NONCONVEXITIES IN ECOLOGICAL MANAGEMENT PROBLEMS

by

W. A. Brock and D. Starrett

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1. INTRODUCTION

This article gives a fairly complete theoretical treatment of a deterministic version of the Carpenter et al. (1997) lake management model. This model is meant to be a general abstraction of optimal management problems for ecosystems where there is a phosphorous load placed by anthropogenic activities (fertilizers for farming and gardening) that contributes to stock stored in a lake. When that variable becomes too high, it sets off an internal positive feedback mechanism which impairs the ecosystem's ability to absorb and biodegrade loadings. Furthermore, there are disamenities from degraded lake quality whose flow is proportional to the level of the stock variable. Here we will emphasize a fully optimizing version in which the manager can measure the stock and can control the loadings as function of stock. He chooses these loadings to best trade off the conflicting interests of farmers and lake users.

We show how methods of infinite horizon optimal control theory developed for problems with convex-concave dynamics of the state variable may be applied to this problem. A reasonably complete analysis turns out to be quite subtle.

Before we begin, we must say a few words about the style of exposition we shall use in this article. We shall be getting into problems of global analysis where there are many possibilities to examine and where the analysis will get bogged down quickly if we insist on writing in the completely rigorous "ε-δ" style. Hence, we shall write in the style of global analysis where arguments which can obviously be made rigorous will be explicated with geometrical devices like phase diagrams. Furthermore, to avoid technical boundary conditions, we shall assume right hand and left hand "Inada" conditions and we shall assume existence of optima to infinite horizon control problems even though this has not been proved.

Existence theory is essentially trivial in discrete time given literature like that treated in Becker and Boyd (1997, see especially the references to Boyd's work on existence in continuous time problems). Existence theory is more technical in the continuous time case but is treated in Carlson, Haurie, and Leizarowitz (1991).

2. THE STYLIZED LAKE ECOLOGY

Following Carpenter et al. (1998), consider the following model of the dynamics of a lake where the limnology is boiled down to a one-dimensional ordinary differential equation,
\[
\frac{dx}{dt} = a - bx + f(x) = F(a,x), \quad x(0) = \text{given.}
\]

Here, \(x\) represents the state of the system, and corresponds to the stock of phosphorous suspended in algae (which we will take as proportional to undesired turbidity of the water in the lake), \(a\) is phosphorous inputs from the watershed (sometimes referred to as the "loading"), \(b\) is the rate of loss per unit stock (sedimentation, outflow, and sequestration in organisms other than algae), and \(f(x)\) is internal loading. This function is assumed to be "S-shaped;" that is, for low stocks of phosphorous, additions tend to be stored in the lake bed so there is a relatively low marginal return to the water, whereas for higher stocks this marginal return increases, only to fall again when maximal suspension is approached. This equation represents a "minimal state variable" approximation to the complex foodweb of a real lake. The reduction and an argument that a convex-concave "recycling curve", \(f(x)\) is a good abstraction capturing real world positive feedback processes that are triggered inside a typical lake when P-load becomes too high is given by Carpenter et al. (1998) and the references in that paper. See Cottingham and Carpenter (1994, p. 2129) for a much more disaggregated model of the state variable dynamics which consists of a set of differential equations to represent the food chain of the lake. As we said before, one might think of (2.1) as an "aggregative reduced form" for the true disaggregated model.

3. BEHAVIOR UNDER CONSTANT LOADING

We can get some intuition for the functioning of this model and set a benchmark for comparing with optimal management by considering behavior under constant loading, that is the dynamics of equation (2.1) when the loading rate from outside sources \((a)\) is held constant. Depending on the relationship between the purification parameter \((b)\) and the feedback function \((f(x))\), various qualitative behaviors are possible; we distinguish three distinct cases.

Case I: Unique equilibrium

When the purification rate is greater than the maximal feedback rate \((b > \max f(x))\) there will be a unique long run equilibrium steady state for any given constant loading rate and the steady state will be a continuous increasing function of the loading. Furthermore, convergence to the steady state will be monotone from any starting stock of phosphorous. Referring to figure 1, if the loading is \(a\) and \(x\) is above \(\hat{x}\), \(bx-a\) is larger than \(f(x)\) so \(dx/dt\) is negative and \(x\) will fall toward the unique steady state \(\hat{x}\). Similarly, the reverse dynamics holds if initial stock is below \(\hat{x}\). Here the nonconvexity appears inessential in that behavior is qualitatively as it would be in a fully convex version. However, we will see that this need not be true in the optimizing version—that is for some combinations of parameters consistent with Case I, there will be multiple local optima. As we will see, this situation becomes more likely in the other cases.

Case II: Multiple equilibria with reversibility
When the purification rate is smaller than the maximal feedback rate, there will be multiple long run steady states for some values of the constant loading. Case II corresponds to the situation when, in addition, \( f(x) \) is less than \( bx \) for all \( x \) (see figure 2). Then for initial loadings between \( L_1 \) and \( L_2 \), there will be two locally stable long run steady states, one involving a relatively low \( x \) "clear water" state (referred to as oligotrophic in the ecology literature) which is highly valued by the lake's users, but also a relatively high \( x \) "turbid water" state (referred to as eutrophic in the ecology literature) which is disliked by the lake's users but valued by agricultural interests because it allows them high fertilizer use. Which state is reached for a given constant loading depends on the initial stock of phosphorous, high initial stocks leading to a eutrophic long run equilibrium and vice versa (as indicated by arrows in figure 2). These two "basins of attraction" of the lake will play a major role in the following discussion. In this situation, history matters in that for a given constant load the long run steady state will depend on the initial level of \( x \). However, all long run outcomes are reversible in this case. A sufficiently low load (below \( L_1 \)) will always restore the clear water type state if it is maintained sufficiently long.

We will show that a fully optimizing version of case II can display a wide variety of qualitative behaviors. There can be a single locally optimal path or as many as you like, depending on parameters. Further, we will show that history may or may not matter in the optimizing solution.

Case III: Multiple equilibria with irreversibility

If \( f(x) \) lies above \( bx \) for sufficiently high \( x \), then behavior is as in case II except that once the phosphorous level exceeds a certain critical value, there is no policy that can ever again achieve lower values. In figure 3, this critical level is \( \hat{x} \). If the initial stock is larger than that, then even if we reduce the external loading to zero, the phosphorous stock will converge to a eutrophic long run steady state. In our optimizing version there will be a unique locally optimal policy if the initial stock is above \( \hat{x} \). However, if it starts below, there may still be multiple local optima—that is it might be desirable to cross the threshold from below in an optimal management scheme. We will try to show when and how this happens.

Notice that the function \( f(x) \) is the source of the potential hysteresis in all of these cases. A plausible mechanism to generate an effect like \( f(x) \) is the theory put forth in Sheffer (1997, Chapter 3 and Figures 3.19 and 3.22). Further, the model's "prediction" that lakes may be seen to flip from low turbidity to high turbidity states in response to small changes in loading factors seems to be confirmed by observation.

4. AN OPTIMIZING FRAMEWORK

There is a community of individuals who share the lake and its watershed. These individuals have conflicting interests. Some are producers whose activities release loadings into the catch basin of the lake, which contribute to the term \( a \) in (2.1). Others are users of the services of the lake and are injured by the "dumping" activities of the others in that they care
about the overall quality of the lake which we proxy by the stock $x$—higher stocks meaning lower quality.

Utility (measured, for example, as increases in profitability of economic operations) is generated for the producers of $a$ from agriculture, lake side cottages, developments, etc. Disutility is generated by $x$ and must be measured by willingness to pay for a clean water environment.

We formulate the objective of a Social Planner who acts in the interest of the community as a whole as follows,

$$\begin{align*}
W(x) &= \text{Max} \{ \int exp(-rt)\{U(a,x)\}dt \}, \text{ subject to (2.1).} \\
&= 0
\end{align*}$$

Here the maximum is taken over the set of piecewise continuously differentiable functions \{a(t)\}. We assume that the utility $U(a,x)$ is concave in $a$ and is decreasing in $x$. This generates a tradeoff which the social planner must optimize. Here $r>0$ is the real rate of discount on future utility. Since this rate is generally thought to be quite small (on the order of 1-2\%) we will focus attention on situations where it is small relative to the parameter $b$.

The plan of this part of our paper is as follows. We shall study the optimal solutions of (4.1) and use the first order necessary conditions of optimality to characterize these solutions. We shall spend a fair amount of time on the special case $U(a,x)=U_1(a)+U_2(x)$ because, here, we can "separate" the interests of the Affectors (those interests like agriculture and developers whose activities damage the resource) from the interests of the Enjoys (those interests like fishers, viewers, and swimmers whose activities do not injure the resource). We take the terms "Affectors" and "Enjoys" from the work of Scheffer et al. (1998). We shall also spend some time on the case where $U_1$ is linear because it generates a particularly simple dynamics that motivates the more general behavior. Linearity can be thought of as representing a reduced form restricted profit function of a constant returns to scale Affect sector. This case is mostly of pedagogical interest since with a fixed amount of land in proximity to the lake, we would expect diminishing returns.

5. THE MOST RAPID APPROACH SPECIAL CASE

Here we study the linear, separable case $U_1 = Aa$, $U_2 = -c(x)$, $c^>0$, $c^">0$. For the deterministic case linear $U$ generates a Most Rapid Approach Problem (MRAP) where one operates the control variable $a$ to push the state variable $x$ to the maximum of a certain function of $x$ as we shall see. Thus we now study the problem:

$$\begin{align*}
W(x) &= \text{Max} \{ \int exp(-rt)\{Aa-c(x)\}dt \}, \\
&= 0
\end{align*}$$
\[ \frac{dx}{dt} = a - bx + f(x). \]

We assume that \( f(x) \) is convex-concave and is increasing in \( x \) with \( f(0)=0, \ f'(0)=0, \ f'(\infty)=0, \) with unique \( x_* = \text{argmax} \ f, \ f'>0 \ for \ x<x_*, \ f'<0 \ for \ x>x_* \). Notice that if we define \( AP(x) = f(x)/x, \ MP(x) = f'(x) \), then just as in elementary economics of increasing returns production functions, we have \( MP>\text{AP} \ (MP<\text{AP}) \) for \( x<x_* \ (x>x_*) \) with \( \text{MP}=\text{AP} \) for \( x=x_* \). Furthermore, \( \text{AP}(0)=\text{MP}(0) \) and \( \text{AP}(\infty)=\text{MP}(\infty)=0. \)

Now, for any finite \( T \), substitute for "a" from (5.4) and observe that

\[ \int_0^T \exp(-rt) \{Aa-c(x)\} dt = A[\exp(-rT)x(T)-x(0)] + \]
\[ \int_0^T \exp(-rt) \{A[rx+bx-f(x)]-c(x)\} dt. \]

Hence if we restrict ourselves to the set of functions \( x(.) \) that satisfy the transversality condition:

\[ \exp(-rT)x(t) \rightarrow 0, \ T \rightarrow \infty, \]

then (5.5) suggests that we control \( a(.) \) to move \( x(t) \) to most rapidly approach

\[ x^* = \text{argmax} \ \pi(x) = \{A[rx+bx-f(x)]-c(x)\} \]

Indeed, since the objective is now linear in control \( a \) we know from the work of Spence and Starrett (1975) that even if there are restrictions on the control, the optimum from any starting point is to most rapidly approach some local maximum in (5.7). We can think of this objective as trading off long run interests of Affectors and Enjoyers. Affectors get a flow benefit of \( rA \) per unit from the stock per unit time while Enjoyers pay cost \( c(x) \). In addition the stock provides a net purification benefit of \( bx - f(x) \) which acts like a productivity term in the objective.

Notice that it is easy for the mathematics in (5.7) to yield "unpleasant" solutions. For example, if the turbidity disutility \( U_2(x) = c(x) \) were bounded from below, (5.7) would shoot \( x \) off to infinity and turn the lake into a dumping ground for the \( a>0 \) polluting "stakeholders." This is so because the polluters are willing to pay \( A>0 \) for each extra unit of \( x \)-dumping while the willingness to pay to prevent such dumping embodied in the disutility is falling to zero.

Part of the reason for this anomaly derives from our assumption of linearity (marginal benefit from pollution assumed not to decline in pollution level). However, it seems sensible to posit an unbounded cost which will assure an interior solution. If \( x(t) \) can be moved immediately to \( x^* \), we have from (5.54)

\[ W(x) = -ax(0)+(1/r)[a[rx^* + bx^* - f(x^*)] - c(x^*)]. \]
An initially pristine lake is one with \( x(0) = 0 \). We may differentiate \( W \) to obtain how much an innovation would be worth that increased "b" by one unit,

\[
(5.9) \quad dW/db = (1/r)a x^*.
\]

The innovation allows the lake to "biodegrade" one more unit of \( x \) per unit time. This extra loading capacity is worth \( a \) to the polluters each period. When this amount is capitalized over all periods at rate \( r \) we obtain (5.9). Similar sensitivity analysis can be done with for other parameters of interest.

The analysis provides more insight into the nonlinear case if we consider constraints on the control variable that will prevent instantaneous adjustment. Nonnegative values of a naturally prevent such downward adjustment and there may well also be upper bounds (say \( a < M \)) which similarly prevent upward adjustment. Then, as in Spence and Starrett (1975) we must search among local optima of \( \pi(x) \). Examining this function, we see that there may well be multiple local optima. To see this, examine the first order conditions for a maximum:

\[
(5.10) \quad A[r + b - f'(x)] = c'(x).
\]

Since for \( x > x_m \), \( f'(.) \) is decreasing and \( c'(x) \) is increasing, we can make particular choices for these functions that make the left and right sides of (5.10) equal as often as we like (see figure 4). \(^1\) (We will be somewhat more rigorous in demonstrating similar properties in the nonlinear case.) As always, generically, the equalities will alternate between local maxima and local minima. Since we assume \( f'(0) = c'(0) = 0 \), it is easy to see that the first intersection must correspond to a local maximum; also, since \( f'(\infty) = 0 \) and \( c'(\infty) = \infty \), the last intersection is also a local maximum. Of course, multiplicity need not always happen for all choices of parameters and functions, but we see that arbitrarily large numbers of local optima cannot be ruled out a priori. This observation will carry over to the general optimization problem as we shall see.

Let us consider the simplest generic multiplicity case where \( \pi(x) \) has two local maxima \( \alpha \) and \( \gamma \) separated by a local minimum \( \beta \). The remaining problem is to determine which of the local maxima is the optimal long run steady state from a given starting position given the restrictions on controls. The global maximum may not always be the long run optimum, as it may take too long to reach it from the other local optimum and this matters if we discount enough.

\[^1\]It is interesting to note that a concave version of this problem for which a sufficiency theorem holds requires that \( f \) be convex rather than concave. The reason is that the state variable here is a "bad" which reverses sign conventions. Of course, when this theorem holds there can be only one local optimum satisfying transversality and indeed with \( f(.) \) convex, there is only one solution to (5.10). However, the nonconvex element is fundamental to our problem and a concave segment of \( f(.) \) is well established empirically.
We will not try to solve this problem completely here. However, the basic qualitative analysis is easy to give and will carry over to our nonlinear general case. As long as the controls are bounded so that our single state variable must move continuously, then a simple application of the Bellman principle implies that it must move monotonically in an optimal trajectory and the set of states for which a particular local maximum is the long run optimum must be connected. (The set associated with a particular local maximum will be referred to as its basin of attraction.) Consequently, either the global maximum is always optimal or there is a knife-edge point between the local optima such that the global optimum goes to $\gamma$ for all starting stocks to the right of the knife-edge and to $\alpha$ for all stocks to the left. We refer to this knife-edge point as the Skiba point since he identifies and studied it in the related renewable resource problem (see Skiba (1978)).

We turn now to the general nonlinear case where we will find a similar qualitative picture. We will try to discover what particular features of the problem lead to multiplicities (or lack thereof) and will discuss ways of determining Skiba points that result.

6. GENERAL ANALYSIS OF THE LAKE PROBLEM

Consider the general separable-utility problem:

\[
W(x) = \max \left\{ \int_0^\infty \exp(-rt)\{u(a) - Bc(x)\} dt \right\},
\]

Subject to $\frac{dx}{dt} = a - bx + f(x)$, $x(0) = x$.

We assume: $u' > 0$, $u'(0) = \infty$, $u'' \leq 0$, $c(0) = 0$, $c' > 0$, $c'(0) = 0$, $c'' \geq 0$. Verbally, utility to Affectors is concave (diminishing marginal utility), costs to Enjoysers are convex (increasing marginal cost), with B representing a parameter that determines the relative importance of the two groups. The assumptions of $f(.)$ are as in the previous section.

We use optimal control theory to characterize optimal solutions. Using $p$ as the current value costate variable, we set up the current value Hamiltonian:

\[
H(a, x, p) = u(a) - Bc(x) + p(a - bx + f(x)).
\]

Then the maximum principle tells us that optimal loading is set to satisfy

\[
a > 0 - u'(a) = -p, \quad a = 0 - u'(a) < -p,
\]

defining a function $a(p)$. The Hamiltonian dynamics then are written as the coupled pair of differential equations:

\[
\frac{dx}{dt} = a(p) - bx + f(x), \quad \frac{dp}{dt} = \{r + b - f(x)\}p + Bc'(x).
\]
The equations (6.2)-(6.4) constitute necessary conditions for a solution to (6.1) in that for the optimal policy we must be able to find a function p(.) such that these equations are satisfied for all time.

Parenthetically we remark that the assumption, \( u'(0) = +\infty \) implies that \( a=0 - p = +\infty \), so we may ignore the strict inequality case in (6.2). We shall do this in what follows. Also, due to this fact, it is sometimes convenient (particularly in doing phase diagrams) to change variables and work in \( \{a,x\} \) space rather than \( \{p,x\} \) space. This is accomplished by logarithmically differentiating (6.2) with respect to time, obtaining:

\[
(6.5) \quad \frac{dp}{dt}/p = (da/dt) \left[ \frac{u''(a)}{u'(a)} \right].
\]

Then, substituting into (6.4). We get the corresponding pair of equations:

\[
(6.3') \quad \frac{dx}{dt} = a - bx + f(x) = X(a,x),
\]

\[
(6.4') \quad \frac{da}{dt} = \left[ r + b - f'(x)\right] \left[ \frac{u'(a)}{u''(a)} \right] - B \left[ c'(x)/u''(a) \right] = A(a,x,B).
\]

Note for future reference that \( \partial A/\partial B = -c'(x)/u''(a) > 0. \)

We shall proceed to do a general analysis of this problem by carrying out the following steps: (i) Investigate sufficient conditions for a unique steady state in these equations and find sufficient conditions for multiple steady states. It turns out to be easy to construct examples where there are as many steady states as you want. (ii) Conduct a dynamic analysis using linearization around steady states, \( p(x)=W(x) \), and phase diagrams in order to piece together the value function \( W(x) \) following the strategy of Skiba (1978), Brock and Dechert (1983), (1985) (exposed in Brock and Malliaris (1989)).

7. A UNIQUE STEADY STATE

It turns out that there can be a unique locally optimal steady state in either case one or case two as identified earlier (but not in case three). However the outcome is quite parameter dependent and with particular choices, there can be multiple steady states in any of the cases. In this section, we restrict ourselves to cases one and two. We show first that for any other parametrization consistent with these cases, there is a breakeven \( B \) (call it \( B^* \)) such that there exists a locally optimal steady state in the eutrophic (high \( x \)) basin of attraction if an only if \( B \) is below \( B^* \). It will then follow almost immediately that if \( B \) is above \( B^* \), there is a unique local (and therefore global) optimal path converging to a steady state in the oligotrophic (low \( x \)) basin of attraction. We will also see that this path is saddle-value stable. In later sections, we will explore the behavior with multiple steady states.

To identify steady states, we look for a joint solution to the equations \( dx/dt = da/dt = 0 \), namely:
\[(7.3') \quad a - bx + f(x) = 0,\]
\[(7.4') \quad [r + b - f'(x)][u'(a)] - B[c'(x)] = 0.\]

These equations define curves in a,x space where \(dx/dt = 0\) and \(da/dt = 0\) respectively. Note that in this situation, both of these define a uniquely as a function of x and we label these \(a_{i}(x)\) and \(a_{j}(x)\) for convenience. Since \(bx > f(x)\), all \(x > 0\) in cases one and two, \(a_{i}(x)\) will be strictly greater than zero for all positive \(x\) in these cases. This is the important contrast with case three for purposes of this section. Any candidate for an optimal steady state must satisfy \((7.3'), (7.4')\), since \((6.3'), (6.4')\) are Hamiltonian necessary conditions.

Recall the following features of \(f(.): \quad x_{\ast} \) is the unique inflection point where \(f'' = 0\). Where they exist (case two), \(x_{i}\) and \(x_{j}\) are the two interior points where \(f'(x) = b\).

**Lemma 1**

In case two, there exists a \(B\) such that for all larger \(B\), and any \((a,x)\) with \(x \geq x_{i}\) and \(X(a,x) = 0\), then \(A(a,x,B) > 0\).

**Proof:**

Examining \((6.3'), (6.4')\), we can see that the conditions \(X(a,x) = 0\), \(A(a,x,B) > 0\) are equivalent to

\[(7.5) \quad [r + b - f'(x)][u'(bx-f(x))] - B[c'(x)] < 0.\]

Now, for \(x \geq x_{\ast}\), using concavity of \(u(.)\), convexity of \(c(.)\) and nonnegativity of \(f(.)\),

\[ [r + b - f'(x)][u'(bx-f(x))] - B[c'(x)] \]
\[ < [r + b][u'(b x_{\ast}f(x_{\ast}))] - B[c'(x_{\ast})]. \]

Hence, recalling that \(\partial A/\partial B > 0\), the conditions of the lemma are satisfied by choosing

\[ B = [r + b][u'(b x_{\ast}f(x_{\ast}))] / c'(x_{\ast}). \]

Q.E.D.

**Corollary 1:**

In case one, there exists a \(B\) such that for all larger \(B\), and any \((a,x)\) with \(x \geq x_{\ast}\) and \(X(a,x) = 0\), then \(A(a,x,B) > 0\).

**Proof:**

Observe that all the steps of Lemma 1 go through in case one if \(x_{\ast}\) is substituted for both \(x_{i}\) and \(x_{j}\). Q.E.D.
The significance of lemma and corollary for us is that when $B$ is large enough there cannot be any candidate steady states with $x$ bigger than $x_i$ (case 2) or $x_\infty$ (case 1). This accords with our intuition that if enjoyers matter enough the social planner will not want to go toward the "high $x$" basin of attraction. Next we establish something of the reverse for low $B$.

Lemma 2:
Pick any $\tilde{x}$ bigger than $x_\infty$ for which $r + b > f'(\tilde{x})^2$. Then in either case one or case two, if $\tilde{a}$ is such that $X(\tilde{a}, \tilde{x}) = 0$, there exists $B > 0$ with $A(\tilde{a}, \tilde{x}, B) < 0$.

Proof:
We require

$$r + b - f'(\tilde{x})[u'(b \tilde{x} - f(\tilde{x}))] > B[c'(\tilde{x})].$$

Since the left side of (7.5) is strictly positive and $c'(\tilde{x})$ is finite, this inequality can be satisfied for positive $B$. Q.E.D.

Using these lemmas, we can show that there is a breakeven relative importance of the parties such that if the enjoyers are relatively more important there will be a candidate steady state in the low-$x$ basin of attraction whereas if they are less important there will always be a candidate steady state in the high-$x$ basin of attraction.

Proposition 1: (note: This and later propositions to be read twice--with and without braces)
In case 1 (case 2), there exists a $B^* > 0$, such that

$$B < B^* \text{ if and only if there is a candidate steady state with } x > x_\infty \{x_i\}$$

Proof:
We set $B^*$ to be the smallest $B$ that will satisfy lemma 1 (corollary 1). Since $B$ sets a positive lower bound, there must be such a positive least upper bound.

Q.E.D.

Proposition 2:
In case 1 (case 2), if $B > B^*$, there will exist a unique candidate steady state with $x < x_\infty \{x < x_i\}$.

Proof:
From Corollary 1 (lemma 1) we know that

$$A(a_i(x_\infty), x_\infty, B) > 0 \{A(a_i(x_i), x_i, B) > 0\}$$

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This must be possible since we assumed that $f'(x)$ goes to zero in $x$. 10
Further, since \( a_i(0) = 0 \), using (6.4') and the assumption \( c'(0) = 0 \), we see that

\[
A(a_i(0),0,B) < 0.
\]

Hence by continuity, there exists \( 0 < x^* < x_\text{lin} \) such that \( A(a_i(x^*),x^*,B) = 0 \). By definitions of functions \( a_i(.) \) and \( A(.) \), \( x^* \) satisfies the steady state equations (7.3'), (7.4'). Further, since \( a_i(.) \) is monotone increasing and \( a_i(.) \) is monotone decreasing in \( x \) on the interval \([0,x_\text{lin}]\) \( \{[0,x_\text{lin}]\} \) there can be only one such candidate steady state in that interval. And from proposition 1, there are no such candidates outside the interval. Q.E.D.

In the remainder of this section we focus on the situation in which there is a unique steady state (\( B > B^* \)). The interpretation of this steady state in terms of long run optimization is very similar to what we found in the MRAP case. (7.4') tells us that the marginal consumption plus investment benefit of the stock to Affectors \((r + b - f(x))u'(a)\) must be set equal to the marginal cost to Enjoyers \((B(c'(x)))\). The only difference now is that marginal utility to Affectors is no longer constant and in the long run optimum, \( a \) must satisfy (7.3') in order that the steady state level of \( x \) be maintained.

Notice that for the "Ramsey" case, \( r=0 \), steady states, \( (a^*,x^*) \) solve the problem:

\[
(a^*,x^*) = \text{argmax} \quad \{U(a,x), \text{ s.t. } 0=a-bx+f(x)\}.
\]

Substitute out \( a=f(x)-bx \) from the contraint into the utility to obtain

\[
x^* = \text{argmax} \quad \{U(f(x)-bx,x)\}.
\]

In our case of separable utility this objective becomes \( u(f(x)-bx) - Bc(x) \) and looks even more like a benefit minus cost objective.

Now however, we care not just about the steady state level of \( x \) but also on how quickly we get there: The value per unit of \( a \) depends on the rate of loading so the total benefit to Affectors depends on how loads are distributed over time. So we turn next to transitions.

The behavior away from steady states is most easily (though not entirely rigorously) analyzed using phase diagrams. These are constructed by plotting the functions \( a_i(.) \) and \( a_i(.) \) And deducing the direction of motion in all regions of the \( a_i x \) space between these curves using (6.3') and (6.4'). In the following discussion, refer to figures 5 (for case 1) and 6 (for case 2). Shapes of the curves differ in the two cases.

From (7.3') we deduce:

\[
(7.7) \quad \frac{da_i}{dx} = b-f'(x).
\]
Thus, $a_\ast$ is monotone increasing for case one, whereas it increases to $x_1$, decreases from there to $x_2$, and increases thereafter in case 2 (as we have drawn).

Turning to (7.4'), we see that (generally) in case 2, $a_\ast$ will intersect the $x$ axis. This happens as long as $f'$ is bigger than $r + b$ on some interval (represented in the diagrams as $[x, \bar{x}]$). Implicitly differentiating to find shapes elsewhere, we find

$$da/dx = [Bc''(x) + u'(a)f''(x)]/[(r+b-f'(x))u''(a)].$$

Outside the above mentioned interval the denominator of (7.8) is always negative. The numerator is positive for $x < x_\ast$, and also for sufficiently large $x$ (since $f''$ goes to zero). However in an intermediate range, it is clearly ambiguous. Indeed, we know that for $B$ sufficiently close to $B_\ast$, $a_\ast$ must have a rising segment since at $B_\ast$ the two curves must intersect and $a_\ast$ is strictly positive (and strictly increasing in case 1). Since, in addition, computational simulations generally exhibit an increasing segment, we have drawn the diagrams in that way. This feature will have interesting implications as we will see below.

We have drawn in arrows to indicate the direction of motion off the curves of stationarity as determined by equations (6.3') and (6.4'). For example, since $\partial A/\partial a > 0$. We know that $dx/dt$ is positive above $a_\ast$ and negative below. Similarly we find that $da/dt$ is positive above $a_\ast$ and negative below. Thus we get the indicated qualitative dynamics in the four "quadrants" of our diagrams.

Examining these diagrams, we see that the unique steady state is saddle value stable (this can be demonstrated more rigorously using linearizations, and we will utilize that approach later). Under standard regularity conditions (cf. Hartman (1973, See Chapter IX on Invariant Manifolds and Linearizations)), there will exist a unique one dimensional local stable manifold of points satisfying these differential equations and converging to the steady state. Hence by running time forward and backwards along solutions of the costate/state equations that lie on the stable manifold near the steady state, one may construct an entire global stable manifold. Since the costate variable $p(x)$ must be the derivative of the value function, points $(x,p(x))$ on the global stable manifold give $(x, V(x))$ for a "candidate" value function $V(x)$.

Are there other solutions (besides that lying on the global stable manifold of the unique steady state depicted in figures 5 and 6), of the costate/state equations that might be optimal? This question has not been definitively answered even in the general literature on optimal control. All we know in general is that if there is an optimum it must satisfy the costate/state equations for all $x$. However, arguments parallel to those in Skiba (1978) and Ekelund and Scheinkman (1986) suggest that we look for solutions of the costate/state equations that satisfy the transversality condition at infinity (TVC$\to$).

$$\text{(TVC$\to$)} \exp(-rt)p(t)x(t)-0, t-\to\infty.$$
Solutions of the costate/state equations that lie on the global stable manifold satisfy (7.9) as long as \( r > 0 \). This is because \( p \) and \( x \) converge to finite limits and the negative exponential converges to zero. However, since transversality is not known to be strictly necessary or sufficient here, one generally finds an auxiliary argument to rule out other paths.

If we assume \( c(x) \) increases sufficiently rapidly relative to \( u(a) \) this will eliminate \( (x(t), p(t)) \) such that \( x(t) \to \infty, p(t) \to 0, a(t) \to -\infty \) by a comparison argument. Solutions \( (x(t), p(t)) \) such that \( x(t) \to 0, a(t) \to 0, p(t) \to -\infty \) as \( t \to \infty \) can also be eliminated by a comparison argument provided, \( c(0) = 0 \) and \( u(0) < 0 \). Hence the only candidate solution remaining in this case is the one depicted in the figures. Consequently there is a one dimensional stable manifold which extends over all values of \( x \), converging to the steady state and these paths constitute the optimal transition programs. As mentioned before (and now seen in the diagrams) the steady states must lie in the low-\( x \) basin of attraction and paths converge monotonically in \( x \). What is more surprising and new is that they need not converge monotonically in \( a \). One might expect that if the initial stock is too high relative to the long run optimum, the manager would initially set the loading quite low (thus lowering the stock quickly to benefit Enjoyers who experience increasing marginal cost) and let it rise over time to the long run optimum.

However, this behavior is not generally optimal here, and the reason has to do with the shape of \( f(.) \). When \( x \) is quite large, there is saturation and lowering \( x \) does not give the feedback benefits it will in intermediate ranges. Thus, it may be optimal to start with relatively high levels of \( a \) (possibly even higher than the long run optimum) to benefit Affectors, and lowering these only when the region of maximum feedback benefit is reached.

It is interesting to note that parameters \( r \) and \( B \) affect only the \( da/dt = 0 \) curve which makes for easy comparative dynamics. For example, an increase in the discount rate will raise the \( a_x(.) \) curve but not affect the \( a(.) \) curve. Consequently the long run equilibrium level of \( x \) will rise—that is, with more impatience, the planner worries more about the immediate benefits to Affectors and less to the long run benefits of Enjoyers.

Similarly, we can deduce comparative dynamics by considering (for example) a surprise increase in \( B \) due to a new surge of wealthy recreationists locating on the shores of the lake. This event causes the \( da/dt = 0 \) curve to fall and does not change the \( dx/dt = 0 \) curve. Let the lake be at the old steady state. Draw the old stable manifold on Figure 5 (or 6) and draw the new stable manifold which lies below it. The new optimal loading a drops immediately to locate on the new stable manifold and slowly rises as \( x \) falls to the new, lower steady state level of \( x \). We italicize "rises" because one might expect "a" to keep decreasing for a while. In the real world there would be adjustment costs to rapid change in "a" which may deliver a result more like intuition.
8. MULTIPLE STEADY STATES

Once we pass from the situation of a unique steady state, there generally will be more than one locally stable manifold satisfying all necessary conditions and transversality. Our strategy will be to use phase diagrams of the current value costate/state equations to identify all solutions that satisfy the transversality condition at infinity. Given these, we can use the Hamilton-Jacoby property: \( p(x) = W(x) \) to construct a finite collection of "candidate" value functions \( \{ V_i(x), i=1,2,\ldots,n \} \). The optimal value function is given by

\[
W(x) = \max_i \{ V_i(x) \}.
\]

The points where the maximizing "i" changes constitute the switch (Skiba) points that correspond to those we discussed briefly in the MRAP section. We will try to characterize these later. Generally, each of the "i’s" above will correspond to a locally stable steady state, so first we explore the potential multiplicity of these.

We return to an analysis of the steady state equation:

\[
[r + b - f(x)]u' &= \text{Bc}'(x).
\]

If we let \( R(x) \) represent the function on the left hand side of (8.2), then the argument given in section 5 can be used to show that if (and only if) \( R(\cdot) \) has a monotone increasing segment in an interval where it is strictly positive, a \( c'(x) \) can be found under which there are arbitrarily many solutions to this equation. We demonstrate this somewhat more rigorously here.

Lemma 3:

Assume \( R'(\cdot) \) is continuous and there is an \( x = A > 0 \) such that \( R(A) > 0 \) and \( R'(A) > 0 \). Then there is a convex increasing cost function \( c(x) \) such that (8.2) has countably many positive solutions. Under the plausible right hand and left hand "Inada" conditions \( R(0) > c(0) \) and there is \( x_m \) such that \( x > x_m \rightarrow R(x) < c'(x) \), we have (generically) \( R \) intersecting \( c' \) from above for the first steady state and \( R \) intersecting \( c' \) from above for the last steady state. Hence, generically there are an odd number of steady states if the number is finite.

Proof:

Notice that the only requirement for convexity of \( c(x) \) is that \( c'(x) \) be nondecreasing. Now \( R'(A) > 0 \) implies there is an open interval \( I = (A - \varepsilon, A + \varepsilon) \), \( \varepsilon > 0 \) such that \( R(x) \) is strictly increasing on \( I \). Now construct a nonnegative non decreasing function \( c'(x) \) on \( [0, \infty) \) such that it intersects \( R(x) \) on \( I \) a countable number of times. This can be done because \( R(x) \) is strictly increasing on \( I \). The second part of the Lemma is obvious. Q.E.D.

Of course, assumptions on \( R(\cdot) \) are not fundamental since they are not stated on the primitives of the problem. We now seek such primitive assumptions. Note first that \( R(\cdot) \) is not well defined unless \( bx > f(x) \) so we restrict to cases 1 and 2 for the moment. Further, \( R(\cdot) \) will be
positive if and only if \( r + b > \frac{f(x)}{x} \) so our search will be restricted to intervals where this inequality holds. Assuming only that \( f(\cdot) \) and \( u(\cdot) \) are twice differentiable, \( R(\cdot) \) is differentiable and we have:

\[
R'(x) = [r + b - f(x)]u''(\cdot)[b - f(x)] - u'(\cdot)f''(x).
\]

**Proposition 3:**

There always exists parameter choices consistent with case 2 such that there are a countable number of candidate steady states in a neighborhood of \( x_i \).

**Proof:**

In case 2, \( r + b - f(x_i) > 0 \) and \( b x_i - f(x_i) > 0 \), so \( R(x_i) > 0 \). Further, since \( f''(x_i) < 0 \) and \( b = f(x_i) \), we see that \( R'(x_i) = -u'(\cdot)f''(x_i) > 0 \). The proposition then follows from Lemma 3.

Q.E.D.

Unfortunately, we are unable to get such a clean result for cases 1 and 3. In case 1, the first term in \( R'(\cdot) \) is always negative so the multiplicity result will depend on the relative absolute magnitude of \( u'' \) and \( f' \). For case 3, \( x_i \) lies in the interval where (8.2) cannot have a solution and again, wherever \( R \) is positive and \( f'' \) is negative, the first term in \( R' \) is negative. Clearly, we could state sufficient conditions on \( f'' \) and \( u'' \) that would guarantee multiplicities in cases 1 and 3, but we do not do so explicitly here.

To analyze the behavior of dynamics away from steady states, we now employ the standard linearization methodology. It is convenient to revert to the state/costate formulation but defining \( Q = -p \) so that all variables take on nonnegative values (recall that since the state variable is a "bad" in this problem, the natural costate is negative). In these units, the equations of motion take the form:

\[
\begin{align*}
(8.3) \quad & \frac{dQ}{dt} = (r + b - f(\cdot))Q - Bc'(x), \\
(8.4) \quad & \frac{dx}{dt} = a(Q) - bx + f(x),
\end{align*}
\]

where \( a(Q) \) is implicitly defined by \( u'(a) = Q \).

The linearized dynamics is defined by the jacobian coefficients:

\[
J_{11} = r + b - f(\cdot); \quad J_{12} = -Bc'(\cdot) - u'(\cdot)f'(\cdot); \quad J_{21} = 1/u''(\cdot); \quad J_{22} = f(\cdot) - b.
\]

Let \( |J| \) denote the determinant of \( J \). The eigenvalues are given by solutions to \( |J - \lambda I| = 0 \), and solving this quadratic equation, we find:

\[
(8.5) \quad \lambda = \frac{r \pm \sqrt{r^2 - 4|J|}}{2},
\]

where
\[ |J| = - r(f'-b) - (b-f) + (1/u')(Bc'' + u'''). \]

From this we see that the sign of \(|J|\) is critical to local dynamics. If \(|J|\) is negative then both eigenvalues are real with one greater than zero and one less than zero. This is the case of saddle value stability where we expect a one dimensional local stable manifold converging to the associated local steady state. On the other hand, if \(|J|\) is positive (and \(r\) relatively small) then the roots are imaginary and we expect cyclic dynamics. We will amplify on these possibilities below.

It turns out that for strictly concave \(u(.)\), there is a very simple characterization of the sign of \(|J|\). Let the \(dx/dt = 0\) and \(dQ/dt = 0\) curves be represented by \(Q_0(x)\) and \(Q_1(x)\) respectively. Then we have:

**Lemma 4:**

Assuming \(U'' < 0\), evaluated at any steady state, \(\text{Sign } |J| = \text{Sign } (dQ_0/dx - dQ_1/dx)\).

**Proof:**

From (8.3) and (8.4) we derive:

\[ \frac{dQ_0}{dx} - d \frac{Q_1}{dx} = (b-f)u'' - Bc''/(r+b-f') - Bc'f'/(r+b-f')^2 \]

Consequently, since \(u'' < 0\) and at any steady state, \(r+b-f' > 0\),

\[ \text{Sign } (dQ_0/dx - dQ_1/dx) = \text{Sign } -(b-f)'(r+b-f') + Bc''u'' + Bc'f'/(r+b-f')u'' \].

Then, substituting for \(Bc'/(r+b-f')\) from the steady state condition (8.2) and comparing with the formula for \(|J|\), the lemma follows. Q.E.D.

From this lemma, we can infer the following:

**Proposition 4:**

For strictly concave \(u(.)\), candidate steady states generically will alternate between two types, type 1 where there will exist a one dimensional local stable manifold converging to the steady state and type 2 which are totally unstable. Further the first and last steady states (assuming there are more than one) are of type 1.

**Proof:**

Generically (8.6) will be nonzero and the Sign will alternate between positive and negative at successive intersections. Therefore, from Lemma 4, \(|J|\) will alternate in sign at successive steady states. When \(|J|\) is negative we know we are at a type 1 steady state. When \(|J|\) is positive, we can see from (8.5) that either the eigenvalues are real in which case they must both be positive, or they are imaginary with positive real part–in either case totally unstable. By our assumptions on \(u(.)\), \(f(.)\) and \(c(.)\), \(Q_0(0) < \infty\) and \(Q_0(0) = \infty\), so the first intersection is of type 1 and since we
already established that the number of candidate steady states is generically odd, the last intersection must be of type 1 as well. Q.E.D.

Note that there is some remaining ambiguity concerning the qualitative behavior at type two intersections. If the interest rate is sufficiently small, we can see from (8.5) that eigenvalues will be imaginary and the dynamics must be an unstable spiral. However, unless \( r \) is actually zero, there is the possibility that roots will be real (both positive). We comment on this further when constructing phase diagrams.

Once there are multiple candidate steady states, there are almost certain to be multiple solutions to the necessary conditions that satisfy transversality, at least for some values of the state variable. Each of these can be thought of as a local maximum (or local minimum) of the value function \( W(.) \). The main tool that we have for choosing the global maximum among these is a generalization of that used by Skiba (1978). It is based on use of the Hamilton-Jacoby equation for time autonomous problems:

\[
(8.8) \quad \max H(x,Q(x),a) = rW(x),
\]

where \( Q(x) \) represents the optimal stable manifold. Thus, for any candidate locally optimal trajectory \( Q_i(x), a_i(x) \), we can associate a candidate value function \( V_i(x) = H(x,Q_i(x), A_i(x))/r \) and the global optimum starting at \( x \) will be defined by \( i(x) = \arg \max V_i(x) \).

Proposition 5 (Candidate Value Function Comparison)

Consider two candidate value functions \( V_i(x) \) and \( V_j(x) \). Suppose

\[
(8.9) \quad Q_i(x) \frac{dx_i(x)}{dt} \geq Q_j(x) \frac{dx_j(x)}{dt}
\]

(which obviously holds if \( \frac{dx_i}{dt} = 0 \), then \( V_j(x) \geq V_i(x) \) with strict inequality if \( u(a) \) is strictly concave at \( a_j \) or (8.9) holds with strict inequality.

Proof:

Evaluated at any chosen value of the state variable,

\[
r(V_i - V_j) = u(a_j) - u(a_i) - Q_i \frac{dx_i}{dt} + Q_j \frac{dx_j}{dt}
\]

\[
\geq u(a_j) - u(a_i) - Q_j \{ \frac{dx_i}{dt} - \frac{dx_j}{dt} \}
\]

\[
= u(a_j) - u(a_i) - u'(a_i) \{ a_j - a_i \}
\]

\[
\geq 0,
\]

where the first inequality follows from assumption (8.9) and the second from concavity of \( u(.) \).
If (8.9) is strict, the first inequality is strict whereas the second is strict if \( u(.) \) is strictly concave at \( a \),

Q.E.D.

9. THREE CANDIDATE STEADY STATES.

We know that the simplest generic multiplicity situation will involve three candidate steady states so we take up this situation first. Three candidate steady states can arise in all three of our cases and as in section 5, we label these states \( \alpha, \beta \) and \( \gamma \). We give a relatively complete discussion of case II first and then indicate modifications that must be made in the other cases. There are two types of ambiguity in the qualitative features of phase diagrams that cannot be resolved without further assumptions on functional forms. The first involves the sign of the slope of the stable eigenspace at the largest steady state and the second involves whether or not the two stable manifolds extend through the entire state space.

The first is illustrated by a comparison of Figures 7a-b which depict local dynamics around the largest equilibrium (we draw these in \( a,x \) space for visual convenience). Fig 7a applies when the largest equilibrium occurs in a region where \( da/dt = 0 \) is decreasing whereas, Fig. 7b applies if that equilibrium occurs where it is increasing. The operational distinction between these situations involves whether the optimal loading will be a monotone increasing or decreasing function of the phosphorous stock in this region.

The second ambiguity is illustrated by comparing Figures 8a-c. For Case II we can always associate the lowest equilibrium with an oligotrophic state and the highest with a eutrophic state since the lowest occurs below \( x \) and the highest above \( x \). In Fig. 8a the oligotrophic stable manifold extends over the entire state space, in Fig. 8c, the eutrophic stable manifold extends back to the lowest states and in 8b, both manifolds originate from the unstable equilibrium. Using Proposition 5, we can now state the following:

Proposition 6

(1) If Fig. 8a applies, the oligotrophic state is long run optimal from any starting state. (2) If Fig. 8c applies, the eutrophic state is long run optimal from any starting state. (3) If Fig. 8b applies, there is a Skiba point (S) in the interval \([.\) such that the oligotrophic state is long run optimal from starting points below S and the eutrophic state is optimal for starting points above.

Proof:
To establish (1) apply proposition 5 at any state greater than or equal to \( \gamma \) with \( i \) representing the eutrophic manifold and \( j \) the oligotrophic manifold (\( Q_i < Q_j \) and \( dx/dt < 0 \)) and use connectedness property of optimal paths. To establish (2) apply the proposition to any state less than or equal to \( \alpha \) with the reverse orientation. Statement (3) is obvious since the oligotrophic manifold must be optimal to the left of \([.\), the eutrophic manifold to the right and there can be only one switch point.

\[^3\text{We have drawn these consistent with Fig. 7a but the same analysis applies to Fig 7b.}\]
Actually, we can show that for Fig. 8b, the Skiba point is strictly inside the interval [1] if \( u(.) \) is strictly concave. This is because we can use Proposition 5 in the same way as above to show that the oligotrophic manifold is strictly optimal at the left point of the interval and the eutrophic manifold similarly strictly optimal at the right point.

Obviously, it is quite important for policy to know which of these three situations apply. We can say using a continuity argument that if \( B \) is sufficiently close to \( B^* \) (from below) then situation 8a must apply since at \( B^* \) the oligotrophic manifold is strictly optimal. Beyond this, we are unable to say anything definitive without knowing more about functional form.

We comment briefly on modifications for cases I and III. The story is pretty much unchanged for case I except that there is no obvious way of locating the positions of the and \( \gamma \) relative to \( x_m \) except for \( B \) close to \( B^* \) where again a continuity argument will guarantee that \( \alpha \) is less than \( x_m \). For case III, the \( \frac{d}{dt} = 0 \) curve cuts the horizontal axis so situation 8a cannot hold. This makes sense since if the phosphorous level is sufficiently high we know there is no policy that will return us to the oligotrophic state. However, either of the other situations could apply.

10. FIVE OR MORE CANDIDATE STEADY STATES

The situation with more than three multiplicities and be analyzed in much the same way as when there are three, using proposition 5. In a sense (except in case III), all possibilities can occur. That is (for five candidate steady states) there can be zero, one or two Skiba points. When there are zero, any of the three locally stable manifolds can be the global optimum and when there is one, it can occur between the first and second stable candidates or between the second and third and any pair of manifolds can have basins of attraction. We illustrate in Fig. 9a-c by drawing the dynamics (this time for case I) in which the middle locally optimal steady state is always long run optimal (9a), where the first and third have basins of attraction (9b) and where all locally stable steady states have a basin of attraction (9c). We leave it to the reader to fill in other cases.

One could proceed formally by induction to analyze seven or more candidate steady states and show that again, all combinations remain possible but we leave this exercise to the sufficiently interested reader.

11. Conclusions

In this paper, we have characterized the types of behavior that can be optimal in the lake model of Carpenter et al (1997) and shown how these behaviors change as functions of key parameters such as \( B \). However, particularly in cases where there are multiple candidate steady states we are unable to resolve some key ambiguities such as the existence and location of Skiba points and the resulting sizes of basins of attraction. Unfortunately, these ambiguities seem to be
characteristic of the class of nonconvex problems of which ours is an example. When there are multiple local optima, it is rarely possible to choose the best among them except by direct numerical comparison.

A second ambiguity that we have uncovered involves the number of candidate steady states. Drawing intuition from the constant loading model, we might expect at most two basins of attraction and it is not clear whether the possibility of more than two is likely to be observed. We know that for some standard functional forms for \( f(.) \) and \( c(.) \) there will not be more than two such basins but whether or not such behavior is robust or not is an open question.
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lake model -- case 1: unique equilibrium with constant loading

\[ bx - a \]

\[ f(x) \]

\[ f'(x_m) \]

\[ b > f'(x_m) \]

Phase diagram (\( B > B^* \) = unique equilibrium)

\[ x = 0 \]

\( a_1(x) \)

\( a_2(x) \)

\[ q = \cdot \]

\[ x = \cdot \]
lake model -- case 2: 2 basins of attraction with reversibility (constant loading)

**Fig 2**

\[ f(x) \]
\[ bx - a \]
\[ f(x) < bx \quad \forall x \]
\[ f'(x) = b < f'(x_m) \]
\[ \frac{1}{x} = \frac{1}{x_m} = 0 \]

low basin stable equilibrium

high basin stable equilibrium

**Phase diagram (B > B^* - unique equilibrium)**

**Fig 6**

\[ q_2(x) \]
\[ (\dot{a} = 0) \]
\[ q_1(x) \]
\[ (\dot{x} = 0) \]

\[ \gamma_2(x) \]
\[ (\dot{a} = 0) \]
potentially optimal manifold
that cannot be optimised

Fig 8a

Fig 8b

Fig 8c