SOME MATHEMATICAL TOOLS FOR ANALYSING COMPLEX-NONLINEAR SYSTEMS

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These notes sketch some basic structure of dynamical systems theory that will be useful for analyzing the models that appear in this book as well as extending results in this book and elsewhere. There are three types of basic dynamical systems that get used a lot in applied science: Flows (differential equations), Maps (difference equations), Recursive Estimators and Stochastic Approximation Theories.

For recursive estimators and stochastic approximation theories, there is an associated ordinary differential equation (ODE) that governs the limiting behavior. A general theorem of Benaim and Hirsch (cf. Fudenberg and Levine (1998)) states that if the associated ODE is "Morse/Smale" (this roughly means that the limit behavior of the ODE is a finite number of steady states and periodic cycles) then the associated ODE essentially controls the limiting behavior. See Fudenberg and Levine (1998) for references and uses of this type of work. There are ugly examples where this type of control of the limiting behavior of the dynamical system by the associated ODE is lost in "non Morse/Smale" cases. This kind of theory is used a lot in economics, especially in the modeling of dynamic learning and adaptation. See Fudenberg and Levine (1998) for typical examples of this type of theory in use. We shall concentrate on differential equations and difference equations here.

It is a common strategy to use the "genericity dogma" to cut a swath through the mind boggling complexity of this subject. Genericity is a precise way to formulate the notion of "hairline" cases. A "generic property" is "one which holds for most functions in the function space under consideration." (Robinson (1995, Chapter X.).)

We illustrate a use of the "genericity dogma" by one of the classical textbooks in dynamical systems theory. Guckenheimer and Holmes (1983, p. 149) state that "...all bifurcations of one-parameter families at an equilibrium with a zero eigenvalue can be perturbed to saddle-node bifurcations. Thus one expects that the zero eigenvalue bifurcations encountered in applications will be saddle-nodes. If they are not, then there is probably something special about the formulation of the problem which restricts the context so as to prevent the saddle-node from occurring...."

We sketch below (up to "genericity") "all" of the "local" bifurcations you can get for the one parameter case for Flows and Maps. This sketch will follow Kuznetsov (1995) and Robinson (1995). Global bifurcations, such as homoclinic bifurcations, will be treated later. While concepts such as homoclinic bifurcations may seem arcane, the article by Brock and Hommes (1997) shows how homoclinic bifurcations arise naturally when there's a coevolutionary tension between expensive rational expectations and cheap naive expectations. Scheffer (1998) shows how various bifurcations, including homoclinic bifurcations, arise naturally in the context of aquatic ecological models which play a central role in the current volume.

We shall indicate below how one deals with two and higher dimensional parameter cases.

These notes are organized as follows. First we review local bifurcations of flows and maps up to their normal forms. Second, we discuss the notion of fast and slow variables and the center manifold idea which is one way of making this notion precise. Third, we present a template for conducting an analysis of a dynamical system. Fourth, we give a brief discussion of
singular perturbation theory which is a precise way of doing a simplified approximate analysis of dynamical systems with slow and fast variables. This discussion gives a brief treatment of both deterministic cases and stochastic cases. Fifth, we give an analysis of global bifurcations with most attention given to the homoclinic bifurcation which arises naturally in dynamical systems adaptive agent modeling of information paradoxes and related informational/learning phenomena in economics. Sixth, we give a brief discussion of applications of these tools to time stationary costate/state dynamical systems arising from infinite horizon optimal control problems with time-invariant objective, dynamics, and discounting. Seventh, we give a brief and intuitive discussion on how one might apply some of these tools to study infinite horizon optimal control problems with time-invariant objective, dynamics, and discounting, but where the dynamics are now infinite dimensional (e.g. governed by partial differential equations). Eighth, we include a summary and suggestions for future research section that contains some very speculative ideas for future research. Before we begin, we emphasize that these notes are written to help readers use these tools to formulate useful optimal management problems and to produce insights from these problems with a high degree of confidence, but not the perfect degree of confidence that a rigorous treatment would provide. Many references are given to the literature to increase the user's degree of confidence to the level obtained from a rigorous mathematical treatment.

1. FLOWS AND MAPS: ONE DIMENSIONAL PARAMETER CASE

Flows are differential equations

\begin{equation}
\frac{dx}{dt} = f(x, \alpha), \quad f: \mathbb{R}^n \to \mathbb{R}^n, \quad \alpha \in \mathbb{R}.
\end{equation}

Steady state \( \tilde{x}(\alpha) \) solves

\begin{equation}
0 = f(\tilde{x}(\alpha), \alpha).
\end{equation}

The linearization of (1.1) at steady state \( \tilde{x}(\alpha) \) is given by,

\begin{equation}
\frac{dy}{dt} = J(\tilde{x}(\alpha), \alpha)y, \quad J(\tilde{x}(\alpha), \alpha) = \frac{\partial f}{\partial x},
\end{equation}

evaluated at (\( \tilde{x}(\alpha), \alpha \)). Calculate the eigenvalues of the Jacobian matrix J. You'll get \( n(+) \) with positive real part, \( n(-) \) with negative real part, \( n(0) \) with zero real part. The hyperbolic case is when \( n(0)=0 \). Generically you'll get at most two eigenvalues with zero real part (a pair of complex conjugates).

Up to the linear level the qualitative behavior of (1.1) is given by putting (1.1 L) into Jordan canonical form. To do this locate the eigenvalues of J and the corresponding space of eigenvectors for each eigenvalue. In the case where all eigenvalues are real and all are unique (no repeated eigenvalues) you'll get J into diagonal form when you do this. Just write out the equations that define eigenvalues and stack them to locate the change of units matrix that puts J into diagonal form. In the case where all eigenvalues are unique and some are complex, you reduce J into a form with 2x2 matrices on the diagonal for each pair of complex roots. In general you can reduce J to a block diagonal matrix and that's what is called the Jordan canonical form.

This kind of reduction gives you the local qualitative behavior in the hyperbolic case when the eigenvalues of J are unique and all have nonzero real parts. We are interested in the taxonomy of local behavior at bifurcation values of alpha up to "hairline" cases. These notes rely on the
references below to give you a catalog up to "hairline" cases. We shall use
the "genericity dogma" to ignore "hairline" cases. The definition of
"hairline" will depend on the class of perturbations that are used. In the
definition of "hyperbolicity" the $C^1$-class of perturbations was used ($g$
is close to $f$ if $|f-g|_0 + |f-g|_1$ is small where $|h|_1 = \sup (|D^r h|)$. Here $D^r h$ denotes
the $r$'th derivative of $h$. In general $C^r$ close demands that the functions and
their first $r$ derivatives be close.

One must be careful when using the "genericity dogma" in economics. Do
not throw out homogeneous of degree one production functions, homogeneous of
degree zero demand functions, and excess demand functions that satisfy Walras
Law and homogeneity of degree zero. Use the "economic restrictions" to
"reduce" these functions (for example, choose a "numeraire" and measure
everything relative to that numeraire) and apply the "genericity dogma" to the
reduced systems.

Also one must be careful in applying genericity theory in ecology.
Consider Murray's (1993, p. 100) discussion of synchronized insect emergence
in the cases of 13-year locusts and 17-year cicadas. These models have enough
"special" structure that formulation of a proper notion of genericity takes
some skill. Despite these caveats: notions of genericity are very useful in
prioritization of cases for analysis if done properly.

MAPS

The word "Maps" ("flows") is just a synonym for "difference equations"
(differential equations) in many references. We use these words as synonyms
here. Consider,

\[(1.2) \quad x(t+1) - x(t) = f(x(t), \alpha). \quad \text{(Also written } x \mapsto x + f(x, \alpha))\]

A bifurcation point $\alpha = \alpha_0$ is a point where structural stability is lost. It is
signalled by the real part of an eigenvalue of the Jacobian matrix passing
through zero for flows and passing across the unit circle for maps. Replace
"crossing negative real-part half of complex plane" with "crossing boundary of
unit circle in complex plane" to carry analysis forward from flows to maps.
Rescale, WLOG, so that bifurcation takes place at steady state $x=0$, $\alpha = 0$.

CENTER MANIFOLD IDEA

The general one parameter $n$ dimensional case reduces generically to the
one parameter two dimensional case by the following device. Look at the
linearization for flows. I use "root" interchangeably with "eigenvalue." Generically at a bifurcation point you'll have one real root that's zero or a
pair of complex roots with zero real parts. All other roots will have
positive real part or negative real part. One can locate changes of units and
a new state space so that local analysis of bifurcations reduces to analysis
in a new one dimensional state space for the real root (a new two dimensional
state space for the case of two complex roots). Intuitively these spaces are
spanned by the eigenvector for the real root and the real and complex parts of
the eigenvector for one of the complex roots in the case of complex roots with
zero real part.

Intuitively, the center manifold "trick" "works" because one imagines
rescaling time so that motion on the space spanned by eigenvectors of nonzero
real part roots is so fast relative to motion on the space spanned by
eigenvectors of the zero real part roots that all local qualitative analysis
is captured by restriction to the space spanned by zero real part eigenvectors
(Kuznetsov (1995, pp. 138-9)).
Turn now to an intuitive approximation strategy used in applied mathematics that is related to center manifold theory in rigorous analysis. Suppose there are two scales of time, "slow" and "fast." It is sometimes a useful approximation to assume the fast variables have already "equilibrated" for each value of the slow variable. For example, suppose $x_2$ is fast. Let $\epsilon$ be a small positive number and consider the pair of differential equations

\[ (1.3) \quad dx_1/dt = f_1(x_1, x_2), \quad \epsilon dx_2/dt = f_2(x_1, x_2), \quad x_1(0) = x_{10}, \quad i=1,2. \]

Replace the differential equation system above with the system

\[ (1.4) \quad dx_1/dt = f_1(x_1, x_2(x_1)), \quad 0 = f_2(x_1, x_2(x_1)), \]

where the solution $x_2(x_1)$ is produced by the implicit function theorem. Under regularity conditions, and by considering "inner" and "outer" solutions and "matching" them (Murray (1993, p. 113-114.)), one can expand the system (1.3) in a "Taylor" series around $\epsilon = 0$ and deliver accurate approximations. Indeed the system (1.4), which we call the "$x_2$-slaved" system (since $x_2$ is "slaved" by the current value of $x_1$), is a good approximation to the system (1.3) outside any fixed $\delta$-neighborhood provided that $\delta$ is big enough relative to $\epsilon$ and $\epsilon > 0$ is small enough. Center manifold theory can be viewed as a way of making the mathematics of these intuitive procedures more precise.

Center manifold theory is used to reduce the general $n$-dimensional state space problem to a one or two dimensional state space problem. Therefore our classification problem for one dimensional parameters and $n$-dimensional state spaces reduces to the classification problem for one dimensional parameters and 1 or 2 dimensional state spaces depending on whether the bifurcation eigenvalue is real or complex.

We now observe that there are two classes of one parameter bifurcations for flows: (i) A real root passes through zero, (ii) A pair of complex roots have real part which passes through zero with positive speed (Andronov/Hopf bifurcation).

For maps we have three classes: (i) A real root passes through +1 ("fold" bifurcation); (ii) A real root passes through -1 ("flip" bifurcation); (iii) A pair of complex roots pass across the unit circle with positive speed ("Nelmark/Hopf" bifurcation).

For each of these, there may be either "soft" (also called "supercritical") loss of stability or there may be "hard" (also called "subcritical") loss of stability. The soft loss of stability is a case where small changes in $\alpha$ through the bifurcation value $\alpha_0$ do not cause big moves in the state variable. I.e., there is not a "blue sky" i.e. "catastrophic" move in the state as there is in hard loss of stability. Hard loss of stability leads to "hysteresis cycles" which are explained below.

Consider for example, a pair of differential equations with $\alpha$ as slow variable and $m$ as fast variable,

\[ (1.5) \quad \epsilon dm/dt = T(m, \alpha) - m; \quad d^2\alpha/dt^2 = P(\alpha, dm/dt), \quad 0 < \epsilon << 1. \]

Let both $m$ and $\alpha$ be one dimensional. Notice that the differential equation for $\alpha$ in (1.5) is second order so that a cycle is possible. Let the solution for $\alpha$ raise and lower $T$ over the cycle so that there is one equilibrium when $\alpha$ is low, three at medium level, and one when $\alpha$ is high. Start the system at a low level of $\alpha$ and follow the steady state $m(\alpha)$ on the fast dynamics. Notice
that the system "snaps" to the high steady state value of \( m \) when the lower one is "lost" due to a "flip" as the cycle slowly raises \( T \). Notice how the system stays at the high level of \( m \) as the cycle slowly lowers \( T \) back to its old level. As \( T \) slowly falls further until the high steady state is lost, we finally get a "snap" back to the old low level area of \( m \)-space.

This is a very simple example of a "biological oscillator" (Murray (1993, Chapter 6, Figure 6.6a,b)). We have already seen this device used in this book in the analysis of lake systems with a slow variable "mud" and a fast variable "phosphorous sequestered in algae." Economic examples include "switching" type business cycle models with duration dependence when the \( m \)-part of (1.5) is replaced by two dynamics (one dynamic for recession and another dynamic for expansion), noise is added (which may have different conditional variances), and the "\( \alpha \)-part" is some slow moving variable (e.g., a set of "institutions", "business styles", "expectations and confidence," "habits" slowly adapting to a recessionary phase or slowly adapting to an expansionary phase). Turn to normal forms for deterministic flows and maps.

By changes of units and/or time rescaling, under "nondegeneracy conditions" which "generically" hold, each bifurcation may be locally represented by a dynamical system called the "normal form." We follow Kuznetsov's (1995) exposition of normal form theory here.

NORMAL FORMS FOR FLOWS

Normal forms for each bifurcation type for flows are given below. Without loss of generality we assume the steady state equals zero and the bifurcation value \( \alpha_0 = 0 \). There are only two "generic" one parameter bifurcations for flows: (i) Fold and (ii) Hopf

FOLD BIFURCATION (A real root of the Jacobian passes through 0 at \( \alpha_0 \) at positive speed. Also called "saddle-node."")

NORMAL FORM: \( \frac{d\eta}{dt} = \beta \eta^2 \)

HOPF BIFURCATION (A pair of complex roots has real part which passes through zero, (but complex part remains nonzero) at positive speed.)

NORMAL FORM:

\[
\begin{align*}
\frac{d\eta_1}{dt} &= \beta \eta_1 - \eta_2 \\
\frac{d\eta_2}{dt} &= \eta_1 + \beta \eta_2 \\
\end{align*}
\]

\[= \pm \left( \eta_1^2 + \eta_2^2 \right) \eta_1 \]

NORMAL FORMS FOR MAPS

There are three "generic" one parameter bifurcations for maps: (i) The Fold (a real eigenvalue passes through +1 at positive speed); (ii) The Flip (a real eigenvalue passes through -1 at positive speed); (iii) The Neimark/Sacker, also called "Secondary Hopf" (a pair of complex eigenvalues pass across the unit circle in the complex plane at positive speed).

The normal form for the Fold is the same as for flows when the map is written in first difference form.

NORMAL FORM FOR FOLD: \( \eta_{t+1} - \eta_t = \beta \eta_t^2 \)
NORMAL FORM FOR FLIP: \( \eta_{t+1} = -(1+\beta)\eta_t + \eta_t^3 \)

NORMAL FORM FOR NEIMARK/SACKER: The normal form theory is similar to that for flows but is more delicate. The formulae are messier and non "resonance" conditions must be imposed. The theory is workable, however. See Guckenheimer and Holmes (1983, p. 161) for the messy details.

GENERAL PRINCIPLES

Let us now pause, recap, and state general principles in laying out an analysis of a dynamical system.

In a subject like economics or ecology it is useful to stratify your variables according to the natural stratification of time scales. It is also helpful to write out a path diagram and label which variables on it are "fast" and which are "slow". See Puccia and Levins (1985) for a large catalog of path diagrams in ecology. Parenthetically we remark that path diagrams (called "signed directed graphs" SDG's by Logofet (1993, p. 100, Figure 10)) only give qualitative information which is not nearly enough to characterize quantitative properties like stability. Available conditions on a matrix A for its SDG to characterize stability are very strong (cf. Logofet (1993, Chapter 5)). Indeed research programs in both ecology and mathematical economics that attempted to investigate local asymptotic stability by exploiting sign pattern information of the Jacobian matrix, A, of the linearized system at a stable state have had very limited success. See Logofet (1993), especially Chapter 5 for a review of results in both literatures. However, path diagrams are useful for depicting the interaction structure.

Returning to stratification of time scales, rank these "lumps" of time scales from fastest to next fastest to...next slowest to slowest. Use variables on the slow scale above each scale as the slow moving bifurcation parameter \( \alpha \). Similar stratification should be done on spatial scales as well when dynamical systems are written for "spatial sites" and coupled across spatial sites in model building. In this way one gets a hierarchy of time scales to structure one's analysis.

We shall talk later about bifurcation classification analysis for cases where \( \alpha \) is k-dimensional for k>1. It will be seen that one can go much further with bifurcation analysis for k=1 than one might think by applying one dimensional parameter theory to arcs through k-dimensional parameter space.

We may now lay out a fairly general dynamical systems analysis for flows. Follow these steps. First, find the steady states, and linearize the system. If the state space is two dimensional or below, sketch a phase diagram and sketch the null clines on it. The "null cline" for a differential equation is the set of points where the rate of change for that equation is zero. The steady states are where the null clines cross.

Second, obtain the Jacobian matrix of the linearized system, and its eigenvalues and eigenvectors. Locate values of \( \alpha \) where zero real parts occur. Third, at these bifurcation values locate which of the generic class of bifurcation types is pertinent. Sketch out the normal form for the pertinent type and phase diagram it.

Fourth, ascertain for each bifurcation value whether the loss of stability is "soft" or "hard".

Fifth, do a global analysis. For global analysis one looks first for the possibility of cycles. Hopf cycles are produced locally. For each cycle construct a Poincare' section (this is an n-1 dimensional manifold, "P," typically an n-1 dimensional "plane" that is transverse to the cycle), the
Poincare' map (i.e. the first return map $M$ of points near that cycle on $P$ to $P$ induced by the flow), and apply the analysis of maps to ascertain the stability type of that cycle. Then do a bifurcation analysis of the cycle following the analytic program for maps. Both rest points and limit cycles are examples of invariant sets under the dynamics. A related program of analysis can be carried out for more complicated invariant sets such as tori and "strange" attractors. See Robinson (1995, Chapter 7) for the details.

Except for global phenomena like homoclinic bifurcations, this template is a fairly complete analytic outline for the case of one dimensional bifurcation parameters. The reader is advised to see Kuznetsov (1995) Chapters 8 and 9 for two dimensional bifurcation parameters as well as a discussion of higher dimensions. An important tool used to get results for higher dimensional bifurcation parameters is the generalization of one dimensional results along one dimensional arcs in parametric bifurcation space.

SLOW/FAST VARIABLES

As we have seen in other chapters of this book, ecological analysis puts a lot of emphasis on slow and fast variables. We give a brief guide to the literature on this topic here. We follow Murray (1993, p. 112-114) to give a short exposition of deterministic singular perturbation theoretic methods of dealing with (1.3). Then we follow Skorokhod (1989) for a brief treatment of the stochastic case.

Suppose one conjectures an analytic solution to (1.3) of the form

$$x_i(t, \epsilon) = \sum_{n=0}^{\infty} x_{1n} \epsilon^n, \quad i=1,2,$$

where the sum runs from $n=0,1,2,\ldots$.

Substitutes (1.6) into (1.3) and equates powers of $\epsilon$ to obtain a series of differential equations. One obtains for $\epsilon=0$,

$$\frac{dx_{10}}{dt} = f_1(x_{10}, x_{20}), \quad 0=f_2(x_{10}, x_{20}), \quad x_i(0)=x_{1i}, \quad i=1,2.$$

Notice that (1.7) is the same as the $x_2$-slaved system (1.4) above. It is clear that (1.7) will not satisfy the initial conditions $x_i(0)=x_{1i}$, $i=1,2$, except in exceptional cases because the second equation in (1.7) is an algebraic equation, not a differential equation. In order to see immediately the source of the problem suppose the implicit function theorem can be applied to produce a local solution $x_2(x_1)$ such that $0=f_2(x_1, x_2(x_1))$. It would be a lucky accident if $0=f_2(x_{10}, x_2(x_{10}))$ and $x_2(x_{10})=x_{20}$. To put it another way, this trouble is signalled by the loss of a derivative in (1.3) when $\epsilon=0$. Hence we can not expect to get a uniformly valid approximation out of (1.7) for the solution of (1.3). General problems of the form (1.3) when a small parameter $\epsilon$, $0<\epsilon<<1$, multiplies a derivative are called singular perturbation problems. To put it yet another way, if the order of the system drops when $\epsilon=0$, one can not expect the initial conditions to be satisfied.

Suppose we use a time scale of $s=t/\epsilon$ near $t=0$ so that $\epsilon dx_2/dt = dx_2/ds$.

Then a small domain in $t$ near 0 becomes a large domain in $s$ as $\epsilon-->0$. Murray (1993, p. 115) shows how to adapt the expansion procedure (1.6) by using time scale $s$ to get a good approximation for the "boundary layer" near $t=0$. He then shows how to "paste" this approximation together with the approximation from (1.6) which is good outside a neighborhood of $t=0$ to get a good approximation overall. For larger times $t$, one just uses (1.7) as an
approximation, (which is called the nonsingular or outer solution) since for 0<ε<<1, the second equation of (1.3), εdx_2/dt=f_2(x_1,x_2), has almost been forced to equal zero. i.e. εdx_2/dt=f_2(x_1,x_2) moves rapidly in time from its initial condition, x_2(0)=x_20, to x_2(x_1) such that f_2(x_1,x_2(x_1))=0 treating x_1 as approximately constant. This is why the x_2-slaved system (1.4) is a good approximation outside any δ-neighborhood of t=0 provided δ is large enough relative to ε>0 and ε is small enough. It is beyond the scope of these notes to go into further detail here. Murray (1993) not only has an excellent treatment but also applies the method throughout his book.

Turn now to extensions of singular perturbation theoretic stochastic approximations to the stochastic case. We follow Skorokhod (1989, Chapter II). Consider the system

\[ dx_1 = f_1(x_1, x_2) dt + b_1(x_1, x_2) dz_1, x_1(0) = x_{10}, \]

\[ dx_2 = \left(1/\epsilon\right)f_2(x_1, x_2) dt + \left(1/\epsilon^{1/2}\right)b_2(x_1, x_2) dz_2, x_2(0) = x_{20}, \epsilon > 0 \]

where z_1 are standized wiener processes (i.e. the mean of each dz is zero and the variance is dt), and the stochastic differential equations (1.8), (1.9) are Ito equations. Notice the scaling of mean and variance in ε in (1.9). We seek the stochastic analog of approximation (1.7) above. Consider the special case where f_2, b_2 in (1.9) do not depend upon x_1. We need the following "auxiliary" stochastic differential equation.

\[ dy = f_2(y) dt + b_2(y) dw, y(0) = x_{20}. \]

Suppose we place sufficient conditions on (1.10) to obtain a unique ergodic invariant measure, ρ(dy). For the case b_2=0, this invariant measure is just a deterministic steady state, y*, of (1.10). Skorokhod (1989, p. 138) locates sufficient conditions for system (1.8), (1.9) such that x_1 converges in distribution to the process X_1 that solves

\[ dX_1 = F_1(X_1) dt + [B_1(X_1)]^{1/2} dz_1, X_1(0) = x_{10}, \]

where F_1(X) = ∫ f_1(X, y) ρ(dy), B_1(X) = ∫ b_1^2(X, y) ρ(dy).

Let us explain (1.11) a bit more. In the case that ρ(.) is a point mass at a steady state y* of (1.10) (the case b_2=0), we have F_1(X) = f_1(X, y*), B_1(X) = b_1^2(X, y*). This is a natural generalization of the "x_2-slaved" system (1.4) above. Let us explore it. For the special case b_1(X, y*) = b_1(y*) constant in X, the invariant measure of (1.11) (Bhattacharya and Majumdar (1980)) is given by

\[ \Pr(X = X) = \exp\left(2/b_1(y*) \right)^2 \int_{X} F_1(x) dx / Z, \]

\[ Z = \int \Pr(y) (2/b_1(y*) \right)^2 \int_{X} F_1(x) dx dy. \]

It is worthwhile to make a quick side comment about simulated annealing using (1.12). Notice that as b_1→0, the probability mass in (1.12) converges to X* = argmax{∫_{X} F_1(x)}. This is related to simulated annealing theorems which
locate sufficient conditions on noise perturbed gradient processes,

\[ (1.13) \quad dX = VU(X)dt + \sigma(t)dZ, \quad dZ \text{ Wiener}, \]

such that the solution \( X \) to the Ito stochastic differential equation converges in some sense to \( X^* = \arg\max \{ U(X) \} \) as \( t \to \infty \). If \( \sigma \) were constant, formula (1.12) shows immediately that \( \{X\} \) must converge to \( X^* \) if \( \sigma \) is taken to zero. It takes a lot more work to find the rate at which \( \sigma(t) \) must be taken to zero with \( t \) to get convergence, but we have indicated intuitively why such theorems might exist. This type of "simulated annealing" process prevents the gradient process (1.13) from getting trapped at local maxima. Return now to the main theme of approximating fast/slow variable stochastic systems with analogs of \( x_2 \)-slaved systems (1.4).

One can follow Bhattacharya and Majumdar (1980) and compute invariant measure formulae for the general case which look rather like (1.12) but messier. The important thing for us to note is the analytical tractibility of the invariant measure for the \( x_2 \)-slaved system which, in turn, is a usable approximation for the original system under Skorokhod's conditions. This device of approximating by the \( x_2 \)-slaved system is handy in some applications. The general case (1.8) and (1.9) is treated in Skorokhod with analogous results but more complicated formulae. Since the point of these notes is to survey useful mathematical tools for the working ecologist and economist, we refer the reader to Skorokhod for the details. Turn now to a discussion of global bifurcations for deterministic dynamical systems.

GLOBAL BIFURCATIONS

An appendix gives a brief review of global bifurcations. An important global bifurcation is the homoclinic bifurcation. The paper by Brock and Hommes (1997) shows how homoclinic bifurcations arise naturally in information paradox problems in economics. Let us explain the logic in some detail because we later indicate how the same logic may appear in a general class of complex adaptive evolutionary dynamical systems models and, potentially, in some other dynamical decision making settings.

Many dynamic information collection problems in economics have the following character. If no one collects the information then it is worthwhile for me to collect it. But if everyone collects it, it is not worthwhile for me to collect it. I.e. in a game theoretic context there is no equilibrium in pure strategies. In the Brock/Hommes framework, hereafter, BH, the context is expensive but "rational" expectations (the standard rational expectations theory) co-evolving with less "rational" but cheaper expectations. Consider, for example, the parable of a hog market. If everyone in a hog market pays a fee and gets rational expectations, I can "informationally free ride" on the work of the others because the market reveals the value of my hogs. Hence, I have an incentive not to collect the information to form rational expectations. I just guess next period's price, I expect that I will get last period's market price for my hogs. If everyone else is using last period's price to forecast this period's price, then it is worthwhile for me to pay resources and find out the determinants of next period's demand for hogs. Now let us expose when homoclinic tangencies and homoclinic bifurcations might appear.

Suppose the hog market is locally unstable at equilibrium when all farmers are using naive expectations, i.e. they use last period's price to predict next period's price. This is the classic cobweb instability which happens when supply is more elastic than demand at the equilibrium point.
Suppose all are using naive expectations and the system is slightly off the equilibrium. Then a small cobweb instability exists and gradually builds up to bigger oscillations.

At some point in time, depending on the size of the cost that must be paid to obtain rational expectations, farmers will regret not having paid the fee to obtain rational expectations (perfect foresight in this case). I.e. as the oscillations increase in amplitude the net fitness measure for rational expectations increases and this causes more farmers to switch to rational expectations. The intensity of choice measures how sensitive farmers are to differences in fitnesses. Hence as the intensity of choice increases the unstable manifold of the co-evolving evolutionary dynamical system described by the population of farmer expectational types moves towards the stable manifold and complex dynamics associated with homoclinic tangencies and bifurcations arise.

The population dynamics ends up going through phases where the system is near equilibrium and most of the population ends up reverting to naive expectors causing instability to slowly build up to the point where rational expectors begin to appear. The system is driven back towards equilibrium when the population fraction of rational expectors gets large enough. The phase lengths follow complex dynamics and some properties of these complex dynamics are described below.

Let us use this story to speculate about complex adaptive systems in general. Consider any dynamic evolutionary system that has the following six properties:

(i) There is a set of strategies which we shall call "simplistic" strategies which do almost as well on measures of net fitness as another set of strategies which we call "sophisticated" strategies, if the system is near a steady state;

(ii) If most of the population is using simplistic strategies, steady states are locally unstable;

(iii) Net fitnesses of sophisticated strategies exceed those of simplistic strategies if the instability becomes large enough per time step;

(iv) If most of the population is using sophisticated strategies, the system moves towards a steady state;

(v) Population dynamics of organism types evolve according to relative fitnesses;

(vi) There is a tuning parameter $\beta$ to play the role of bifurcation parameter where population movement towards higher fitnesses is quicker when $\beta$ increases.

Let us reason about this system. If it is near a steady state, organisms are less fit in net terms if they pay costs of carrying sophisticated strategies, when there is no reward for doing so. Hence the population evolves towards simplicity. But now instabilities in the dynamics begin to grow. At some point these instabilities cause enough movement in the dynamics per unit time that it is worthwhile to pay the costs of sophisticated strategies. Now organisms that use sophisticated strategies are more fit in net terms in this fast moving world. But as most of the organisms become sophisticated this causes the dynamics to move towards a steady state. Once the dynamics gets close to a steady state they slow down and barely move. The scenario repeats. The importance of $\beta$ is this. As $\beta$ increases organisms move faster towards the most fit strategy. This action considered together with
the forces behind the phases described above is linked to moving the stable and unstable manifolds of the dynamics until they either become nearly tangential at which point much of the complex dynamical phenomena of a homoclinic bifurcation appears (Brock and Hommes (1997)).

Another potential example which is a subject for future research is the following. Consider a manager of an ecosystem who must forecast its state variable and take action that influences the dynamical time evolution of that state. She faces two strategy choices. The simple strategy is to predict that next period's state is the same as last period's state and take her action accordingly. The sophisticated strategy is to take a lot of observations on various relevant quantities (perhaps by hiring specialist scientific teams) and obtain a prediction of next period's state that is more accurate. This costs a fee $F$ per period. Meanwhile the system's underlying dynamics are unknown to her and keep changing in ways that keep her from using a simple running history to fit a model and predict the ecosystem state. Call the first strategy "passive" management and the second one "active" management. Let her keep a fitness measure on each management type. Let her use a random utility model with intensity of choice $\beta$ to choose a management style each period with the deterministic part of the utility set equal to the fitness measure. If $\beta=\infty$ she always chooses the management style with current best fitness measure. If $\beta=0$ she puts probability 1/2 on each choice.

Let us reason through how this coupled human and eco-system dynamic is likely to behave. If the system is near a steady state it is not changing much and will continue to change very little. Hence the fitness record for active management is going down relative to passive. So she starts choosing passive most of the time if $\beta$ is large (all of the time if $\beta=\infty$). Suppose that the action she chooses while passive leads to unstable dynamics (due, perhaps to a slow variable that passive managers find too expensive to monitor). Thus instability builds up and continues to build up until the performance of the passive strategy becomes so bad that it would have paid to employ the active strategy each period even though it costs $F$ per period. Now while instability is raging the manager is using active mode most of the time if $\beta$ is large (she is in active mode all the time if $\beta=\infty$). Now the action chosen is driving the system towards a stable state and the scenario repeats. This system looks enough like the Brock and Hommes (1997) system that one might expect rather similar behavior.

We have given examples of homoclinic bifurcations in adaptive learning systems in economics. Homoclinic bifurcations play a prominent role in ecology also. See Scheffer's book (Scheffer (1998)) on shallow lake ecology for examples.

An appendix to these notes gives a more detailed explanation of homoclinic bifurcations and their consequences. Turn now to a brief discussion of the application of these tools to optimal management problems.

2. APPLICATIONS TO OPTIMAL CONTROL AND DYNAMIC GAMES

Many of the chapters in this book derive the dynamical system under study from an optimal control problem or a dynamic game. If one works in the costate/state space generated by the first order necessary conditions for optimal control, then one can do a local analysis and look for bifurcations w.r.t. a parameter by following the template above. We follow Brock and Malliaris (1989, Chapter 5), Cass and Shell (1976), and Carlson, Haurie, and Leizarowitz (1991) to do a brief sketch of approaches to stability analysis of optimal control systems here.

Let $\rho$, $W(x)$, $q=q(x)=\partial W(x)/\partial x$ denote the rate of discount on future payoff in the optimal control problem, the current state valuation function of the state vector $x$, and the (current value) co-state vector. I.e. letting $x$, $v$
denote n-dimensional state and m-dimensional control vectors, we have

\[(2.1) \quad W(x) = \max \{ \int \exp(-\rho t) U(x(t), v(t)) dt \mid \text{s.t. } dx/dt = T(x(t), v(t)), x(0) = x, \}\]

where the maximum is taken over the set of admissible control functions \(v(.)\) (usually taken to be the set of measurable controls or the set of piecewise continuous controls) and the integral \(\int\) is from \(t=0\) to \(t=\infty\). Let

\[(2.2) \quad H(q, x, v) = U(x, v) + qT(x, v), \quad H^0(q, x) = \max \{ H(q, x, v) \mid \text{over } v\}, \]

and let \(q^*(.)\), \(x^*(.)\) be the optimal costate/state functions for (2.1). Then letting subscripts denote partial derivatives, the first order necessary conditions of optimality for (2.1) may be written thus,

\[(2.3) \quad \dot{q}/\dot{t} = \rho q - H^0_x, \quad \dot{x}/dt = H^0_q. \]

If \(U\) and \(T\) are concave, under modest additional assumptions the following proposition of Brock and Scheinkman (1976) (See Cass and Shell (1976) for related theorems by Cass, Shell, and Rockafellar) is true. Let \(I\) denote the \(n\) by \(n\) identity matrix and define the \(2n\) by \(2n\) matrix \(B(q, x)\) by letting the first \(n\) rows be given by \([-H^0_{xx}, (\rho/2)I]\) and the next \(n\) rows be given by \[(\rho/2)I, \quad H^0_{qq}\].

**Global Asymptotic Stability Result:** If \(B(q, x)\) is positive definite for all \((q(t), x(t))\), then the optimal control \(x^*(.)\) is globally asymptotically stable at a unique optimal steady state \(x^*\).

The result is proved by differentiation of the "Lyapunov" function \(V = (\dot{q}/\dot{t})(dx/dt)\) which is nonpositive for concave problems and showing that \(B\) positive semi-definite implies \(V\) weakly increases.

Notice that \(B\) is always positive semi-definite for \(\rho = 0\) and is positive definite if the product of the smallest eigenvalue of \(-H^0_{xx}\) and the smallest eigenvalue of \(H^0_{qq}\) is greater than \(\rho^2/4\). Notice that \(H^0_{qq}\) is always positive semi-definite since \(H^0\) is convex in \(q\) by standard optimization theory. The matrix \(-H^0_{xx}\) is always positive semi-definite if \(H^0\) is concave in \(x\).

The Brock and Scheinkman (1976) result was used in the chapter by Dechart and Brock in this volume to locate sufficient conditions for global asymptotic stability of optimal solutions to lake management problems where the dynamics were concave in states and controls. Basically the result shows that if the discount rate \(\rho\) is small enough relative to the product of measures of Hamiltonian curvature in \(x\) and \(q\), we should expect global asymptotic stability of optimal management policies. Other results giving sufficient conditions for global asymptotic stability from the literature in both continuous time problems and discrete time problems are reviewed in Brock and Malliaris (1989) and McKenzie (1986).

Since eigenvalues of the linearization of the co-state/state dynamical system come in "Poincare' pairs", \(\lambda, \rho - \lambda\), the \(2n\) degree characteristic polynomial can be factored into a product of two \(n\) degree polynomials. See Dechart and Brock's chapter in this book for application of this technique, a very brief sketch of bifurcation analysis for a version of the lake management problem, and general references. See the wide ranging survey by Nishimura and Sorger (1996) for bifurcation analysis in dynamic economic models, most of which are either discrete time or continuous time infinite horizon models. Bifurcation analysis for both closed loop equilibria and even the easier case
of open loop equilibria appears much less well developed for dynamic games. Basar and Olsder (1999) give a general treatment of dynamic noncooperative game theory. Engineering literature on linear quadratic optimal regulator theory can be exploited as in Brock and Malliaris to give a local analysis. Let us give a very brief sketch of this approach here.

The linear quadratic optimal control problem that best approximates (2.1) at an optimal steady state can be written thus (Brock and Malliaris (1989, Chapter 5), Carlson, Haurie, and Leizarowitz (1991))

\[(2.4) \quad C(x_o) = \inf \{ \int \exp(-pt) [x'(Q-SR^{-1}S')x + u'Ru]dt \}
\]

s.t. $dx/\text{dt} = [F-GR^{-1}S']x + Gu, x(0) = x_o$

where $Q = -U_{xx'}, R = -U_{vv'}, S = -U_{xv}, F = T_x, G = T_v$ are all evaluated at the optimal steady state under consideration. Here "inf" denotes "infimum" which is the minimum if it is actually attained. The state valuation function $C(x)$ is quadratic in $x$, i.e. $C(x) = x'Px$, for a positive definite matrix $P$. The Bellman equation is

\[(2.5) \quad \rho C(x) = H(q, x), \quad q = \partial C/\partial x = 2Px,
\]

which is equivalent to a quadratic matrix equation for $P$ which, in turn, is called a Ricatti equation. Here $H$ denotes the current value minimized Hamiltonian.

If $u_1 = \exp(-(\rho/2)t))u, x_1 = \exp(-(\rho/2)t)x$, then (2.4) may be written in the undiscoutned form

\[(2.6) \quad \inf \{ \int x_1'(Q-SR^{-1}S')x_1 + u_1'Ru_1 \} dt \quad \text{s.t.} \quad dx_1/\text{dt} = [F-(\rho/2)I-GR^{-1}S']x_1 + Gu_1
\]

Problem (2.6) is a standard Optimal Linear Regulator Problem (OLRP) from control engineering. Consideration of this problem gives rapid insights into the forces that contribute to local asymptotic stability of the optimal dynamics near the optimal steady state that (2.4) approximates.

Kwakernaak and Sivan (1972, Section 3.4 and Theorem 3.8, p. 247) do the following: (i) It is shown that if $\lambda$ is an eigenvalue for the co-state/state system so also is $-\lambda$. (ii) They locate sufficient conditions (which are easy to check, for example they automatically hold if the $G$ matrix above is square and invertible and the eigenvalues are unique) for the co-state/state system to have no eigenvalues with zero real parts, (iii) They locate sufficient conditions which are easy to check (for example they hold if $Q$ and $R$ are positive definite matrices) for optimal $x_1, u_1$ to converge to zero as $t \to \infty$. Results (i) and (ii) are useful in bifurcation analysis of steady states of general optimal control problems. Result (iii) gives bounds on how fast an optimal path can diverge. Roughly speaking, an optimal path can not diverge faster than $\exp[(\rho/2)t]$ as $t \to \infty$.

A main theme of this book has been the analysis of problems that have nonconcave dynamics (e.g. the variations on the lake management problem as well as games on lakes). In this case tools for analysis of global asymptotic stability that exploit concave dynamics are of limited use. However, tools based upon linear quadratic approximations at an optimal steady state can still be used to understand local asymptotic stability and to supplement local bifurcation analysis of the costate state equations at that steady state. Notice that the presence of multiple optimal steady states does not require
the presence of nonconcave dynamics (Wirl and Feichtinger (1999)) but nonconcave dynamics can easily produce multiple solutions of the first order necessary conditions.

Here is the main difficulty presented by nonconcave problems. For concave problems, the first order necessary costate/state conditions plus the transversality conditions at infinity are sufficient for optimality under modest regularity conditions. See, for example, Kamien and Schwartz (1991) for a textbook treatment of this well known basic theory. See Carlson, Haurie, and Leizarowitz (1991) for a treatment that includes overtaking optimality as well as problems of optimal control of infinite-dimensional dynamical systems (e.g. integro-differential equations and partial differential equations) under their heading of "distributed parameter systems" (Carlson et al. (1991, Chapters 8 and 9). For nonconcave problems we can have many solutions of the costate/state equations that satisfy the transversality conditions and not all of these (in fact typically only very few of them) are serious candidates for optimal solutions.

Brock and Dechert (1983) gave an early treatment of time stationary nonconcave infinite horizon optimal control problems that used the convexity of the optimized Hamiltonian in the costate to screen out solutions that had multiple initial costate candidates for a given state vector. One idea used in that paper is the simple idea that a convex function takes maxima on a convex set at its extreme points and the Bellman equation, \( \rho W(x)=H(q,x) \) suggests that one wants \( H \) as large as possible over the set of candidate solutions of the costate/state equations to get the candidate solutions with the largest value function \( W \). Clearly, since \( H \) is convex in \( q \), one wants the costate to be as far away from the minimum of \( H \) in \( q \), i.e. the \( dx/dt=0 \) locus as possible. This idea plus shape restrictions on candidate value functions gleaned from the phase portrait of the costate/state equations is enough to characterize the optimal solutions in some one dimensional state cases as we have seen in this book.

For one dimensional state vector problems, this technique works well for screening candidate solutions. It is less useful for multidimensional state problems, but is still of some use. As we said before, there are analytically tractable related techniques available for one dimensional state problems (e.g. those used in various chapters in this book).

Many dimensional state vector problems with nonconcave dynamics create challenges for global analysis for pencil and paper mathematics. However some insight may be had by examining the general class of problems where utility or payoffs is linear in the control vector. In this case the typical solution is to approach a target as rapidly as possible subject to problem constraints (cf. Brock and Starrett, this volume, and Kamien and Schwartz (1991) for a textbook treatment). As in Kamien and Schwartz, we shall call such problems, Most Rapid Approach Problems (MRAP). We indicate application of MRAP theory to spatial settings after the brief treatment of spatial problems below.

3. SPACE

The study of optimal management problems for systems governed by spatial dynamical systems is an analytical challenge because the underlying dynamical system has high dimension (infinite dimension in many cases). Hence this section of the paper will consist of speculations on how one might still get insights by use of simple tools available to economists and ecologists without training in the mathematics of infinite dimensional dynamical systems. The spirit of this section is rather in the style of "physics" where the objective is to teach creativity in using minimal mathematics to garner most of the potentially available insights by trading off analysis of the full problem for analysis of strategically chosen "analog" problems.
One approach to a serious treatment of space would treat the optimal control of systems governed by partial differential equations (e.g. Carlson et al. (1991, Chapter 9), Lions (1971), Raymond and Zidani (1998)). The infinite dimensional dynamical systems generated by partial differential equations systems such as those popular in "spatial" problems appearing in mathematical biology and mathematical ecology (cf. Levin, Kareiva, Pacala, Tilman, et al. in (Tilman and Kareiva 1997), and Murray (1993)), require a lot of fairly intricate mathematics if they are treated at the level of rigor demanded in a serious mathematical treatment. However, if one is prepared to "act like an applied engineer or applied physicist" and be satisfied with an heuristic treatment that yields the correct conclusions with high probability then it is quite easy to work with these systems.

The first order necessary conditions of optimal control of systems governed by partial differential equations or integro-differential equations can be derived by adapting the perturbation arguments used in classical texts such as Athans and Falb (1966, Chapter 5). I.e. one writes down a perturbation of the optimal control, which, in turn generates a perturbation of the state through the state dynamics. The difference in the objective evaluated at the perturbed control and the optimal control can be expanded in a Taylor type series to the first order. The (adjoint) dynamical system that must be satisfied by the costate variable as well as the maximum principle itself emerge naturally from the condition that all perturbations must not increase the objective at an optimum. Let us illustrate by a scalar example.

Example 1: Optimal control of a one dimensional diffusion system.

Consider the following infinite horizon time stationary "toy" spatial optimal control problem

\[ W(x) = \text{maximum} \int \text{exp}(-\rho t) \, U(x(t,a),v(t,a),a) \, dt \, da, \]

\[ \text{s.t. } \partial x/\partial t = T(x(t,a),v(t,a),a) + D \partial^2 x/\partial a^2, \quad x(0,a) = x_0(a), \quad a \in [0,2\pi] = \mathbb{C}, \]

with 0,2\pi identified. I.e. the set \mathbb{C} is a circle which is parameterized by the parameter a which runs from 0 to \pi to 2\pi = 0, as a increases. Here the state x is a scalar and v is a finite dimensional control vector. The time integral runs from t=0 to t=\infty and the space integral runs over the circle \mathbb{C}. In the case that x is a vector, the operator D\partial^2(.) must be replaced with a vector analog where D is a matrix.

The general theory in Raymond and Zidani (1998, p. 1858) adapted to (3.1) and (3.2) implies that the current value costate p(x,a) satisfies

\[ \partial p/\partial t = \rho p - \partial U/\partial x - p \partial T/\partial x - D \partial^2 p/\partial a^2, \]

for p=p(x(t,a),a) for optimal solution x(.,.). Of course the optimal values of p,x agree for a=0,2\pi for all t. The optimal control v maximizes U+pT.

The pair of Partial Differential Equations (PDE's) (3.2), (3.3) are difficult to analyze. Of course if one knew the functional form of the optimal control in terms of (x,a), one could do a linearization of (3.2) at a "steady state" and use Fourier series as does Murray (1993, p. 382) to do a tractible local asymptotic stability analysis. The difficulty lies in coping with the costate dynamics (3.3).

However, in the special case where U,T are the same for all a, we can find steady states (p^*,x^*), linearize the costate/state PDE's at each steady state, and follow Golubitsky and Schaeffer (1985, p. 315) (using the cosine
instead of the sine) to find the eigenvalues and eigenfunctions. The eigenfunctions associated with the smallest eigenvalues (just like the standard and familiar D=0 case) are solution candidates.

We illustrate this procedure with a simple special case, \( T=Fx+u, \ U=Qx^2-u^2 \). Put \( z=(p,x) \), and write the costate/state system in the matrix notation

\[
(3.4) \quad \frac{\partial z}{\partial t} = Az + (\text{Diag}) \frac{\partial^2 z}{\partial a^2}, \quad z(0,a)=z_0(a), \quad z_0(0)=z_0(2\pi),
\]

where the first row is \([\rho-F,2Q]\) and the second row of \( A \) is \([1/2,F]\). Here Diag denotes a 2x2 matrix with first row \([-D,0]\), second row \([0,D]\). Try solutions of (3.4) of the form

\[
(3.5) \quad z(t,a)=z_0(a)\cos(2\pi ka)\exp(\lambda t).
\]

Notice that (3.5) matches the initial condition in (3.4) at \( t=0 \), and preserves matching at the "endpoints" of the circle \( a=0,2\pi \) for all \( t \). Insert (3.5) into (3.4) and observe that (recalling \( \sin'(x)=\cos(x), \ \cos'(x)=-\sin(x)\)),

\[
(3.6) \quad |A-\lambda I-(2\pi k)^2\text{Diag}|=0
\]

must hold for (3.5) to be a solution. Here \( I \) denotes the 2x2 identity matrix. Inspecting the matrix \( A \) in (3.6) we see that introduction of \( D \) amounts to replacing \( F \) by \( F_k=F-(2\pi k)^2D \) in \( A \). This suggests that an increase in \( D \) could stabilize an optimally controlled system that is unstable.

Here is an example. First let \( D=0 \) and let

\[
(3.7) \quad F>0, \ \rho>2F.
\]

It is easy to check that both eigenvalues of the costate/state system are positive. The optimal solution space is the linear space generated by the eigenvector of the smallest eigenvalue which is positive for case (3.7). Now let \( D>0 \). The smallest eigenvalue \( \lambda \), call it \( \lambda_k \), of (3.6) is given by

\[
(3.8) \quad \lambda_k=(\rho-B^{1/2})/2, \quad B=(\rho-2F_k)^2+4Q.
\]

The largest such \( \lambda_k \) occurs when \( k=1 \). Hence, if \( \lambda_k \) in (3.8) is negative at \( k=1 \), all \( \lambda_k \) defined by (3.8) will be even more negative for non zero integral \( k \) not equal to one. It is easy to see, using (3.7) and the definition of \( F_k \) that \( D \) can be increased from \( D=0 \), to a level \( D_c \) such that \( \lambda_1<0 \) for \( D>D_c \).

Construct the optimal solution space using eigenvectors associated with the smallest eigenvalues.

Although this treatment has been very cursory and brief, the reader may fill in details with Golubitsky and Schaeffer (1985) and Raymond and Zidani (1998) at her side. We make a final point to help the reader appreciate the form of the costate equation in (3.3) above.

Consider a spatial version of the familiar neoclassical growth model, i.e. the special case where \( T(x,v)=f(x)-c, \ U(x,v)=u(c) \), where \( f(x) \) is the same production function at all sites \( a \), and \( u(c) \) is the utility function of consumption which is the same at all sites \( a \). This system is close to that of Carlson et al. (1991, Chapter 9). The costate state equations for the homogeneous spatial form of the neoclassical growth model are given by
(3.9) \( \partial p/\partial t = pp' - D\partial^2 p/\partial a^2 = rp - D\partial^2 p/\partial a^2, \quad r = p - f' \).

(3.10) \( \partial x/\partial t = f(x) - c(p) + D\partial^2 x/\partial a^2. \)

In order to motivate the form of the costate equation (3.9) with a purely economic arbitrage argument think of the following two choices which must yield the same return in equilibrium: (i) at site a, take \( p(t,a) \) "dollars" put them into a "bank" at date t which pays interest at rate r and close out the bank account at date \( t+\Delta t \) which yields principal plus interest \( p(t,a) + rp(t,a)\Delta t \); (ii) at site a, leave \( p(t,a) \) at that site which changes to \( p(t+\Delta t,a) + \text{net diffusion gain at date } t+\Delta t, \text{ where net diffusion gain is } D\{p(t,a+\Delta a) - p(t,a); - 2p(t,a)\}. \) Equate these two sums at date \( t+\Delta t, \) recall the requirement \( da^2 = \Delta t, \) and let \( dt \) go to zero to obtain (3.9). Hence, (3.9) may be viewed as a common sense arbitrage equation generalized to space. The same type of reasoning can be used to derive spatial costate equations in much more general cases. Turn now to "analog" problems which capture some of the flavor of control problems of PDE's but use more elementary tools of analysis.

We shall follow "analog" approaches here that should expose much of the insights that are available as well as expose the added insight of adding "space" for this particular example.

Two "analog methods" suggest themselves. The first analogy is to replace \( \partial t, \partial a \) by small positive quantities and approximate the \( \int \) by a Riemann sum and approximate (3.2) by a difference equation. Under appropriate regularity conditions one would expect to obtain the original system in the limit as \( (\partial t, \partial a) \) are taken to zero appropriately. We abuse notation and continue to use \( \int \) and \( (\partial t, \partial a) \) for the difference scheme.

McKenzie (1986) reviews literature on convergence theorems (e.g. "turnpike" theorems) for discrete time deterministic systems, some of which parallels the Brock and Scheinkman (1976) result for continuous time systems. That is to say, for systems with concave payoffs and concave dynamics, for a fixed temporal step size, \( \Delta t > 0, \) if the discount factor \( \beta(\Delta t) = \exp\{-\alpha(\Delta t)\} < 1 \) is close enough to unity (how close depends upon a "curvature quantity" that parallels the Hamiltonian curvature in the Brock and Scheinkman (1976) result), we have global asymptotic stability of optimal solutions to a unique optimal steady state. This is a direct parallel for discrete time systems which can be applied to the discrete approximation to (3.1) and (3.2) provided that the payoff and the dynamics are concave. This leads to the conjecture that a parallel result should be available for the original system (3.1) and (3.2). It would appear worthwhile to develop such results in future research on the optimal control of systems for applications at the juncture of economics and spatial ecology.

The second type of analogy is less justified for the mathematical analysis of the original PDE system but still yields some insight. Here we replace the circle C with the finite set of points on it, \( 0 = a_1 < a_2 < \ldots < a_n < a_{n+1} = 2\pi = 0, \) and replace the \( \int \) over a in C in (3.1) with a sum over a. Replace the term, \( D\partial^2 x/\partial a^2, \) with the term

(3.11) \( D\{[x(a_{i+1}) - x(a_{i})] - [x(a_{i}) - x(a_{i-1})]\} \)

at site \( a_i. \) Notice that for the case \( a_{i+1} = a_i = \partial a = \text{constant}, \) for all i, this is the numerator of the simplest approximation to \( D\partial^2 x/\partial a^2. \) In fact it is probably a misuse of the word, "analog" because it suggests too strong a relationship between the solution of this problem and the original problem.

Consider the following dramatic example from Golubitsky and Schaeffer
(1985, p. 309) where it is shown that the steady state \( x=0 \) is locally asymptotically unstable for the case \( D=0 \), but is \textit{locally asymptotically stable} for \( D \) positive and large enough for one dimensional reaction-diffusion equations

\begin{equation}
\frac{\partial x}{\partial t} = T(x) + D \frac{\partial^2 x}{\partial a^2}, \quad T(0) = 0, \quad T'(0) > 0,
\end{equation}

with appropriate boundary conditions. Let us sketch the reasoning here. Suppose 0 is a steady state for (3.12), linearize to obtain (abusing notation by letting \( x \) denote the linearization)

\begin{equation}
\frac{\partial x}{\partial t} = T'(0)x + D \frac{\partial^2 x}{\partial a^2}.
\end{equation}

Try solutions of (3.13) of the form

\begin{equation}
x(t,a) = x(0,a) \cos(2\pi ka) \exp(\lambda t),
\end{equation}

where \( k \) is an integer, insert (3.14) into (3.13), and observe that we need,

\begin{equation}
\lambda_k = T'(0) - D(2\pi k)^2.
\end{equation}

Hence, the family of solutions is given by

\begin{equation}
x(t,a) = x(0,a) \cos(2\pi ka) \exp(\lambda_k t).
\end{equation}

All of these are asymptotically stable if

\begin{equation}
\lambda_1 = T'(0) - D(2\pi)^2 < 0,
\end{equation}

because all other \( \lambda_k \) are negative for \( k \) not zero if \( \lambda_1 < 0 \). Thus we see how diffusion can stabilize a one dimensional locally unstable system. Golubitsky and Schaeffer (1985, Chapter 7) also conduct a bifurcation analysis.

However, for example, the best one can do for stabilization by diffusion by increasing \( D \) in the simplest "analog"

\begin{equation}
\frac{dx_1}{dt} = T(x_1) + D(x_2 - x_1), \quad \frac{dx_2}{dt} = T(x_2) + D(x_1 - x_2),
\end{equation}

is to turn the two positive eigenvalues of the linearization of (3.18) at zero into one positive and one negative as \( D \) is increased. Thus, care must be taken in analogizing. In view of this problem, we shall use the word "allied" instead of "analog."  

In any event, for the second allied problem we pose the following problem. Let

\begin{equation}
x_i(t) = x_{i1}(t,a_1), \quad v_i(t) = v_{i1}(t,a_1), \quad U_i(...,a_1) = U_{i1}(...,a_1),
\end{equation}

\begin{equation}
T_i(...,a_1) = T_{i1}(...,a_1), \quad x_o = (x_{10}, x_{20}, ..., x_{n0}, x_{n+1,0} = x_{10}).
\end{equation}

\begin{equation}
W(x_o, D) = \max \{ \exp(-pt) \sum_i U_{i1}(x_i(t), v_{i1}(t)) dt \},
\end{equation}

\begin{equation}
\text{s.t.} \quad \frac{dx_i}{dt} = T_{i1}(x_i, v_{i1}) + D([x_{i+1} - x_i] - [x_{i} - x_{i-1}]), \quad x_i(0) = x_{i0}, \text{ all } i.
\end{equation}

We are now in a position to view the difference space makes in a setting where
we can use simple available mathematical tools without any investment in optimal control theory for PDE systems. Let us make the following observations for the second allied system although it will be clear how their parallels could be developed for the first analogy system.

First, if $D=0$, $W(x_0, 0) = \sum_i W_i(x_{i0})$, where $W_i(x_{i0})$ is the current value state valuation function for problem $i$. I.e. the $D$-coupled system (3.21) (3.22) breaks into a set of uncoupled independent optimal control problems for $D=0$. We can apply the existing set of tools reviewed in Brock and Malliaris (1989) as well as the bifurcation tools outlined above to analyze each one of these problems separately. E.g. local asymptotic stability of optimal steady states can be analyzed by doing bifurcation analysis, using Lyapunov functions for OLRP's following Brock and Malliaris (1989), using methods from engineering OLRP theory to study the location of the eigenvalues for the linearized costate/state system, etc.

Second, global asymptotic stability can also be studied using the Lyapunov function methods reviewed in Brock and Malliaris. Hence, for concave problems of the form (3.21), (3.22) above, one can expect a unique optimal steady state and global asymptotic stability of optimal solutions to that steady state for small enough positive $\rho$ under modest regularity conditions. Turn now to a speculative discussion on future research.

4. SUGGESTIONS FOR FUTURE RESEARCH

Since it is beyond the scope of these notes to do any serious analysis of optimal control of the spatial systems we posed above, we content ourselves with making a set of comments about some directions and strategies for future research.

First, some natural questions raised by the presence of $D$ in (3.22) are the following:

(i) For a fixed and known control function what is the impact of an increase in $D$ on the stability of (3.22)? At the risk of repeating, in the one dimensional reaction-diffusion case Golubitsky and Schaeffer (1985, p. 309) show how an increase in $D$ stabilizes a locally unstable system. They also emphasize how an increase in $D$ increases stability for a stable system in that case. But multi-dimensional systems are much more complicated as they illustrate with a well known two dimensional reaction-diffusion system called "the Brusselator" which exhibits striking patterns especially near bifurcation points (Golubitsky and Schaeffer (1985, p. 312)). So we should expect related phenomena such as, for example, diffusion induced (Turing type) instabilities to crop up in the optimal control case.

In particular if each equation is stable at some steady state when $D=0$, what happens if $D$ increases from zero? It may be possible to get some analytical results by computing the Ricatti equation for the matrix of the approximating OLRP, call the positive definite matrix root, $P(D)$, and expanding this matrix in a matrix Taylor series around $D=0$. Since $P(0)$ is a diagonal matrix where each element solves a simple quadratic equation in one variable, this exercise is analytically tractable. However, given the speed of today's algorithms for computing Ricatti equations and given their current popularity in macroeconomics which has produced wide availability of software in economics, it is probably best to turn to the computer in this type of exercise.

(ii) If the optimally controlled system is stable at some optimal steady state for the case $D=0$, what happens if $D$ is increased from zero? In particular could an analog of diffusion driven instability (i.e. Turing
instability as in Murray (1993, Section 14.2) arise in the optimally controlled case where \( x \) is a vector and the operator \( D \partial^2/\partial a^2 \) is a vector diffusion operator? To put it another way, could a stable system when no control is applied be turned into an unstable system when optimal control is applied (as in Brock and Malleris (1989, p.156)), but this phenomena is due to the presence of diffusion rather than due to differential state costs and/or differential control costs as in Brock and Malliaris? To put it another way, we are suggesting that it may be interesting to investigate the interaction between the forces that cause instability of the optimal control and the conditions that cause Turing type instabilities.

(iii) Arguments for why "space matters" in ecology (even in aggregative ecology where a lot of averaging over space occurs to produce the dynamical quantities of interest) are given in Tilman and Kareiva (1997). However, optimal management of ecosystems consisting of heterogeneous patches with coupled dynamics (e.g. optimal management of an ecological mosaic) is not considered in Tilman and Kareiva. An example of a "new" wrinkle raised by space is the following. Suppose some of the optimal paths in the \( D=0 \) system are locally unstable at the corresponding optimal steady states and others are locally stable. What happens to this pattern when \( D>0 \)? This issue can be studied by considering the general OLPR approximation for \( D>0 \).

Second, the general message of stability theorems for concave problems where the sufficient conditions involve the product of Hamiltonian curvatures in state and costate being greater than \( \rho/4 \) is this. Under modest regularity conditions, an infinite horizon discounted optimal control problem with time invariant concave objective in state and control with time invariant dynamics concave in state and control has a critical value \( \rho_c > 0 \) such that for \( \rho < \rho_c \) the optimal path is globally asymptotically stable to a unique optimal steady state.

This statement is independent of dimension of the system (so long as it is finite), the degree of coupling via diffusion, integro-differential equations or whatever, whether the model is continuous time or discrete time, or even the presence of stochasticity (e.g. Arkin and Evstigneev (1987), Malliaris (1982) for work on stochastic turnpike properties). Of course the critical value of \( \rho \) will depend upon parameters and features of the problem. See Carlson et al. (1991) and McKenzie (1986) and Arkin and Evstigneev (1987) and Marimon (1989) for a general treatment of stability theory for infinite horizon deterministic and stochastic optimal management problems that should be generalizable to spatial setups (even, possibly, infinite dimensional dynamical systems) provided that appropriate concavity assumptions are made.

Third, it may be useful to investigate generalization of this theory to infinite horizon optimal control problems over space/time where the spatial dynamics are governed by partial differential equations or integro-differential equations of the form

\[
W(a)=\max(\int\exp(-\rho t)U(x(t,a),v(t,a),a)dt),
\]

(4.2) \ s.t. \ \frac{\partial x}{\partial t}=T(x(t,a),v(t,a),a)+D\partial^2 x/\partial a^2, \ x(0,a)=x_0(a), \ a \ in \ [0,2\pi]=C, \ or,

(4.3) \ \frac{\partial x}{\partial t}=T(x(t,a),v(t,a),a)+\int D(a-a')x(t,a')da', \ x(0,a)=x_0(a),

for \( a \ in \ [0,2\pi]=C \), where \( D(z) \) is a kernel function as in Murray (1993, Chapters 9 and 16). Notice that (4.3) captures nonlocal interaction effects whereas (4.2) captures local effects. The linearization techniques described
above are available for \(4.3\) as well as for \(4.2\).

Potential applications of systems (4.1)-(4.3) include (i) Management of fisheries over space where there may be depensation (Allee) effects (e.g. Clark (1990), Murray (1993)); (ii) Optimal control of diffusing pests such as spruce budworm (e.g. Murray (1993, Section 14.7) articles in Tilman and Kareiva (1997)); (iii) Simultaneous management of a number of lake ecosystems like those analyzed in this volume but connected by diffusion processes; (iv) management of gene flow of pest resistance in spatial settings (e.g. a fusion of the optimal control model of the gene pool of Brock/Xepapadeas in this volume with the approach of Antonovics, Thrall, and Jarosz in Tilman and Kareiva (1997)); (v) Regional development theory as in Carlson et al. (1991, Chapter 9); (vi) Optimal control models of rational expectations equilibrium and social optimum building on Krugman's "Turingesque" adaptive expectations models in Krugman (1996). The reader should note the use made of the linearization techniques (as in Murray (1993)) for systems of the form (4.3) by Krugman (1996) which shows the power of such techniques in delivering interesting analytical results on forces for endogenous economic agglomeration. The reader of these notes is sure to think of many more applications.

Fourth, a rich collection of dynamic spatial processes are treated by Durrett and Levin (1998), Bolker and Pacala (1999), as well as articles in Tilman and Kareiva (1997). These works suggest formulation of natural economic "toy" models for the study of optimal management of resources in space/time by coupling economic dynamics and management objectives onto the spatial dynamics and using some of the tools reviewed in these notes to get some insights. For example, the remark by Durrett and Levin (1998, p. 31) "that the spatial distribution of individuals stabilized an otherwise unstable situation" resonates with the observations we made above about the role of diffusion in space across sites stabilizing unstable underlying individual site dynamics. Of course the Durrett/Levin context is different because it involves using a mean field approximation to a stochastic dynamic model, although some of the analytics are related to the analytics covered in Murray ((1993), especially his use of an "energy function" citing Berding (1987), as in his equation (14.52), to measure "heterogeneity" of pattern and used in his Section 14.9).

As an aside, we stress that the points made about the importance of robustness of conclusions to the use of synchronous versus asynchronous updating made by Durrett and Levin and their references to May/Nowak and Huberman/Blance stand for optimal management models in economics as well.

Fifth, MRAP theory of optimal control can be generalized to systems controlled by integro-differential equations and partial differential equations. Although assumptions sufficient for MRAP are strong, the insights gained should be worth having. Spatial ideas could enter via spatial structure of constraints on control sets.

Sixth, the paper by Brock and Xepapadeas in this volume stressed the usefulness of the comparison of three regimes: (i) Privately Optimal Management Problems (and the resulting equilibria), called POMP, (ii) Socially Optimal Management Problems, called SOMP, and (iii) Nature's equilibria. It is standard in theoretical economic welfare analysis to compare regimes (i) and (ii). The third regime, Nature's solution, was added not only because of intrinsic ecological interest, but also to capture the idea that there may be unobserved services or unobserved connections that Nature is providing that contribute to human welfare.

One could imagine current work on spatial ecology like that in Tilman and Kareiva (1997) being combined with economic modeling to produce new insights on topics such as: (i) the optimal design and placement of "trap crops" in integrated pest management applications as well as other economic entomology.
applications; (ii) optimal deployment of disease control in a spatial
Kermack-McKendrick (SIER) model and other spatial epidemiological models
(Holmes, and Ferguson, May, and Anderson in Tilman and Kareiva (1997))

Seventh, the article by Brock, Maler, and Perrings in Gunderson and
Holling (2001) stressed the location of useful sufficient conditions for the
possibility of implementing social optima by the use of decentralized
instruments (e.g. externality taxes, or auction of "rights") even if the
ecosystem dynamics are complex and nonconcave provided that the economic
structure superimposed upon the ecosystem possessed a type of "separability"
property. These types of results should be generalizable to the optimal
management of spatially dynamic systems.

Eighth, nothing has been said in these notes about empirical issues
raised by space and the usefulness of these tools and dealing with them. This
relates to understanding mechanisms behind patterns observed in biology and
ecology (e.g. Brown and West (2000), Holling (1992)). For example, Tilman,
Lehman, and Kareiva state "...In each case the species becomes clumped, but
the extent of this clumping, the size of the clumps, and the average distance
between clumps depend upon the mortality rate, the colonization rate, and the
distance...over which dispersal occurs. A field biologist encountering a
clump of individuals of a species in nature, and noting a nearby habitat in
which the species was rare or absent, might immediately wonder what
environmental characteristics caused such clumping. These results show that
there need not be any environmental causes...the appropriate null model...might be a clumped dispersion..."

This same difficult problem of empirically identifying the relative
contributions of endogenous structuring forces versus exogenous structuring
forces occupies a central place in economics as well as in ecology. We
believe that the formulation of useful theory at the interface of ecology and
economics will benefit from more use of tools like those put forth in these
notes to better understand endogenous patterning mechanisms in
economic/ecological contexts. This enhanced understanding could then be
combined with new econometric work on identification of "true" endogenous
interactions (those that possess a type of "social multiplier", i.e. an
endogenous positive feedback mechanism) from "spuriously" endogenous
interactions which commonly appear because of left out effects such as
correlated unobserved heterogeneity. See Brock and Durlauf (2000) for a
detailed review of the literature on this problem in social science.

Turn now to an appendix on global bifurcations.

APPENDIX ON GLOBAL BIFURCATIONS

We discuss here, two global bifurcations that are of special interest in
economics. These are heteroclinic and homoclinic bifurcations. There are
others which may be of use also but they are not discussed here. A
comprehensive discussion is in (Kuznetsov (Chapter 6)).

We follow Brock and Hommes (1997) which, in turn, follows Palis and
Takens (1993) for part of the exposition concerning homoclinic orbits and
homoclinic bifurcations that follows. A homoclinic [heteroclinic] bifurcation
occurs when the stable manifold and the unstable manifold of a steady state
(i.e. a rest point) [two steady states (i.e. two rest points)] of a dynamical
system pass through each other as the bifurcation parameter passes through a
critical point. We need to define concepts such as stable and unstable
manifolds. Although we shall focus on maps on the plane (in order to exposit
the results from Palis and Takens (1993) below) many definitions and concepts
are similar for continuous time systems and for higher dimensional systems.

Let \( F: \mathbb{R}^2 \to \mathbb{R}^2 \) be a smooth (i.e. differentiable) map and \( x_{t+1} = F(x_t) \) the
corresponding dynamical system. Let \( p \) be a "nondegenerate" saddle fixed point of \( F \) where the Jacobian \( JF(p) \) has two real eigenvalues \( \lambda \) and \( \mu \) such that \( |\lambda| < 1 \) and \( |\mu| > 1 \). The local stable and unstable manifolds of \( p \) are defined by

\[
(A1) \quad \mathcal{W}_\text{loc}^s(p) = \{ x \in U \mid \lim_{n \to \infty} F^n(x) = p \}, \quad \mathcal{W}_\text{loc}^u(p) = \{ x \in U \mid \lim_{n \to \infty} F^n(x) = p \},
\]

where \( U \) is a neighborhood (typically thought of as small) of \( p \). These objects are defined for \( n \)-dimensional maps and flows the same way.

Stable manifold theory locates sufficient conditions for existence and various degrees of smoothness of local (and global) stable and unstable manifolds of points, cycles, and general invariant sets. The theory also locates sufficient conditions for \( \mathcal{W}^s_\text{loc}(p) \) and \( \mathcal{W}^u_\text{loc}(p) \) to be smooth curves tangent to the stable and unstable eigenspaces of \( JF(p) \) (Kuznetsov (1995)). The global stable and unstable manifolds are defined by

\[
(A2) \quad \mathcal{W}^s(p) = \bigcup_{n=0}^{\infty} F^{-n}(\mathcal{W}^s_\text{loc}(p)), \quad \mathcal{W}^u(p) = \bigcup_{n=0}^{\infty} F^{n}(\mathcal{W}^u_\text{loc}(p))
\]

If \( F \) is a diffeomorphism (i.e. a smooth map with a smooth inverse) the stable and unstable manifolds are curves without self-intersections because \( F \) is one to one. If \( F \) is not invertible, i.e. many to one, then \( \mathcal{W}^u(p) \) can have self-intersections and \( \mathcal{W}^s(p) \) can have several components.

A homoclinic point \( q \neq p \) is an intersection point of the stable and unstable manifolds \( \mathcal{W}^s(p) \) and \( \mathcal{W}^u(p) \). This intersection will not be a tangency, but will be a "transversal crossing" for the nondegenerate case which we concentrate on here. If \( \mathcal{W}^s(p) \) and \( \mathcal{W}^u(p) \) are tangent at \( q \) then \( q \) is called a point of homoclinic tangency; if \( \mathcal{W}^s(p) \) and \( \mathcal{W}^u(p) \) intersect transversally at \( q \) we say that \( q \) is a point of transversal homoclinic intersection. The concept of homoclinic points associated with a periodic saddle point \( p \) is defined in the same way.

Consider a one-parameter family of maps \( F_{\alpha} \), each possessing a saddle fixed point \( p_{\alpha} \). Here \( \alpha \) may be driven by another dynamical system or differential equation whose speed is much slower than the \( F \)-system. We say that \( F_{\alpha} \) exhibits a homoclinic bifurcation at \( \alpha = \alpha_0 \) if

(i) for \( \alpha < \alpha_0 \), \( \mathcal{W}^s(p_{\alpha}) \) and \( \mathcal{W}^u(p_{\alpha}) \) have no intersection point \( q \neq p_{\alpha} \).

(ii) for \( \alpha = \alpha_0 \), \( \mathcal{W}^s(p_{\alpha}) \) and \( \mathcal{W}^u(p_{\alpha}) \) have a point of homoclinic tangency.

(iii) for \( \alpha > \alpha_0 \), \( \mathcal{W}^s(p_{\alpha}) \) and \( \mathcal{W}^u(p_{\alpha}) \) have a transversal homoclinic intersection point \( q \neq p_{\alpha} \).

Replace the argument, \( p_{\alpha} \), in \( \mathcal{W}^s \) with another fixed point \( q_{\alpha} \). Then we say that \( F_{\alpha} \) exhibits a heteroclinic bifurcation at \( \alpha = \alpha_0 \) if

(i) for \( \alpha < \alpha_0 \), \( \mathcal{W}^s(q_{\alpha}) \) and \( \mathcal{W}^u(p_{\alpha}) \) have no intersection point \( q \neq p_{\alpha} \).

(ii) for \( \alpha = \alpha_0 \), \( \mathcal{W}^s(q_{\alpha}) \) and \( \mathcal{W}^u(p_{\alpha}) \) coincide.

(iii) for \( \alpha > \alpha_0 \), \( \mathcal{W}^s(q_{\alpha}) \) and \( \mathcal{W}^u(p_{\alpha}) \) have no intersection point \( q \neq p_{\alpha} \).

See Kuznetsov (1995, p. 56, Figure 2.14) for a heteroclinic bifurcation for a planar system.
Several complicated dynamical phenomena occur in one-parameter families $F_\alpha$ exhibiting a homoclinic bifurcation. First, a transversal homoclinic point has the following two properties:

(1) There are wild oscillations of the stable and unstable manifolds and

(2) The map $F_\alpha$ has infinitely many Smale-type horseshoes.

Notice that if $q$ is a transversal homoclinic point, then each point $F^n(q)$, $n \in \mathbb{Z}$, is also a transversal homoclinic point. Hence, both the unstable and stable manifolds accumulate onto themselves and so-called homoclinic tangles arise. These homoclinic tangles already indicate the complexity of the dynamics and some form of sensitivity to initial conditions.

A more precise formulation of the horseshoe property (2) is the following: There exists an integer $N > 0$ and a rectangular region $R$, such that the image $F^N_\alpha(R)$ has the form of a horseshoe folded over the region $R$. When this happens, we say that the map $G = F^N_\alpha$ has a (full) horseshoe. Smale (cf. Brock and Hommes (1997) and, especially Palis and Takens (1993)) proved that this geometric configuration implies that the map $G$ has an invariant Cantor set on which the dynamics is equivalent to a two-sided subshift on two symbols. The invariant Cantor set contains infinitely many periodic points, an uncountable set of chaotic orbits and the map $G$ has sensitive dependence on initial conditions in the invariant Cantor set. Notice however that a horseshoe is not an attractor of the dynamical system. Moreover, the invariant Cantor set usually has Lebesgue measure zero. In that case for typical orbits erratic behaviour may only be a transient phenomenon and almost all orbits may still converge to a stable steady state or a stable cycle. Hence, in the presence of horseshoes, the typical long run dynamical behaviour may still be regular.

There are three additional dynamic phenomena associated to a homoclinic bifurcation that we need to mention. For complete details and references see Brock and Hommes (1997) and, especially, Palis and Takens (1993). Suppose the saddle fixed point $p_\alpha$ of $F_\alpha$ is dissipative. I.e. the product of the eigenvalues $\lambda$ and $\mu$ of $JF(p_\alpha)$ satisfies $|\mu| < 1$. This basically says that the map is a volume shrinker. Then under a set of generic regularity conditions (e.g. the homoclinic tangency is quadratic (see Palis and Takens (1993, p. 141)), in the parameter interval $(\alpha_0, \alpha_0 + \varepsilon)$ we have the following three conclusions:

(3) There exist strange attractors for a set of $\alpha$-values of positive Lebesgue measure.

(4) We have the Newhouse phenomenon, i.e. coexistence of infinitely many stable cycles for a residual set of $\alpha$-values.

(5) There exist cascades of infinitely many period doubling bifurcations.

Property (3) is perhaps the most important and implies complicated, chaotic long run dynamical behaviour for a large set of parameter values. This property explains why, in numerical experiments, "strange attractors" are frequently observed.

One may ask whether properties D1-D5 already occur before the homoclinic bifurcation, i.e. in the parameter interval $(\alpha_0 - \varepsilon, \alpha_0)$. This turns out to be an
important question for some economic applications like Brock and Hommes's (1997) cobweb model with rational versus naive expectations. In that case the bifurcation parameter is the intensity of choice $\beta$ which is a rough measure of "force of evolution" or "selection pressure" in that setting. As the intensity of choice goes to infinity, the BH system gets arbitrarily close to a homoclinic tangency for a parameter $\alpha$ close to a "critical value" $\alpha_0$, but a transversal homoclinic intersection between the stable and the unstable manifolds of the steady state never occurs.

Nevertheless properties D1-D5 do occur for $\alpha < \alpha_0$ when there are homoclinic bifurcations between the stable and unstable manifolds of periodic saddle points, or when horseshoes associated to these periodic saddles are created. Whether properties D1-D5 already occur for $\alpha < \alpha_0$ depends upon the global configuration of the stable and unstable manifolds of the steady state and upon the magnitude of the corresponding contracting and expanding eigenvalues. See Brock and Hommes (1997) for details on this problem.

A heteroclinic connection occurs when part of the stable manifold of one steady state is part of the unstable manifold of another steady state. For example, a heteroclinic connection between steady states $P$ and $Q$ for a planar system of differential equations occurs when one side of the stable manifold for steady state $P$ is a side of the unstable manifold for steady state $Q$. A very useful example for ecologists and biologists of this phenomenon is illustrated by Murray (1993, p. 279, Figure 11.1) where a travelling wave solution $u(x,t)=U(z)$, $z=x-ct$, of the (rescaled) Fisher equation

\[(A3) \ \partial u/\partial t = u(1-u)+\partial^2 u/\partial x^2,\]

is a heteroclinic connection of the two "steady states" $(U,V)=(0,0)$, $(U,V)=(1,0)$ of the system

\[(A4) \ \ U'=V, \ V'=[-cV-U(1-U)]/V.\]

Another example of a heteroclinic connection that is important for this book is given by Wagener (2000). Recall the lake problem treated by Brock and Starrett, Müller, Xepapadeas, and de Zeeuw in this volume. Let $P,Q$ denote the low-$x$, high-$x$ steady states in control/state space. Let $u(a)=\log(a),c(x)=cx^2$, $dx/dt=a-bx+f(x)$, and $f(x)=x^2/(1+x^2)$. Recall from Figure 8 of Brock and Starrett that there are three major cases: (1) [(iii)] If the projection of the stable manifold of $P$ [Q] onto the $x$-axis covers the positive real $x$-axis, (Figure 8a of Brock and Starrett) then this is the manifold of optimal solutions; (ii) The case where neither manifold's projection covers the positive $x$-axis and there is a "Skiba" point where the optimal manifold switches.

Wagener indicates that if a parameter $\alpha$ moves the phase portrait from case (1) to case (iii) then there is an intermediate value of the parameter, call it $\alpha_c$, such that for $\alpha=\alpha_c$ there is a heteroclinic connection from $Q$ to $P$ which is displayed for a particular example in his paper. Here the Right Hand Side (RHS) of the stable manifold of $P$ is the same as the LHS of the unstable manifold of $Q$. Thus the solution manifold over the $x$-domain to the right of $P$ consists of, at least part of, the coinciding part of the stable manifold of $P$ and the unstable manifold of $Q$, but to the right of $Q$, the solution manifold is the RHS of the stable manifold of $Q$. Wagener also shows how to use several computational tools that are available in the computational dynamical systems literature to compute objects of interest such as solution manifolds for specific examples.
FOOTNOTES

1. This paper is prepared for a volume tentatively titled ECOLOGICAL AND ECONOMIC MODELING, which is being edited by Editor Karl Coran Maler of the Beijer Institute of Ecological Economics of the Swedish Academy of Sciences and other editors. I thank participants at a conference in Belize for many stimulating and helpful comments. I also thank the NSF under Grant SES-9911251 and the Vilas Trust for essential financial support. None of the above are responsible for errors or shortcomings in this paper.

ANNOTATED REFERENCES

(This well known book gives a thorough treatment of stochastic "turnpike" theorems. The methods developed and reviewed in this book will be valuable for the treatment of optimal management problems in stochastic spatial settings, especially when the payoff function is concave and the dynamics possess some useful concavity. The book also gives an initial foreshadowing of some of the research on optimal control problems on networks which is currently very active at the Central Economic Mathematics Institute in Moscow, Russia.)

(A well known and useful text on optimal control theory.)

(This book gives a detailed treatment of dynamic games where each player optimizes something like an optimal control problem facing the dynamical strategies of the other players. Much of it is linear quadratic, but this is still very useful for linear-quadratic approximations of nonlinear games around steady states)

(This article by Bhattacharya and Majumdar gives a very nice discussion of computing invariant measures and applying the results to economics. They also show how to compute invariant measures for general n-dimensional systems provided certain conditions hold that are more general than simple existence of a "potential".)

(This paper uses moment equation methods to study endogenous spatial pattern formation in a context that stresses the role of specializations for colonization, exploitation, and tolerance in partitioning the endogenous spatial structure in the environment.)

(This paper contains an early treatment of nonconcave optimal control problems. It presents techniques for usefully searching the space of candidate optimal solutions of the first order necessary conditions and transversality conditions.)

(This paper surveys work on econometrically and statistically measuring interactions in various contexts. Many of the methods should be applicable to empirical problems in ecology as well as to economics.)

(This article shows how a more costly but more adaptive (e.g. rational expectations) strategy in competition with a less costly but less adaptive strategy (e.g. static expectations) where fitness of a strategy is defined to be the adaptibility measure minus the cost can lead to dynamic phases of apparently random length. This is caused by a homoclinic bifurcation with respect to a parameter which measures how responsive players are to differences in net fitness when they select strategies. To put it another way this paper demonstrates that "near" homoclinic bifurcations are common in dynamic versions of information paradoxes. It also contains a review of phenomena that occur when a homoclinic bifurcation is present.)


(This book gives a review of methods of stability analysis for deterministic optimal control problems that appear in economics. The book also treats increasing returns problems that look much like the lake problem analyzed in the current volume as well as general methods of comparative dynamics and analysis of tax distortions.)

(This article develops the link between curvature of the optimized Hamiltonian, the discount rate, and global asymptotic stability of optimal management trajectories in infinite horizon deterministic optimal control problems. Malliaris (1982) gives a review and guide to the literature for related results for stochastic optimal control problems.)

(This book covers a lot of recent work on scaling laws and other empirical patterns in biology and ecology. It also serves as a good entry point into this area.)

(This book gives a quite general treatment of infinite horizon optimal control problems and turnpike theorems in the continuous time case. Sufficiency theory is covered with some material on nonconcave problems. It also contains some material on optimal control of systems governed by integro-differential equations and partial differential equations.)
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(This neat little book exposites self-organizing forces in the economy using two central concepts: (i) Order from Instability (e.g. Turing's diffusion induced instability of a flat structure) and (ii) Order from Random Growth (e.g. Zipf's Law)).

Kwakernaak, H., Sivan, R., (1972), LINEAR OPTIMAL CONTROL SYSTEMS, New York: John Wiley and Sons. (This book treats linear optimal control theory for both continuous and discrete time as well as the stochastic case.)

Kuznetsov, Y., (1995), ELEMENTS OF APPLIED BIFURCATION THEORY, New York: Springer-Verlag. (This text is very useful for applications as well as pure theory. It is very user-friendly. Kuznetsov works closely with Cars Hommes's Institute, CENDEF, URL: http://www.fee.uva.nl/cendef. Software for computation of bifurcations can be obtained by linking through CENDEF. Both de Vilder and Kuznetsov have produced useful software. The text also treats bifurcation analysis for reaction-diffusion and other infinite dimensional dynamical systems.)

Lions, J., (1971), OPTIMAL CONTROL OF SYSTEMS GOVERNED BY PARTIAL DIFFERENTIAL EQUATIONS, New York: Springer-Verlag. (This is a basic text for optimal control theory of infinite dimensional dynamical systems including systems driven by partial differential equations.)

Logofet, D., (1993), MATRICES AND GRAPHS: STABILITY PROBLEMS IN MATHEMATICAL ECOLOGY, Boca Raton, Florida: CRC Press. (This book reviews qualitative and quantitative stability analysis of matrices and applies the results to ecological models mostly of Lotka-Volterra, Leslie, and trophic cascade type. It contains theorems on links between qualitative sign patterns of matrices and conditions for the real parts of the eigenvalues to be negative. It does a nice job of quantifying how far one can go with sign information alone (which is not much) in making stability statements.)


Marimon, R., (1989), "Stochastic Turnpike Property and Stationary Equilibrium," JOURNAL OF ECONOMIC THEORY, 47(2), 282-306. (This paper contains general convergence results for optimal solutions of stochastic multisectoral infinite horizon optimization problems for the important case where the payoff function is concave and the constraint set is convex. The paper reviews the literature in this large area. If the discount rate is near zero or is actually zero, convergence always occurs to a unique stationary measure (i.e. a unique stochastic steady state) if modest regularity conditions are imposed. These results may generalize to a host of spatial setups.)

(A classic book in mathematical biology. This book covers not only dynamical systems but also reaction diffusion equations, Turing models, Michaelis-Menten theory, and much more. A partial list of topics includes insect outbreak models (e.g. spruce budworm problem of Ludwig et al.), delay differential equations models, predator-prey models, synchronized insect emergence models, biological oscillators and switches, singular perturbation theory (e.g. mathematical methods for dealing with slow/fast variable systems), reaction diffusion systems, central pattern generators, biological waves, travelling waves, animal coat patterns as application of reaction diffusion mechanisms, neural activity modeling, morphogenesis modeling, evolution and morphogenesis, epidemic models, geographic spread of epidemics, Turing type models, and more.)

(This survey is quite complete for work through 1996 inclusive of economists on bifurcation analysis of discrete time and continuous time optimal control models and related systems.)

(This book develops powerful theory that goes far in characterizing dynamical phenomena in two dimensional nonlinear difference equations systems.)

Puccia, C., and Levins, R., (1985), QUALITATIVE MODELING OF COMPLEX SYSTEMS: AN INTRODUCTION TO LOOP ANALYSIS AND TIME AVERAGING, Harvard University Press. (A loop diagram is basically a dynamical version of a path diagram. You draw an arrow from a variable $x_2$ to a variable $x_1$ with a plus (a minus) on the head if $x_2$ increases makes $dx_1/dt$ increase (decrease). P&L draw these for various ecological systems including (carnivore, herbivore, nutrient), a lake ecology ((piscivorous fish, planktivorous fish, plankton,...,bottom nutrient)) (similar to the lake systems treated here), a tidal pool ecology, and many more. P&L conduct qualitative stability analyses much like the qualitative analysis of tatonnement processes in economics. Qualitative techniques may be useful in cataloging the generic bifurcations in these systems and extending the work to the optimal control of systems shocked by noise and muddled measurements with a temporal hierarchy of variables from fast to slow with increasing difficulty of obtaining observations on slower time scales.)

(This article gives detailed derivations of necessary conditions for optimal control of systems governed by partial differential equations, as well as a guide to the extensive recent literature in this area.)

(This is a very useful text which is user-friendly.)

Scheffer, M., (1998), "Ecology of Shallow Lakes," New York: Chapman and Hall. (Dynamical systems theory and bifurcation analysis plays a major role in this study of shallow lake ecology.)

Skorokhod, A., (1989), ASYMPTOTIC METHODS IN THE THEORY OF STOCHASTIC
DIFFERENTIAL EQUATIONS, American Mathematical Society, Providence, Rhode Island.
(This book could be viewed as a generalization of deterministic theorems on "Taylor" expansion type approximations of slow/fast differential equations systems to the stochastic case.)

(A collection of articles on recent developments in spatial ecology modeling.)

(This paper contains results on the lake problem studied in this volume which relates the phenomena of heteroclinic connections and heteroclinic bifurcations in costate/state dynamics along parameter arcs to the familiar appearance of multiple steady states.)

(This paper gives a general treatment of optimal control problems with multiple optimal steady states and relates convexo-concave state dynamics to the general phenomenon of multiple optimal steady states. It also contains many references to recent work in this area.)