) denote the "probability of event \( E \)," and let it be desired to test the proposition

\[
\Pr(X_t | Y_{t-1}) = \Pr(X_t), \text{ using data } \{X_t\}, \{Y_t\},
\]

i.e. \( X_t \) is independent of \( Y_{t-1} \). While one can do this by constructing estimators of \( \Pr(X_t, Y_{t-1})/\Pr(Y_{t-1}) \), \( \Pr(X_t) \) and setting up a test of the null hypothesis: \( \Pr(X_t | Y_{t-1}) = \Pr(X_t) \) using these estimators, one can alternatively set up a test based upon

\[
H_0: \frac{\Pr(|X_t - X_s| < \epsilon, |Y_{t-1} - Y_{s-1}| < \epsilon)}{\Pr(|Y_{t-1} - Y_{s-1}| < \epsilon)} = \Pr(|X_t - X_s| < \epsilon).
\]

Now any object of the form \( \Pr(|Z_t - Z_s| < \epsilon) \) can be estimated by the correlation integral estimator,

\[
U_T = (1/T^2) \sum_{t=1}^{T} \sum_{s=1}^{T} \chi_{|Z_t - Z_s| < \epsilon},
\]

using data \( \{Z_t, t=1,2,\ldots,T\} \), where \( \sum \) runs over \( s, t=1,2,\ldots,T \). Here \( 1\{A\} \) is 1 if event \( A \) occurs, zero otherwise.

For the univariate case Brock, Dechert, and Scheinkman (1987) showed that statistics of the form \( U_T \) are U-statistics and tests for IID can be set up using such statistics. Brock, Dechert, Scheinkman, and LeBaron (1991) later showed that such statistics had the same first order asymptotic distribution on estimated residuals as on the true residuals of IID-driven estimated models on data. This made such statistics useful for model specification tests (cf. Bollerslev, Engle, and Nelson (1994)).

Baek and Brock (1992) extended this analysis for the multivariate case, including proofs of invariance theorems on estimated residuals. Hiemstra and Jones (1994) showed how to test \( H_0 \) by replacing each population value in \( H_0 \) by a corresponding correlation integral estimator like \( U_T \), taking the difference and multiplying it by \( T^{1/2} \), and using the statistician's Taylor series approximation method, which statisticians call "the delta method", and working out the null distribution. Hiemstra and Jones (1994) faced the challenge of dealing with a long and complicated formula for the variance because, unlike Baek and Brock (1992), they did not have an IID maintained hypothesis.

Using this test for "nonlinear causality of \( Y \) for \( X \)" where the word "casuality" here is used in the narrow sense of incremental predictive ability of past \( Y \)'s given, above and beyond, that already contained in past \( X \)'s, they found evidence consistent with the proposition that daily stock returns help predict trading volume and trading volume helps predict daily stock returns.
I.e. they found evidence consistent with bidirectional causality. However, Gallant, Rossi, and Tauchen's (1993) impulse response analysis showed that returns help predict trading volume, they did not find that trading volume helped predict prices. Antoniewicz (1993) using an adaptation of the Brock, Lakonishok, LeBaron (1992) bootstrap test found evidence that trading volume did help predict prices. The efficient markets argument would lead one to expect that any test capable of picking up evidence (that presumably would have been erased by traders' actions) that trading volume helps to predict prices would have to be quite subtle.


Gallant, Rossi, and Tauchen (1993) show that a shock to current volatility leads to an immediate increase followed by a long term decline in future trading volume. However, shocks to current volume have little effect on future volatility. They develop general nonlinear impulse response analysis and apply it to finance. Potter (1991) develops nonlinear impulse response analysis and applies it to macroeconomic time series. Both sets of authors find strong evidence for nonlinear effects that would not be found by standard linear impulse response analysis.

de Fontnouvelle (1984) has developed a heterogeneous agent asset pricing model with endogenous evolution of signal purchasing belief types versus passive investor types (who free ride on the signal purchasers as in Brock and LeBaron (1995)) with infinite lived assets, unlike the finitely lived assets in Brock and LeBaron (1995). His model generates time series output for returns, volume, and volatility that is approximately consistent with some of Gallant, Rossi, and Tauchen's findings on impulse responses as well as Fact 3 on conditional predictability and Fact 4 on autocorrelation and cross correlation structure on returns, volume, and volatility. His model is promising enough for estimation by methods such as the Method of Simulated Moments (cf. Altug and Labadie (1994) for a nice exposition) and specification testing.

FACT 8: Market Micro Structure Effects on Index Returns and Returns on Individual Securities.

Ross (1992) discusses construction of different types of stock market indices and surveys the major indices in use around the world. He discusses the advantage of using value weighted (usually capitalization weighted, i.e., weighted by the value of the capitalization of the company's common stock) indices over other types. The Standard and Poors 500 is a value weighted index. He points out that smaller stocks trade in a less liquid market and trade less frequently than larger stocks. This is called the non trading effect and it induces positive serial correlation in daily index returns. This effect may loom larger in more broadly based indices. An offsetting effect is the "bid-ask" effect which induces negative serial correlation in the returns for individual stocks. This effect looms larger for smaller time intervals and for less frequently traded stocks. Ross points out that in indices where large liquid stocks (which tend to have small bid/ask spreads) dominate the index, the nontrading effect will generally dominate.

An extensive discussion of the impact of different market institutions upon the stochastic structure of returns, volatility, and volume is given in the surveys of Guillaume et al. (1994) and Goodhart and O'Hara (1995).

FACT 9: Evidence of Departures from "Standard" Models
There is evidence from studies of technical trading rules that popular
volatility models such as GARCH and EGARCH over predict volatility following
buy signals, under predict returns following buy signals and over predict
returns following sell signals. Indeed there is some evidence that returns
following sell signals are negative even though volatility following sell
signals is substantial. Evidence for the Dow Jones Index daily data was given
in Brock, Lakonishok, and LeBaron (1992). Evidence for individual firms was

Trading rule specification tests may be one promising way of getting
around the issue of possible "economic" irrelevance of formal statistical
testing. Let us explain. First, any statistical model is an approximation to
the true joint distribution. Hence any approximation will be formally
rejected if the sample size is large enough. The financial analyst needs to
know "how wrong is it" and "in what direction should repair be made." Second,
if one does diagnostic analysis on residuals of a particular specification and
"accepts" the specification if one fails to reject at the 5% level, then this
strategy may unduly throw the burden of proof against alternatives. It is
easy to create examples where the null model is wrongly accepted by this
strategy. Third, trading rule analysis allows one to calculate the potential
profitability value of rejections.

The technique of bootstrapping was applied to measure the statistical and
economic "significance" of the results. We digress here to give a brief
explanation of bootstrapping. We believe that bootstrapping of economically
motivated quantities such as trading rule (both technical and fundamental)
profits, returns, and volatilities represents the emerging scientific trend of
the future in designing specification tests and goodness of fit tests of
predictive models. This methodology will tend to replace traditional
"analytics" in statistics as computers become more powerful and bootstrapping
theory is worked out for more general environments.

BOOTSTRAPPING: A BRIEF EXPLANATION

Bootstrapping is a technique of resampling from IID driven null models to
approximate the null distribution of complicated statistics under such models.
Surrogate data (cf. Theiler et al. (1992)) is an adaptation that preserves the
spectrum or autocorrelation function. Surrogate data techniques have gained
some popularity in testing versions of the linearity hypothesis, but we shall
concentrate on explanation of bootstrapping here.

We shall follow statistics jargon and call the parametric class of
probabilistic data generating mechanisms whose "goodness of fit" we wish to
test on our dataset under scrutiny, the "null" model. Let \( \{y_t, I_t, t=1,2,\ldots, T\} \)
be a T-sample of data. Let bold case letters denote random
variables and/or vectors. Consider the following null model

\[ y_t = H(I_t, e_{t+1}; a), \]

where \( \{e_t\} \) is Independently and Identically Distributed with distribution
function \( F \), i.e., IIDF, where \( F \) is the cumulative distribution function,
\( F(e) = \text{Prob}(e_t < e) \). Here \( a \) is a vector of parameters to be estimated. Estimate
the null model to obtain the estimator \( \hat{a} \). Given \( \hat{a} \), insert the data into the
null model and invert \( H \) to find \( \{\hat{e}_{t+1}\} \). Place mass \( 1/T \) at each \( \hat{e}_{t+1} \)
to create the empirical distribution \( \hat{F}_T \). Suppose we want to calculate the null
distribution of some statistic \( S_T(y_t; F) \), \( Z_T = \{y_1, I_1, \ldots, y_T, I_T\} \). Consider set \( A \)
and the quantity
\[ \Pr\{S_T(Z^b_T; \hat{F}_T) \in A\} \approx (1/B)\sum_{b=1}^{B} \{S_T(Z^b_T) \in A\}, \]
where each \( Z^b_T, b=1,2,\ldots,B \) is a resampling of \( \hat{F}_T \) generated by independent and identically distributed (IID) draws with replacement from \( \hat{F}_T \).

Bootstrap is a useful method if \( \Pr\{S_T(Z^b_T; \hat{F}_T) \in A\} \) is a good approximation to \( \Pr\{S_T(Z^b_T; F_T) \in A\} \) and if \( \Pr\{S_T(Z^b_T; F_T) \in A\} \) is a good approximation to \( \Pr\{S_T(Z^b_T; F) \in A\} \). Glivenko-Cantelli and Edgeworth expansion theorems give sufficient conditions for \( F_T \) to be a good approximation to \( F \). While it is technical one can show that for certain statistics, called "pivotal" statistics, bootstrap can give better approximations than conventional asymptotic expansions. In any event, even for cases where such "incremental accuracy" theorems are not available, bootstrap allows one to use the computer to approximate the distributions for intractible statistics. Standard results on consistent estimation are available in the formal statistics literature which give conditions for \( \hat{a} \rightarrow a, T \rightarrow \infty, \frac{1}{\sqrt{T}}(\hat{a} - a) \rightarrow X \) where \( X \) is some random variable and "\( \rightarrow \)" denotes convergence in distribution. These results can be used to evaluate the quality of convergence of \( F_T \) to \( F \).


I go into some detail on the theory of bootstrapping here because it is sometimes carelessly used in the literature without verifications of the sufficient conditions required for bootstrap to give useful approximations to the null distributions of interest. Results of any bootstrap based method or analogue should be viewed with caution unless the sufficient conditions for convergence have been formally verified.

If formal verification is not possible then computer experiments can be done where one can estimate the null model on a sample of length \( T \), generate artificial datasets from the estimated null model of length \( N \), unleash "fake bootstrapping econometricians" on each length \( N \) artificial dataset, bootstrap the null distribution of statistic \( S \) for this \( N \)-dataset and simply observe whether the \( N \)-null distributions appear to converge as \( N \) increases. Then observe whether this convergence appears to have taken place for \( N=T \).

While Brock, Lakonishok, and LeBaron (1992) discuss convergence quality experiments, one should go beyond their efforts, now that faster computers are available.

**FACT 10:** Evidence consistent with Long Term Dependence and Long Memory Appears Strong in Variance but Weak in Mean for Returns. Care must be taken not to confuse evidence for long memory with evidence for stochastic switching models and other "nonstationarities."

Brock and de Lima (1995) survey a lot of work on testing for long memory in mean and variance, much of which was inspired by Mandelbrot's early work. A rough consensus seems to have emerged that evidence for long memory in mean is weak, at least for individual securities.

However, on the surface, evidence for long memory in variance is very strong. But de Lima has done Monte Carlo experiments (reported in Brock and de Lima (1995)) with a class of Markov switching models which were fitted to financial data by Hamilton and Susmel. It was found that the standard tests
for long memory gave too many false positives for the Hamilton/Susmel models, especially when they were fitted to daily data. Brock and LeBaron (1995) show by example how simple Markov switching models which have nothing to do with long memory can give slow decay in autocorrelation functions in squared returns.

This suggests that extreme care be taken when interpreting evidence for long memory in variance. Apparently this caveat may apply even to vivid evidence such as hyperbolic decay of autocorrelations of objects such as squared returns. Since one common vision of financial markets is that they are infected with nonforecastable nonstationarity with, perhaps, different stochastic dynamical systems operating over time spans of random lengths, buffeted by outside shocks, but yet this might be a short memory process overall, therefore one must proceed with caution in interpreting results from tests for long memory. This same caveat also applies to tests for other features such as nonlinearity and chaos.

Interestingly enough de Lima found that estimates of the Hurst exponent itself were fairly robust. Turn now to a theoretical argument on the applicability of fractional brownian motion models to the stock market.

There has been much recent interest in fractal Brownian Motion models of the stock market. Hodges (1995) has generalized an older argument of Mandelbrot to show that a fractal Brownian motion market with a Hurst exponent that departs very far from 1/2 (the Hurst exponent for the standard Brownian motion) is extremely vulnerable to risk free arbitrage. If H>1/2 there is positive (negative) serial correlation (H<1/2), hence Hodges is able to exploit this serial correlation (and the self similarity of the fractal Brownian motion) to calculate how many transactions are needed to generate large profits per unit of risk borne as a function of the Hurst exponent and transactions costs. He shows that for Hurst values reported in some recent literature, the potential for easily generated, essentially risk free profits, is just too large to be compatible with common sense given the actual costs of transacting.

FACT 11: "Excess" Volatility, Fat Tails, "Inexplicable" Abrupt Changes.

Financial time series generally exhibit excess kurtosis, a fact that has been noticed at least since Mandelbrot's work in the sixties which also raised the issue of nonexistence of the variance. Brock and de Lima (1995) contains a survey of studies of estimation of maximal moment exponents. It appears that the evidence is consistent with existence of the variance but not moments of order four and beyond. However, there is debate over the econometric methods. de Lima's studies which are surveyed in Brock and de Lima (1995) show that many tests for nonlinearity are seriously biased because they require existence of too many moments to be consistent with heavy tailed financial data. This causes some tests to give "false positives" for nonlinearity.

Many studies measure excess volatility relative to the standard set by the discounted present value model

\[ D_t = E_t \{ p_{t+1} + y_{t+1} \} \]

where \( E_t \) denotes expectation conditioned on past prices \( \{ p \} \) and past earnings, \( \{ y \} \). More recent studies attempt to correct for possible temporal dependence in the discount factor "R" but the spirit is the same.

There has also been concern expressed in the media, government, and policy communities about "excess" volatility in stock prices. While much of this concern comes from the October 1987, 1989 crashes, some of the academic
concern follows from the realization that the solution class of (PDV) includes solutions of the form \( p_t = p_0 + b_t \), for any process \( \{b_t\} \), that satisfies

\[ Rb_t = E_t b_{t+1} \]

where \( p_t \) is the "fundamental" solution which is the expectation of the capitalized y's,

\[ (FS) \quad p_t = E_t \{ y_{t+1}/R + y_{t+2}/R^2 + \ldots \}. \]

This has motivated many tests for "excess volatility" that test for deviations of actual prices from the solution predicted by (FS). Econometric implementation of such testing is nontrivial and has given rise to a large literature.

While we do not have space to discuss this literature here, we point out how the models of Brock and LeBaron (1995) can shed light on the issue of excess volatility. Let \( \Delta p_t = p_{t+1} - p_t \) denote the change in price of the risky asset over periods \( t, t+1 \). Define price volatility \( V \) to be the variance of \( \Delta p_t \). Then Brock and LeBaron show that price volatility increases as the market signal precision increases. Here market signal precision is a measure of the average precision of signals about the future earnings of the risky asset. Thus when price is doing a good job in tracking future earnings (high market precision), we have high price volatility. This finding illustrates a beneficial aspect of price volatility.

This shows that price volatility is not something to necessarily be concerned with unless one can identify that such volatility is coming from a harmful source rather than what the market is supposed to be doing when it is working well. Furthermore, this kind of model can easily be generalized to allow signals on a piece of earnings to become less precise the further away in the future the piece is, to allow individuals to purchase signals of higher precision for higher costs, and to allow traders to balance such prediction costs against profits generated by better information. This leads to a dynamic equilibrium of heterogeneously informed traders where the heterogeneity is endogenous.

Such models have been built by Brock and LeBaron (1995) and de Fontnouvelle (1994), (1998). These kinds of models generate data that are roughly consistent with the autocorrelation and cross correlation Fact 4. de Fontnouvelle's (1994) model also is consistent with LeBaron's version of conditional predictability in Fact 3. Brock and LeBaron (1995) show how to use statistical models of interacting systems combined with statistical models of discrete choice over trading strategies to produce sudden jumps in asset market quantities such as returns, volatility, and volume in response to tiny changes in the environment.

Empirical work using such interacting systems models is just beginning. Altug and Labadie (1994) contains an excellent discussion of structural modelling in finance and empirical estimation of more conventional models.

Before continuing, I wish to stress that many models have been built to attempt to understand the Facts listed above. Many of these are reviewed in the excellent survey by Goodhart and O'Hara (1995).


There have been scaling laws reported in finance. The early work of Mandelbrot, which stressed the apparent self similar structure across frequencies of financial time series, has stimulated a large literature on scaling laws of the form \( \ln[Pr(X > x)] = A + B \ln(x) \), often called "Pareto-tall" scaling. The stylized fact is that estimates of \( B \) are similar across
frequencies, across time, and replacement of the event \( \{ X > x \} \), by the event \( \{ X < -x \} \). This is illustrated in Mandelbrot's famous cotton plots. Lorentz and Phillips (1993) have developed formal statistical inference theory for these objects and have applied this theory to several financial datasets.

Scaling laws must be handled with care because many different stochastic processes with very different structure of time dependence can generate the same scaling law. For example, think of limit theorems as \( N \to \infty \) for appropriately scaled sums of \( N \) random variables (the standard Central Limit Theorem with normal limiting distribution) and appropriately scaled maxima of \( N \) random variables (where the limiting distribution is one of the extreme value distributions) stated in the form of "scaling laws."

LeBaron has done computer experiments that show that GARCH processes fit to financial data can generate scaling "laws" of the form \( \ln(\Pr\{X > x\}) = A + B \ln(x) \) that look very good to the human eyeball.

As early as 1940, Feller (1966, see p. 52 and his reference to his 1940 article) had pointed out that the scaling plots used to demonstrate the "universality" of the "logistic law of growth" were infected with statistical problems and may be useless for any kind of forecasting work. He pointed out that logistic extrapolations are unreliable and that other distributions such as the cauchy, normal, can be fitted to the "same material with the same or better goodness of fit," (Feller (1966, p. 53, italics are his)).

A problem with scaling "laws" for financial analysis is this. In financial analysis we are interested in conditional objects such as conditional distributions, whereas scaling laws are unconditional objects. For example, nonlinear impulse response analysis and estimation of conditional objects as in, for example, Gallant, Rossi, and Tauchen (1993) and Potter (1991) is likely to be of more use for financial work, than simply presenting a scaling relationship. Of course, scaling "laws" are useful for disciplining the activity of model building and for stimulating the search for explanations.


This completes the list of facts we wish to concentrate the reader's attention upon. There are more in Guillaume etal. (1994), Goodhart and O'Hara (1995) and Brock and de Lima (1995) as well as their references. In Section two, we shall briefly indicate how dynamic adaptive belief model building may contribute to understanding the economic mechanisms that give rise to these facts.
2. SOME BRIEF EXAMPLES OF ADAPTIVE BELIEF MODELLING

2.1 A BASIC MEAN VARIANCE FRAMEWORK

It is useful to have a basic model of adaptive belief systems in order to organize discussion and to shed light upon the facts listed in Section 1. Although we shall work in a discrete time context with step size \( s = 1 \), one should remember that the ultimate objective is to understand dynamical phenomena for different step sizes. We should also be aware of different behavior that can arise in synchronous adjustment setups versus asynchronous adjustment setups. We shall work with synchronous adjustment here.

Brock and LeBaron (1995) added adaptive dynamics to received asymmetric information models in order to study the dynamics of the market’s precision at tracking the underlying earnings stream that it was pricing. They showed that this kind of model produces dynamics that are roughly consistent with Facts 2, 4, 11. De Fontnouvelle’s models (1994, 1995) produce dynamics that are consistent with Facts 2, 4, 11 as well as Fact 3. Furthermore, because he has long lived assets, he did not have to introduce two scales of time to generate dynamics consistent with these facts, unlike Brock and LeBaron’s (1995) short lived assets model.

However, work by Cragg, Malkiel, Varian, and others discussed in Brock and LeBaron (1995) has stressed that measures of diversity of earnings forecasts are associated with risk in the financial markets. Risk, in turn, is associated with volatility, while volume is associated with diversity. We shall stress diversity of forecasts in the development we sketch here in order to show how complex systems methods can contribute to model building in this area. This complements the use of complex system techniques by Brock, de Fontnouvelle, and LeBaron.

Let \( p_t, y_t \) denote the price (ex dividend) and dividend of an asset at time \( t \). We have

\[
W_{t+1} = R W_t + (p_{t+1} + y_{t+1} - R p_t) z_t,
\]

for the dynamics of wealth where bold face type denotes random variables at date \( t \) and \( z_t \) denotes number of shares of the asset purchased at date \( t \). In (2.1.1) we assume that there is a risk free asset available with gross return \( R \) as well as the risky asset. Now let \( E_t, V_t \) denote conditional expectation and conditional variance based on a publicly available information set such as past prices and dividends and let \( E^{ht}, V^{ht} \) denote conditional expectation and variance of investor type \( h \). We shall sometimes call these conditional objects the “beliefs of \( h \)” Assume each investor type is a myopic mean variance maximizer so that demand for shares, \( z^{ht} \) solves

\[
\text{Max}\{E^{ht}(W_{t+1} - (a/2)V^{ht}(W_{t+1})\}, \text{ i.e.,}
\]

\[
z^{ht} = E^{ht}(p_{t+1} + y_{t+1} - R p_t)/a V^{ht}(p_{t+1} + y_{t+1}).
\]

Let \( z^{st}, n^{ht} \) denote the supply of shares per investor at date \( t \) and the fraction of investors of type \( h \) at date \( t \). Then equilibrium of supply and demand implies,

\[
\sum_{ht} n^{ht} E^{ht}(p_{t+1} + y_{t+1} - R p_t)/a V^{ht}(p_{t+1} + y_{t+1}) = z^{st}.
\]
Hence, if there is only one type h, equilibration of supply and demand yields the pricing equation

\[(2.1.5) \quad R_t^p = E_t^h(p_{t+1} + y_{t+1}^h - aV_{t+1}^h(p_{t+1} + y_{t+1}^h)z_{st}^h)\]

Given a precise sequence of information sets \(\{F_t\}\) (e.g. \(\{F_t\}\) is a filtration) we may use (2.1.5) to define a notion of fundamental solution by letting \(E_t^h, V_{t+1}^h\) denote conditional mean and variance upon \(F_t\).

If one makes specialized assumptions such as \(\{y_t^h\}\) IID or AutoRegressive of order 1 with IID innovations (i.e. \(y_{t+1}^h = a_1^h + a_2^h y_t^h + e_{t+1}^h\), \(\{e_t\}\) IID with mean zero and finite variance), and assumes that \(z_{st}^h = z_s^h\) independent of \(t\), then an explicit solution of (2.1.5) may be given. As an especially tractible example, we examine the fundamental solution for the special case \(z_{st}^h = 0\), for all \(t\). Then we have

\[(2.1.5') \quad R_t^p = E_t(p_{t+1} + y_{t+1}^h)\mid F_t\}.

Note that (2.1.5) typically has infinitely many solutions but (for the "standard" case, \(R > 1\)) only one satisfies the "no bubbles" condition

\[(2.1.6) \lim (E_t^h p_t^h / R_t^h) = 0,

where the limit is taken as \(t \to \infty\).

Proceed now to the general case of (2.1.5). Suppose that a "no bubbles" solution, \(p_t^h\), exists for \(\{F_t\}\) for (2.1.5). Replace \(y_{t+1}^h\) by \(y_{t+1}^h = V_{t+1}^h(p_{t+1} + y_{t+1}^h)z_{st}^h\) in (2.1.5). This move allows us to interpret \(y_{t+1}^h\) as a risk adjusted dividend.

Boilerslve, Engle, and Nelson (1994) review Nelson's work in continuous time settings which shows that conditional variances are much easier to estimate (especially from high frequency data) than conditional means. Motivated by Nelson's work and analytic tractibility, we shall assume homogeneous conditional variances, i.e., \(V_{t+1}^h(p_{t+1} + y_{t+1}^h) = V_t(p_{t+1} + y_{t+1}^h)\), all \(h, t\). Hence \(y_{t+1}^h = V_t(p_{t+1} + y_{t+1}^h)z_{st}^h\) for all \(h, t\).

In order to proceed further, we must make some assumptions. We shall list the assumptions here and assume:

**Assumptions A.1-A.3:**
1. \(V_{t+1}^h(p_{t+1} + y_{t+1}^h) = V_t(p_{t+1} + y_{t+1}^h)\), for all \(h, t\);
2. \(E_t y_{t+1}^h = E^t y_{t+1}^h\), all \(h, t\) where \(E_t y_{t+1}^h\) is the conditional expectation of \(y_{t+1}^h\) given past \(y\)'s and past \(p\)'s (e.g. if \(y_{t+1}^h = a_1^h + a_2^h y_t^h + e_{t+1}^h\), \(\{e_t\}\) IID with mean zero and finite variance, then \(E_t y_{t+1}^h = a_1^h + a_2^h y_t^h\));
3. All beliefs \(E_t p_{t+1}^h\) are of the form \(E_t p_{t+1}^h = E_t^h p_{t+1}^h + f_h(x_{t-1}, x_{t-2}, \ldots)\), all \(t, h\).

Notice Assumption A.2 amounts to homogeneity of beliefs on the one-step-ahead conditional mean of earnings and Assumption A.3 restricts beliefs, \(E_t^h p_{t+1}^h\) to time stationary functions of deviations from a commonly
shared view of the fundamental. Notice that one can simply assume that all beliefs considered are of the form,

\[ E_{ht}(p_{t+1} + y_{t+1}) = E_t p_{t+1} + E_t y_{t+1} + h(x_{t-1}, x_{t-2}, \ldots), \text{ all } t,h; \]

and all of the analysis below is still valid. It seemed clearer to separate the assumptions as we did above. In any case, Assumptions A.1–A.3 will allow us to derive a deterministic dynamical system for the equilibrium path of the system. This will be shown below.

Since we assume homogeneous conditional variances, we may use Assumptions A.2, and A.3 to write (putting \( f_{ht} = f_h(x_{t-1}, x_{t-2}, \ldots) \))

\[(2.1.7) \quad E_{ht}(p_{t+1} + y_{h,t+1}) = E(p_{t+1} + y_{t+1} | F_t) + f_{ht}.\]

Now use (2.1.4) to equate supply and demand and obtain the basic equilibrium equation in deviations form,

\[(2.1.8) \quad Rx_t = D_{ht} f_{ht},\]

where \( x_t = p_t - p_t^\ast. \)

At this stage we shall be very general about the beliefs \( f_{ht} \) which we have expressed in the form of deviations from the fundamental solution \( p_t^\ast \).

Notice that we may deal with a large class of \( y_t \) processes, even nonstationary ones, by working in the space of deviations from the fundamental.

This device allows us to dramatically widen the applicability of stationary dynamical systems analysis to financial modelling. Before we continue we wish to remind the reader that general financial models stress covariances among assets in large markets.

In these more general models (e.g. the Capital Asset Pricing Model (CAPM) of Sharpe, Lintner, Mossin, Black, et al. and the Arbitrage Pricing Model (APT) of Ross) only systematic risk (risk that cannot be diversified away) gets priced. See Altug and Labadie (1994) for a nice general exposition that gives a unified treatment of asset pricing theory and associated empirical methods.

In CAPM/APT models discounting for systematic risk gets built into the discount rate rather than into an adjustment to earnings. Notice that the adjustment to earnings depends on shares per trader, so more traders per share will lower this adjustment. Since we wish to stress the addition of belief dynamics, we shall work with the simpler setup given here. This is enough to suggest what a general development might look like.

Later on we shall make the basic working assumption that we can express the beliefs \( f_{ht} \) as functions of \( (x_{t-1}, x_{t-2}, \ldots, x_{t-L}) \) and parameters. The parameters shall represent biases and coefficients on the lags \( x_{t-1} \) for special cases such as linear cases. We shall examine two types of systems: (i) Systems with a small number of types; (ii) Systems with a large number of types, where the large system limit is taken. But, before we can do any of this kind of analysis we must determine how the \( n_{ht}, h=1,2,\ldots,H \) are formed.

In order to determine the fraction of each type of belief we introduce a discrete choice model of belief types as in Brock (1993a) and Brock and LeBaron (1995). Let
(2.1.9) \[ U_{ht} = U_{ht}^{*} + \mu e_{ht}, \]

where \( U_{ht} \) is a deterministic index of utility generated by belief \( h \) (e.g. a distributed lag of past trading profits on that belief system) and \( \{ e_{ht} \} \) is a collection of random variables which are mutually and serially independent extreme value distributed. For example, \( U_{ht}^{*} \) could be a distributed lag measure of past profits, past risk-adjusted profits, or be related to wealth in (2.1.1). Here \( \mu \) is a parameter to scale the amount of uncertainty in choice. The parameter \( \beta = 1/\mu \) is called the intensity of choice.

de Fontnouvelle (1994) builds on the discrete choice literature in econometrics to give a nice motivation for the presence of the \( \{ e_{ht} \} \) in (2.1.9). Four possible sources for the uncertainty in choice represented by the \( \{ e_{ht} \} \) are: (1) Nonobservable characteristics (these are characteristics of the agents that the modeller cannot observe), (ii) measurement errors (agents make errors in constructing a "fitness" measure for each belief system), (iii) model approximation error (the functions used in the model are wrong, even if the characteristics of the agents are observed by the modeller), (iv) agents do not have the computational abilities assumed by the modeller (i.e. agents are boundedly rational).

Some of these reasons for uncertainty in choice motivate model building that endogenizes \( \beta \) rather than assuming that it is fixed. Brock (1993a) discussed a motivation for it by trading off the cost of making more precise choices against the gain from more precise choice. More will be said about this below.

However, one could imagine that (i) there is learning by doing in choice (the longer an agent has been doing this kind of choosing, the higher \( \beta \) becomes), (ii) some agents have a higher "choice IQ" (choice IQ is a characteristic which varies across agents), (iii) the longer the agent has been choosing strategies in similar settings the better it gets simply by the accumulation of "habits" (this is similar to learning by doing but emphasizes passage of time, rather than passage of volume of experience), (iv) technological change in choice machinery is occurring (faster computers and interlinked databases are making estimation of relevant quantities for more accurate choice possible).

De Fontnouvelle (1994) argues, that in the context of financial modelling, one could imagine that the sources of the \( \{ e_{ht} \} \) could consist of many small nearly independent errors so the Central Limit Theorem would apply. Hence the \( \{ e_{ht} \} \) would be approximately normally distributed in such a situation. We use the extreme value distribution for tractibility, but many of the insights we draw are valid for more general distributions. The extreme value assumption leads to discrete choice "Gibbs" probabilities of the form,

(2.1.10) \[ n_{ht} \propto \text{Prob} \{ \text{choose } h \text{ at date } t \} = \exp(\beta U_{ht})/Z_{t}, \]

(2.1.11) \[ Z_{t} = \sum \exp(\beta U_{jt}), \]

where \( \beta \) is called the intensity of choice. See the book by Manski and McFadden (1981) for a general discussion of discrete choice modelling.

In order to form the "fitness measure" \( U_{ht} \) we need a measure of profits generated by belief system \( f_{ht} \). Let the last type have \( f_{Ht} = 0 \), i.e., the last
type is a "fundamentalist." Measure profits relative to the fundamentalists and leave out risk adjustment for simplicity. We shall say more about risk adjustment below.

The conditional expectation of realized (excess) profits at date \( t+1 \) based upon past prices and earnings available at date \( t \) is given by

\[
\pi_{h,t+1} = E_t \left\{ (E_{ht}(p_{t+1} + y_{t+1}) - R_{t}) (p_{t+1} + y_{t+1} - R_{t}) / aV_{ht}(p_{t+1} + y_{t+1}) \right\}.
\]

We shall use \( \pi_{h,t+1} \) as the fitness measure for \( h \). This is a very idealized modelling of "learning" because it is not clear how agent \( h \) could ever "experience" such a "fitness" measure. We work with this measure of fitness here because it is analytically tractable and leave treatment of more realistic measures to future work. In future work one could introduce a "slow" scale of time for belief change and a "fast" scale of time for trading on beliefs. Then one could form sample analogs of \( E_t(. \) and use these to approximate the population counterparts. Or one could imagine an ergodic stationary setting where each information set is visited infinitely often so an analog estimator could be constructed for \( E_t(.) \). Let us continue with the current development.

Assume homogeneous beliefs on conditional variances, abuse notation by continuing to let \( \pi_{h,t+1} \) denote profit differences measured relative to profits of type \( H \) we obtain

\[
\pi_{h,t+1} = E_t \left\{ f_{ht}(p_{t+1} + y_{t+1} - R_{t}) / aV_{ht}(p_{t+1} + y_{t+1}) \right\}.
\]

Now put \( \sigma_t^2 = V_{ht}(p_{t+1} + y_{t+1}) \) since we have assumed that beliefs are homogeneous on conditional variances. Note that

\[
E_t \{ f_{ht}(p_{t+1} + y_{t+1} - R_{t}) \} = f_{ht} E_t (x_{t+1} - R_{t}) = f_{ht} E_t R_{t+1} = f_{ht} E_t \rho_{t+1}
\]

and

\[
(2.1.15) \quad n_{ht} = \exp(\beta \pi_{h,t-1}) / Z_{t-1} = \exp(\beta / a \sigma_t^2) f_{ht} \pi_{h,t-1} / Z_{t-1}.
\]

Here \( \rho_{t+1} = E_t x_{t+1} - R_{t} \) is just the conditional expectation of excess returns per share, conditioned upon past prices and past earnings.

We shall argue that, Assumptions A.1-A.3 imply that \( x_{t+1} \) is a deterministic function of \( (x_t, x_{t-1}, \ldots) \) so that \( E_t x_{t+1} = x_{t+1} \). Start the equilibrium equation (2.1.8) off at \( t=1 \) using (2.1.15) with all \( x \)'s with dates older than date 1 given by history. Then (2.1.8) implies that \( x_1 \) is given as a stationary function of \( (x_0, x_{-1}, \ldots) \). Hence it is deterministic at time \( t=0 \), in particular \( E_0 x_1 = x_1 \). Continue the argument forward in this manner to conclude that for each \( t \), \( x_{t+1} \) can be written as a deterministic function of \( (x_t, x_{t-1}, \ldots) \). Equations (2.1.8) and (2.1.15) together give us an equilibrium time stationary deterministic dynamical system. We shall analyze some special cases below. Before we do that analysis we wish to make one remark.

In many applications of this kind of modelling, one can put \( U_{h,t+1} \) equal
to a distributed lag of the profit measures, $\pi_{h,t+1}$ and give the system more "memory." In many cases, the more memory the system has, the better it does at eliminating belief systems that deviate away from the fundamental. This can be seen by doing steady state analysis. Also if one writes down continuous time versions of our type of system, one will see a close relationship to replicator dynamics which are widely used in evolutionary game theory. A drawback of more memory (slower decay of past profit measures in the distributed lag) is sluggish adaptation to "regime changes."

2.2. SPECIAL CASES

Consider a special case where all beliefs on conditional variance are the same, are constant throughout time, and outside share supply is zero at all times. Recall that $p_t$ denotes the fundamental and let $x_t = \bar{p}_t - p_t$, $\eta = \beta / a \sigma^2$. Here $\bar{p}_t$ denotes equilibrium price. Consider expectations that are deterministic deviations from the fundamental. Then calculating profits generated by each belief system and using the discrete choice model to determine $(\pi_{ht})$ as above, equilibrium is given by

$$\begin{align*}
(2.2.1) \quad & R_x_t = D_{ht} f_{ht} \cdot n_{ht} = \exp(\eta p_{t-1} f_{ht,t-2}) / Z_{t-1} \cdot \rho_{t-1} = x_{t-1} - R_x_{t-2}.
\end{align*}$$

Rewrite (2.2.1) as

$$\begin{align*}
(2.2.2) \quad & R_x_t = (\Sigma_{ht} \exp(\eta p_{t-1} f_{ht,t-2}) / H) / (\Sigma \exp(\eta p_{t-1} f_{ht,t-2}) / H).
\end{align*}$$

This is a ratio of two "plug in" or analog estimators for the obvious moments on the R.H.S. of (2.2.2). Hence, once we specify distributions for the cross sectional variation in $(f)$ we can take the Large Type Limit as $H \rightarrow \infty$.

Let us be clear about the type of limits we are taking here. In order to get to (2.2.1) the number of traders has been taken to infinity, while holding the number of types $H$ fixed, to get the discrete choice probabilities appearing on the R.H.S. of (2.2.1). After taking the number of traders to infinity first, we take the number of types to infinity second to get what we shall call the "Large Type Limit" (LTL). Before going on, let us say a few words about the notion of LTL.

One might think of the device of LTL as a way to get some analytic results for computer simulation experiments on complex adaptive systems such as those conducted at SFI. The idea is to ask what the typical behavior might be if a large number of types were drawn from a fixed type distribution.

Of course the answers are going to depend on how the parameters of the dynamics of each experiment are drawn from the space of possible experiments, as well as on the experimental design. Since financial markets typically have a large number of players and since equilibrium prices are a weighted average of demands it is not unreasonable to imagine the law of large numbers would operate to remove randomness at the individual level and leave only the bulk characteristics such as mean and variance of the type diversity distribution to effect equilibrium.

The notion of LTL goes even further, however. It recognizes that, at each point in time, financial equilibrium conditions such as (2.2.2) for example, can be typically be written in the "moment estimator" form even after the number of traders have been taken to infinity. This suggests that if type diversity is itself drawn from a type distribution, then large diversity systems should behave like the probability limit of (2.2.2) as $H \rightarrow \infty$. Hence if $H$ is large the $H \rightarrow \infty$ limit should be a good approximation.
The research strategy carried out in Brock and Hommes (1995a,b) was to examine systems with a small number of types and try to catalogue them. The LTL approach is to analyze systems with a large number of types and approximate them with the LTL limit. This gives a pathway to analytic results for two polar classes of systems. We illustrate the usefulness of the LTL device with a couple of calculations.

**LARGE TYPE LIMIT CALCULATIONS**

**EXAMPLE 1:** Let \( \{f_{ht}\} \) be given by \( \{b_{ht}\} \) where \( b_{ht} \) is IID with mean zero and finite variance across time and "space." Then the LTL is given by

\[
(2.2.3) \quad R_x t = \mathbb{E}_{ht} \left[ \frac{\exp(\eta_{t-1} f_{ht}, t-2)}{\exp(\eta_{t-1} f_{ht}, t-2)} \right] = \mathbb{E}_{ht} b_{ht} = 0.
\]

In an LTL with pure bias one might expect \( \mathbb{E}_{ht} b_{ht} = 0 \) since for each type that's biased up, one might expect a contrarian type that's biased down. In systems with a large number of such types it might be natural to expect an "averaging" out of the biases towards zero. Equation (2.2.3) says that \( x_t = 0, t=1,2, \ldots \) for such systems. I.e., the market sets the price of the asset at its fundamental value. Of course if \( \mathbb{E}_{ht} b_{ht} \neq 0 \) not zero then \( R_x t = \mu_b \) for all \( t \).

As an aside, note how the market can protect a biased trader from his own folly if he is part of a group of traders whose biases are "balanced" in the sense that they average out to zero over the set of types. Centralized market institutions can make it difficult for unbiased traders to prey on a set of biased traders provided they remain "balanced" at zero. Of course, in a pit trading situation, unbiased traders could learn which types are biased and simply take the opposite side of the trade. This is an example where a centralized trading institution like the New York Stock Exchange could "protect" biased traders, whereas in a pit trading institution, they could be eliminated.

Consider a modification of Example 1 where

\[
(2.2.4) \quad f_{ht} = b_{ht} + \sum_{i=1}^{L} x_{ht-i+1} + \cdots + x_{ht-L} + w_{ht-L} x_{ht-L},
\]

where \( \{b_{ht}, w_{ht}, x_{ht-1}, \ldots, w_{ht-L}\} \) is drawn multivariate IID over \( h, t \).

I.e. each type has a bias component and \( L \) extrapolative components based upon past deviations from the commonly understood fundamental value. Given a specific distribution such as multivariate normal, it is easy to use (2.2.3) and moment generating functions to calculate the LTL and produce a deterministic dynamical system for \( x_t \).

"QUENCHED" BELIEF SYSTEMS

Another useful way to generate LTL's is to draw \( f_{ht} = b_{ht} + \sum_{i=1}^{L} x_{ht-i+1} + \cdots + x_{ht-L} + w_{ht-L} x_{ht-L} \) at date \( t=0 \) and fix it for all \( t=1,2, \ldots \). Call this draw \( \{b_{ht}, w_{ht}, x_{ht-1}, \ldots, w_{ht-L}\} \) and write

\[
(2.2.5) \quad R_x t = \mathbb{E} \left[ f_{ht} \exp(\eta_{t-1} f_{ht}, t-2) \right] /
\]

\[ \exp(\eta_{t-1} f_{ht}, t-2) \]

21
and use moment generating functions for specific distributions to calculate the LTL. This "quenching" of the form of \( f_{ht} \) at time 0 and then letting time run while the belief types \( f_{ht} \) replicate according to the relative profits they generate probably gives a better representation of the "sea" of belief types evolving in artificial economic life settings like Arthur et al. (1993). Before we do any calculations let us discuss some general issues.

First we must remark, that of course, we still do not yet capture the use of Holland's full blown adaptive complex systems approach to generating belief types as used in Arthur et al. (1993). They introduce crossover and mutation of types as well as replication of types.

Second, consider a second stage attempt to capture the spirit of Holland in the artificial way of building a large set of \( f_{ht} \), where \( b, w \) denote a vector of bias parameters, a matrix of "loadings" on lagged \( x \)'s, all drawn at the starting date \( t=0 \), and \( X_{t-1} \) denotes a vector of lagged \( x \)'s. Even though this does not yet capture Holland's "bottoms up" approach to building belief systems out of elementary building blocks, it goes part way, because one can generate a very large set of \( f \)'s and let them compete against each other as in the above discrete choice approach where the utility of each \( f \) is the profits it generates. At each point in time the fraction of each \( f \)-type is determined by its relative profits.

An interesting recent paper of Dechert et al. (1995) can be applied to make a conjecture about the dynamics of \( f \)-systems like the above. Dechert et al. (1995) show that when dynamical systems of the form

\[
x_t = g(X_{t-1}; \theta)
\]

are parameterized by a family of neural nets then almost all such systems are chaotic provided there are enough lags in \( X_{t-1} \). The probability of chaos is already high when \( X_{t-1} \) only has a modest number of lags.

This finding is suggestive of a possibility for the generic development of complexity as the intensity of choice \( \beta \) increases. Let us indicate a possible line of argument suggested by the work of Doyon et al. (1993). Draw an \( f \)-system at random and increase \( \beta \) (i.e. increase \( \eta \)) for it. One might expect it to be "likely" that the "probability" that the first bifurcation is Hopf with an irrational angle between the real and imaginary part of the complex root pair to "converge" to unity as the number of lags in \( X_{t-1} \) increases. Speculating even more wildly, one might think that it would be quite likely that another Hopf bifurcation might appear after the first one. Furthermore one might argue that for large enough lags "most" of the systems are chaotic. This suggests the possibility of construction of a successful argument that the "generic" route to chaos as \( \eta \) increases is quasiperiodicity.

Of course, this is only suggestive of a research project to formalize and prove such assertions. One possible route might be to follow Doyon et al. (1993), who use Girko's Law on the distribution of eigenvalues of large matrices and computer simulations to argue that one might expect a quasiperiodic route to chaos. Recall that the measure of the set of real eigenvalues is zero under Girko's limit distribution which is uniform over a circle in the complex plane.

Since the measure of the set of complex numbers with rational angle between the complex part and the real part is also zero under Girko's limit distribution, the Doyon et al. (1993) argument is suggestive that the first bifurcation might be "generic Hopf" as the number of lags increases. A reading of Doyon et al. (1993) suggests, due to difficulties raised by discrete
time, that the argument that the first bifurcation is followed by another one which is "generic" Hopf is more problematic. For example, in discrete time, the first bifurcation can be followed "by a stable resonance due to discrete time occurring before the second Hopf bifurcation" (Doyon et al. 1993, p. 285).

However, the problems with this line of argument for our case are several. First, Girkov's Law depends upon the explicit scaling of the system so that convergence of the eigenvalue spectrum of the sequence of Jacobians extracted from the linearizations at the steady state to Girkov's limit occurs. This scaling may not make economic sense. Second, while one may have a chance getting the first bifurcation to be generic Hopf using the Doyon et al. (1993) strategy and still making some economic sense out of it, the second bifurcation is complicated in discrete time settings. Third, the simulations of Doyon et al. (1993) indicate the presence of a large quantity of real eigenvalues, even for large systems.

Another possible research route to exploration of the impact of adding more lags to the dynamics would be to explore Schur-type necessary and sufficient conditions for all eigenvalues of the linearization to lie inside the unit circle. One would expect the probability that a randomly drawn system's linearization to satisfy the Schur conditions to fall as the number of lags increases.

The safest thing that can be said at this stage is this. Exploration of "routes to complexity" of 'randomly' drawn systems is a subject for future research. It would be interesting to formulate and prove a theorem on the quasiperiodic route to complex dynamics like that suggested by the Doyon et al. (1993) work because it would drastically simplify the classification of pathways to complex dynamics in large systems.

Return now to explicit calculations. Consider (2.2.5) with quenched pure biases, $f_{ht}^*_h$, before the LTL is taken,

(2.2.6) $R_x^t = \frac{\sum_b \exp(\eta_{t-1}^b_1) \exp(\eta_{t-1}^b_2)}{\sum \exp(\eta_{t-1}^b_1)} = \phi(\eta_{t-1}^b_1)$.

A direct computation shows that $\phi'>0$. Hence steady states $x^*, y^* = x^* - R_x^*$ must satisfy, $R_x^* = \phi(-r\eta x^*)$, $r=R-1$. Now $\phi(0) = (1/H)\sum_b f_{ht}^*_h$, which is a measure of the average market bias. If this is zero then the only steady state is $x^* = 0$, i.e. there can be no deviation from the fundamental. In general there can only be one steady state. Of course since $R_{t-1} \equiv x_{t-1} - R_{x_{t-2}}$, the steady state may be unstable. Turn now to calculation of LTL's for some specific "quenched" cases.

Consider (2.2.6) but draw b IIDN($\mu_b^*, \sigma_b^2$) across h. Let $\tau = \eta_{t-1}^b_1$. Note that the moment generating function $E(\exp(\tau b))$ and its derivative w.r.t. $\tau$, $E(b\exp(\tau b))$ for the normal are given by

(2.2.7a) $E(\exp(\tau b)) = \exp(\mu_b^* + (1/2)\sigma_b^2 \tau^2)$,

(2.2.7b) $E(b\exp(\tau b)) = (\mu_b^* + \sigma_b^2 \tau) \exp(\mu_b^* + (1/2)\sigma_b^2 \tau^2)$.

Hence we have,

(2.2.8) $R_x^t = E(b\exp(\tau b)) / E(\exp(\tau b)) = \mu_b^* + \sigma_b^2 \tau = \mu_b^* + \sigma_b^2 \eta(x_{t-1} - R_{x_{t-2}})$.

This is a linear second order difference equation which is trivial to analyze by linear methods. The case of economic interest is where $R$ is near one. For
R near one, it is easy to see that (2.2.8) becomes unstable as \( \eta, \sigma_b^2 \) increase. Let \( \alpha = \sigma_b^2 \eta \). Notice that the two roots of (2.2.8) are complex with modulus less than one for small \( \alpha \). As \( \alpha \) increases the roots increase in modulus and pass out of the unit circle in the complex plane. Hence the bifurcation of (2.2.8) under bifurcation parameter \( \alpha \) is Hopf.

For a similar type of example draw b IID with mass points \(-A, +A\) with probability 1/2 each. Calculate as above to obtain,

\[
(2.2.9) \quad R_{x_t} = \text{Atanh}[A \eta (x_{t-1} - R_{x_{t-2}})].
\]

The analysis of the linearization of (2.2.9) is similar to the analysis of (2.2.8). The global analysis of system (2.2.9) may now be analyzed as in Brock and Hommes (1995a,b). Note that the linearizations of (2.2.8) and (2.2.9) have an autoregressive structure of order 2 that is reminiscent of a common structure found in detrended macroaggregates such as detrended real Gross National Product.

Conclusion: In pure bias cases instability tends to appear when \( \eta \) increases and when the variance of the b-distribution increases. Therefore one expects an increase in heterogeneity dispersion to increase chances of instability in these pure bias cases. Depending on one's intuition one might expect an increase in diversity as measured by the variance of the type distribution to lead to an increase in stability. This is so because large diversity of types implies that for any biased type it is more likely that there's another type who would be willing to take the opposite side of the desired trade. Hence an increase in diversity might be expected to lead to an increase in "liquidity" and, possibly, to an increase in "stability." However when types interact with fitness and replication-like dynamics over time an increase in diversity measured by an increase in \( \sigma_b^2 \) in (2.2.8) or an increase in A in (2.2.9) can lead to an increase in instability. This is so because large variance magnifies the system's response to last period's excess returns \( x_{t-1} - R_{x_{t-2}} \).

The same type of calculations can be done to study the Large Type Limit dynamical system for general \( f = b_0 + \sum h_{-1} x_{t-1} + \ldots + \sum h_{-L} x_{t-L} = b_0 + g(x_{t-1}, w) \). One can say more for specific distributions such as multivariate normal where there is a closed form expression for the moment generating function. This kind of calculation allows one to explore the effects of changes in means and variances as well as covariances of bias and lag weights on stability of the Large System Limit dynamical system which can be explored by deterministic dynamical systems methods. In cases where the parameters in the g-part of \( f \) are independent of the bias term, and the g-part is small, then the primary bifurcation is likely to be Hopf as in (2.2.8). We refer to Brock and Hommes (1995b) for more on this.

Before continuing on to a treatment of social interaction in adaptive belief systems, we wish to make an extended remark concerning the use of risk-adjusted profit measures as fitness functions. Put \( r_{t+1} = P_{t+1} + \gamma_{t+1} - R_{p_t} \), let \( \rho_{t+1} = E_r r_{t+1} \), \( \rho_{h,t+1} = E_h r_{t+1} \). Refer to equations (2.1.1), (2.1.2), and (2.1.3) in the following short discussion.

Let \( g(\rho_{t+1}; \rho_{h,t+1}) \) denote the conditional expectation of actual risk-adjusted profits experienced when choice of \( z_{ht} \) was made based upon \( E_{h_t}(.) \). Let \( g(\rho_{t+1}; \rho_{h,t+1}) \) denote the conditional expectation of actual risk-adjusted profits when choice of \( z_{ht} \) was made upon the correct conditional
expectation $E_t(\cdot)$. It is easy to use (2.1.2), (2.1.3), and Assumptions A.1-A.3 to show

$$(2.2.10) \quad g(\rho_{t+1};\rho_{h,t+1}) - g(\rho_{t+1};\rho_{t+1}) = - \frac{1}{(2a \sigma_t^2)}(E_{ht}(q_{t+1}) - E_{t}(q_{t+1}))^2,$$

where $q_{t+1} = p_{t+1} + y_{t+1}$ and $\sigma_t^2 = \nu_t(q_{t+1})$.

Equation (2.2.10) allows us to make three points. First, if memory is introduced into the fitness function for type $h$, i.e.,

$$(2.2.11) \quad U_{ht} = d_{h,t-1} + M_{h,t-1} + d_{h,t-1} = g(\rho_{t-1};\rho_{h,t-1}) - g(\rho_{t-1};\rho_{t-1}),$$

then it may be shown that if $M=1$ (or $\eta$ goes to infinity in the above setting), rational expectations, $E_{t}q_{t+1}$, eventually drives out all other beliefs. Intuitively any belief that is consistently different from rational expectations eventually generates unbounded "value loss" from rational expectations and, hence, its weight, $n_{ht}$ goes to zero for positive intensity of choice $\beta$. Brock and Hommes (1995b) develops this result and other results using risk-adjusted profit measures.

Second, the use of a profit measure that is risk-adjusted as above makes the profit measure used in the fitness function compatible with the goal function that generated the underlying demand functions. This consistency property is not only intellectually desirable but also allows unification of the evolutionary theory with the received theory of rational expectations. Brock and Hommes (1995b) show that LTL-type calculations are messier but still tractable. They also show that rational expectations tend to squeeze out all other beliefs when memory $M$ is close to one, intensity of choice is close to infinity, risk aversion is near zero, or commonly hold beliefs on conditional variance are very small. However, they also show that if there is a cost to obtaining rational expectations which must be renewed each period then this "creates niche space" for nonrational beliefs to compete evolutionarily. Hence, costly rational expectations can lead to complicated dynamics in the space of deviations from the fundamental.

Third, the fact that (2.2.10) always has negative R.H.S. induces a type of quasinorm on the space of belief measures for each $h$ conditional on commonly held information at each time $t$. Since the goal function (2.1.2) is mean/variance therefore this quasinorm only can push conditional mean and conditional variance to their rational expectations values. One can obtain such convergence when, for example, memory $M$ tends to one.

More general goal functions such as the standard Lucas tree model treated in Sandroni (1995) may induce a more powerful quasinorm. The intuition is this. Let $\mu_h(\cdot)$ denote agent $h$'s measure. Agent $h$ makes decisions to maximize its goal (which is the conditional expectation of capitalized utility of consumption in Sandroni's case) and let $d(\mu_h)$ denote this decision. Now evaluate the goal function using the actual measure $\mu(\cdot)$ but with the decision set at $d(\mu_h)$. Obviously agent $h$ does better by using $d(\mu)$, not $d(\mu_h)$.

Hence one can imagine developing an evolutionary approach in more general settings than the mean/variance setting which is exposited here. Provided that the goal function is "rich enough" one could imagine evolutionary forces for rational expectations. Recall that under regularity conditions characteristic functions determine measures and vice versa. In analogy with characteristic function theory one might imagine weak sufficient conditions on a class of goal functions that would deliver strong forces for convergence to
rational expectations when memory is infinite (i.e. \( M=1 \)).

Notice that Araujo and Sandroni (1995) and Sandroni (1995) develop a
theory that locates sufficient conditions for convergence to rational
expectations in a setting where agents who do a better job of predicting the
future achieve higher values of their goal functions. One could
imagine constructing an evolutionary version of their more
general theory parallel to the evolutionary mean/variance theory
outlined here.

Finally, before turning to social interactions, we would like to state
that Marimon's review (1995) gives a broad survey of recent work on learning
dynamics. We urge the reader to study this review in order to gain a proper
perspective on the breadth of work that has been done on learning and
evolution in economics.

2.3 SOCIAL INTERACTIONS IN ADAPTIVE BELIEF SYSTEMS

While there is not space here for an extensive treatment of social
interactions, let us outline a brief treatment since social interactions are
one way to build models consistent with the abrupt change Fact 11.

Specialize (2.1.9) to the case of two choices 1,2 which we recode as
\( \omega=-1,+1 \) and consider the probability model for \( n \) choosers,

\[
\Pr(\omega_1, \ldots, \omega_n) = \exp[\beta(\overline{U}(\omega_1) - C(\omega_1 - \bar{\omega})^2)] / Z,
\]

where \( C > 0 \) is a penalty for agent \( i \) to deviating from the average choice
\( \bar{\omega} = (1/n) \sum_1^n \omega_i \) of the community of agents. Here \( Z \) is chosen so that the
probabilities given in (2.3.1) add up to unity over the \( 2^n \) configurations
(\( \omega_1, \ldots, \omega_n \)). It is easy to rewrite (2.3.1) into the form of a standard
statistical mechanics model, i.e. the Curie-Weiss model, and follow Brock
(1993a) to show that \( \omega \) converges in distribution to the solution of

\[
m = \tanh((\beta/2)dU + \beta J m), \quad J = 2C,
\]

where the root with the same sign as \( dU = U(+1) - U(-1) \) is picked. There are two
roots when \( \beta J > 1 \).

Brock and Durlauf (1995) have developed relatives of this type of model
for application to macroeconomics and to modelling group interactions in
social behaviors which are vulnerable to peer pressure. They have started the
work of development of econometric theory for such models and have started the
work of estimation of such models on data sets.

Let us pause here to take up the issue of endogeneity of the parameter \( \beta \).
In statistical mechanics models, \( \beta \) is the inverse "temperature." In the kind
of models we treat here, \( \beta \) is the intensity of choice parameter which can be
related to individual and population characteristics as in Manski and McFadden

There is another approach to \( \beta \) which is based upon ideas of E.T. Jaynes
and was outlined in Brock (1993a). Consider the maximum entropy problem

\[
\max \{ -\sum p_1 \ln(p_1) \}, \text{ subject to}
\]

\[
\sum p_1 = 1,
\]

\[
\sum p_1 U_1 = \bar{U},
\]

26
where $\bar{U}$ is a target level of utility. W.L.O.G. rank $U$'s as $U_1 \leq U_2 \leq \ldots \leq U_H$. Substitute out (2.3.4), set up the Lagrangian $L = -\sum_1^p \ln (p_1) + \lambda (\bar{U} - \sum_1^H U_1)$, write out the first order conditions and solve them to obtain

$$p_1 = \exp(\beta U_1) / Z, \quad \beta = -\lambda, \quad Z = \exp(\beta U_1),$$

and $\beta$ is determined by (2.3.5).

Now introduce a "production function" of average utility of choice, call it $\bar{U}(e)$, and introduce "effort" $e$ with "cost" $q$ per unit effort. Choose effort level $e$ to solve

$$\text{(2.3.7) Maximize } \bar{U}(e) - q e,$$

let $e^*$ solve (2.3.7), insert $\bar{U}(e^*)$ in place of $\bar{U}$ in (2.3.5) and obtain $\beta = \beta^* = -\lambda^*$. If one puts $\bar{U}(0) = (1/n) \sum_1^n U_1$, we see that $U_n - U_1 = 0$ implies $e^* = 0$ and $\beta^* = 0$. One can show that $\beta^*$ weakly increases when cost $q$ decreases and "choice gain" $U_n - U_1$ increases.

Even though it is trivial, this simple endogenization of $\beta$ is useful as a reasoning tool when coupled with the modelling outlined above. For example, consider agent types where effort cost $q$ is low (e.g. professional traders who have access to large databases, high speed computers with highly trained support staff, as well as years of training and discipline). If there is anything in the reward system of such types that penalizes them from deviation from their group average choice, then (2.3.2) suggests that a small change in $d\bar{U}$ can lead to big changes in the limiting value of $\bar{U}$ in the large system limit.

As another example consider debates about application of interacting systems models to economics because the parameter $\beta$ is "exogenously fixed" and corresponds to "inverse temperature" which has no meaning in economics. The maximum entropy story given above, trivial as it is, is enough to show that some of this debate may be misplaced. Turn now to a general discussion of adaptive belief modelling.

GENERAL THEORY OF ADAPTIVE EVOLUTION OF BELIEFS

During the last few years I have been working alone and with co-authors on developing models of diversity of beliefs and social interactions in belief formation (Brock (1993a), Brock and LeBaron (1995), Brock and Hommes (1995a,b)) not only to help explain the facts listed in the introduction, but also to uncover general complexity producing mechanisms in the economy.

Let us discuss a most interesting pathway to complexity that appears very common in economic situations. I shall call it the Prediction Paradox.

PREDICTION PARADOX

Consider a situation where a collection of agents are predicting a variable (for example, a product price) whose value is determined by the joint aggregate of their decisions (their aggregate supply must equal aggregate demand, for example). Let prediction accuracy for each agent increase in cost. While the cost for some agents may be less than for others, cost increases for all as accuracy increases. If all agents invest a lot in prediction accuracy, then the gains for any one agent may not be enough to cover its cost given that the others are all investing. Vice versa, if no agent is investing in prediction, the gains to any one agent may cover the cost of investing in prediction. We shall
show that a situation like this is a good source of complex dynamics.

There are many examples of this sort of situation in economics. A leading example, relevant to finance and to the real world is the "Index Paradox."

INDEX PARADOX. If everyone invests in index funds in the stock market, it will pay someone to do security analysis and invest in mispriced securities. If everyone does security analysis, then indexers can save on the costs of securities analysis. So it will pay to avoid the costs of security analysis and just put all of one's funds into an index fund.

de Fontnouvelle (1994, 1995) develops adaptive belief models that address the Index Paradox.

We shall use a simple cobweb production setting to show how the prediction paradox can lead to complex dynamics. Think of a collection of firms with anticipated profit function \( \pi^e = p^e q - c(q) \), \( p^e \) denotes price expectation which must be formed to guide the production decision \( q \). Let each firm \( j \) make a prediction \( p^e_j \) and maximize anticipated profits. Equilibrium of demand and supply determines actual price \( p \) and actual profits \( \pi = pq_j - c(q_j) \) for each firm \( j \).

In Brock and Hommes (1995a) predictions are made by choosing a predictor from a finite set \( P \) of predictor or expectations functions. These predictors are functions of past information. Each predictor has a performance measure attached to it which is publically available to all agents. Agents use a discrete choice model along the lines of Manski and McFadden (1981) to pick a predictor from \( P \) where the deterministic part of the utility of the predictor is the performance measure. This results in the Adaptive Rational Equilibrium Dynamics (A.R.E.D), a dynamics across predictor choice which is coupled to the dynamics of the endogenous variables.

The A.R.E.D. incorporates a very general mechanism, which can easily generate local instability of the equilibrium steady state and very complicated global equilibrium dynamics.

For example, under the assumption of unstable market dynamics when all agents use the cheapest predictor, it is shown that in the A.R.E.D. the "probability" of reaching the equilibrium steady state is zero when the intensity of choice to switch between predictors is large enough. This result holds under very general assumptions on the collection of predictors, \( P \). Of course if there are predictors in \( P \) that are not too costly and are stabilizing, they will be activated if the system strays too far from the steady state. In this case the system just wanders around in a neighborhood of the steady state. We illustrate the typical behavior in a two predictor case below.

When \( P \) contains only two predictors, much more can be said. We shall explain a typical result. Agents can either buy at small but positive information costs \( C \) a sophisticated predictor \( H^1 \) (for example rational expectations or long memory predictors such as "Ljung-type" predictors or freely obtain another simple predictor \( H^2 \) (e.g. adaptive, short memory or naive expectations). Agents use a discrete choice model to make a boundedly rational decision between predictors and tend to choose the predictor which yields the smallest prediction error or the highest net profit.

Suppose that if all agents use the sophisticated predictor \( H^1 \) all time paths of the endogenous variables, say prices, would converge to a unique stable equilibrium steady state, whereas when all agents would use the simple predictor \( H^2 \) the same unique equilibrium steady state would occur, but this
time it would be unstable. In the cobweb production setting instability under naive expectations tends to occur when the elasticity of supply is larger than the elasticity of demand.

Consider an initial state where prices are close to the steady state value and almost all agents use the simple predictor. Then prices will diverge from their steady state value and the prediction error from predictor \( H_2 \) will increase. As a result, the number of agents who are willing to pay some information costs to get the predictor \( H_1 \) increases. When the intensity of choice to switch between the two beliefs is high, as soon as the net profit associated to predictor \( H_1 \) is higher than the net profit associated to \( H_2 \), almost all agents will switch to \( H_1 \).

Prices are then pushed back towards their steady state value and remain there for a while. With prices close to their steady state value, the prediction error corresponding to predictor \( H_2 \) becomes small again whereas net profit corresponding to predictor \( H_1 \) becomes negative because of the information costs. When the intensity of choice is high, most agents will switch their beliefs to predictor \( H_2 \) again, and the story repeats.

There is thus one "centripetal force" of "far-from-equilibrium" negative feedback when most agents use the sophisticated predictor and another "centrifugal force" of "near-equilibrium" positive feedback when all agents use the simple predictor. The interaction between these two opposing forces results in a very complicated Adaptive Equilibrium Dynamics when the intensity of choice to switch beliefs is high. Local instability and irregular dynamics may thus be a feature of a fully rational notion of equilibrium.

The Brock and Hommes (1995a) paper makes this intuitive description rigorous and shows that the conflict between the "far-from-equilibrium stabilizing" and "near-equilibrium destabilizing" forces generates a near homoclinic tangency. A homoclinic tangency is created when the increase of a parameter, e.g. the intensity of choice \( \beta \) causes the unstable manifold of a steady state to pass from nonintersection, to tangency, to intersection of the stable manifold of that steady state. This situation is associated with complicated dynamical phenomena. Brock and Hommes (1995a) show that near homoclinic tangencies are typically generated if \( \beta \) is large enough. This generates phases where the market is close to equilibrium which are punctuated by bursts of instability which are quickly quashed by a mass of producers who now find it worthwhile to pay the costs of more accurate prediction.

Brock and Hommes (1995b) applies a similar methodology to adaptive belief evolution and has begun the daunting task of building a taxonomy of "universality classes" of "routes to complex dynamics" in such systems. One could view this type of modelling as the analytic "support" to the computer based artificial economic life work of Arthur et al. (1993).

The discussion of the role of bounded rationality in creating complex dynamics has been theoretical. However, this kind of work suggests tractable econometric settings where traditional rational expectations can be nested within a model where some fraction of the agents choose rational expectations and the rest of the agents choose a form of boundedly rational expectations. In this setting, if one assumes the fraction of each type is constant, one can estimate the fraction of each type as well as other parameters of the model on data and set up a statistical test of the significance of departures from rational expectations. This is especially tractable in linear quadratic stochastic settings.

Brock and Hommes (1995a) suggested this type of research strategy. Baak (1995) and Chavas (1995) have carried out estimation and testing for cattle
and pork data, respectively. They find evidence that purely rational expectations models are not adequate to describe the data. We hope to do some econometric work in finance that nests rational expectations models within heterogeneous expectations models in order to shed light on what kinds of belief systems are consistent with the data. For example, a PhD student of ours, Kim-Sung Sau, has done some preliminary work that suggests evidence consistent with the presence of a type of boundedly rational expectation whose emergence is stimulated by positive runs or negative runs in returns.

Turn now to a short summary and conclusion statement.

SUMMARY AND CONCLUSIONS

We started out this paper with a list of Facts that financial theorizing should attempt to explain. We surveyed some theoretical work which goes part way to explaining some of the Facts, especially the ones concerning the typical shape of autocorrelation functions and cross autocorrelation functions of returns, volatility of returns, and trading volume. We cited surveys that cover theoretic work that contributes to explanation of some of the other facts. We discussed the Facts in enough detail so that the reader can appreciate the caution one needs to display while interpreting evidence from financial datasets.

We sketched some examples of adaptive belief system modelling which we believe have promise into developing into theory which can shed light on the economic mechanisms lying behind the Facts we have listed. We consider the approach to modelling that was sketched here to be in the style of complex systems approach of the Santa Fe Institute which stresses Complex Adaptive System modelling. While much remains to be done, a lot of progress has been made since 1988.
FOOTNOTES

1. This paper was prepared for the book: Arthur, W., Durlauf, S., Lane, D., eds., (1996) THE ECONOMY AS AN EVOLVING COMPLEX SYSTEM II, Redwood City, California: Addison-Wesley. This book contains proceedings of a conference held at the Santa Fe Institute, August 26 to September 1, 1995.

William A. Brock is grateful to the NSF (SBR-9422670) and to the Vilas Trust for essential financial support. He is grateful to his friends Steven Durlauf and Blake LeBaron for many hours of discussions on the contents of this paper, on science in general, and on things complex. He alone is responsible for the views expressed and for any errors in this paper.
REFERENCES


