Variance as a leading indicator of regime shift in ecosystem services

Both authors made equal contributions to this paper.
Many environmental conflicts involve pollutants which are dispersed through space and cause losses of ecosystem services. Greenhouse gas emissions are a specific example. As pollutant emissions rise in one place, a spatial cascade of declining ecosystem services can spread across a larger landscape due to dispersion of the pollutant. This paper considers the problem of anticipating such spatial regime shifts by monitoring time series of the pollutant or associated ecosystem services. Using such data, one can construct indicators that rise sharply well in advance of regime shifts. Specifically, the maximum eigenvalue of the variance-covariance matrix of the multivariate time series of pollutants and ecosystem services, calculated over time windows of length T, increases substantially, starting about 5T to 20T time units before the regime shift. No specific knowledge of the mechanisms underlying the regime shift is needed to construct the indicator. Such leading indicators of regime shifts could provide useful signals to management agencies or to investors in ecosystem service markets.

We are grateful for the support of EPA and NSF.

ecological economics
ecosystem service
indicator
market
pollution
regime shift
threshold
variance
INTRODUCTION

Regime shifts are substantial reorganizations of social-ecological systems (Scheffer et al. 2001, Carpenter 2003, Walker and Meyers 2004, Brock 2006). Although examples are known from a wide variety of systems, regime shifts are difficult to study (Scheffer and Carpenter 2003). They usually involve spatially extensive regions, stochastic processes, events that are infrequent (relative to a human lifetime) and processes at two or more scales of space and time (Carpenter 2003). Because of these challenges we are in the early stages of understanding environmental regime shifts. Nevertheless, regime shifts have important implications for human livelihoods and well-being. The Millennium Ecosystem Assessment (MA 2006) noted that regime shifts (which it called "nonlinear changes") could substantially alter the flow of ecosystem services, i.e. the benefits that people obtain from ecosystems.

Recent research shows that ecosystem dynamics become more variable prior to some regime shifts (Berglund and Gentz 2002a,b, Kleinen et al. 2003, Oborny et al. 2003, van Nes and Scheffer 2003, Brock et al. 2006, Carpenter and Brock 2006). This finding suggests that changes in variance can be a leading indicator of impending regime shift. Brock et al. (2006) explain why this is so for one-dimensional systems. Carpenter and Brock (2006) showed that the variance component related to impending regime shift could be separated from environmental noise, using methods that required no knowledge of the mechanisms underlying the regime shift. This finding suggests that it might be possible to detect impending regime shifts using routine monitoring data, even in the absence of detailed ecological understanding of the underlying nonlinearities. It remains to be seen, however, how variance indicators will perform in the real world of managing ecosystem services.

This paper expands the study of variance indicators of impending regime shift in two main ways. First, we extend the approach to multivariate, spatially-distributed systems. For ease of exposition we analyze a two-dimensional system, but our methods generalize directly to higher-dimensional systems. In particular, we sketch a framework for choosing empirical variance indicators that are likely to be sensitive to impending regime shifts. Second, our model couples ecosystem dynamics to flow of ecosystem services. This suggests a forward-looking management system based on expected future flows of ecosystem services. While this paper focuses on the variance indicators rather than policies for future flows of ecosystem services, it is clear that our findings can be nested in a more complete analysis of ecosystem service markets and policies.

MODEL AND METHODS

Overview of the Model

We consider a minimal model of two regions connected by transport of a pollutant (Fig. 1). The pollutant is released as a side effect of activities that yield economic benefits. However, the pollutant has adverse effects on an ecosystem service, such as production of food or regulation of water quality. Thus the net present value to the society in a region is a balance between polluting
activity and the ecosystem service. The situation is analogous to many situations in modern society (MA 2006). Some examples are (1) greenhouse gas pollution, which produces economic benefits but also produces adverse effects on climate and ecosystems, (2) nitrogen emissions from feedlots and fossil fuel burning, which are linked to economic benefits but also produce adverse effects on air and water quality, and (3) pollution with persistent organic toxins, which are associated with specific economic benefits but also cause unwanted harm to ecosystems and human health.

<FIGURE 1 NEAR HERE>

Within a region, the pollutant can be permanently removed (the decay process of Fig. 1) or sequestered (the sink process of Fig. 1). One process is linear in the pollutant (the decay process) and the other becomes saturated at high pollutant levels (the sink process). The linear process is rate-limited only by the supply of the pollutant. An example is destruction of organic pollutants by UV irradiance. The saturating process is rate-limited by factors other than pollutant supply. An example is sequestration of phosphorus (an important cause of water pollution) in soils, which can be limited by the capacity of the soil to bind phosphate ion. Another example is sequestration of CO₂ in ecosystems, which is limited by the capacity of the plants to assimilate carbon.

The pollutant moves between regions, from the region of higher abundance to the region of lower abundance (mixing process of Fig. 1). Many pollutants are transported across regional boundaries. Atmospheric pollutants such as greenhouse gases are the most familiar example. Water pollutants are transferred among regions that share shorelines on a single body of water. Other pollutants are directly transported by people. For example, manure is a waste product of animal production that causes air and water pollution, and is also be transported from regions of high manure production to regions where it is needed as a fertilizer.

Model Details

Within each region i, dynamics of the pollutant P and ecosystem service S are assumed to follow

\[
\frac{dP_i}{dt} = \left[M_i + D(P_j - P_i) - cP_i - sf(P_j)\right]dt + \sigma dW_i
\]

where

\[
f(P) \equiv \frac{m^q}{m^q + P^q}
\]

and

\[
\frac{dS_i}{dt} = \left(\frac{r}{kP_i} - hS_i\right)dt
\]

M is the emission rate, D is the transport rate between region j and region i, c is the decay rate, s is the maximum sequestration rate, f(P) scales the sequestration to the amount of pollutant, \(\sigma\) is the standard deviation of shocks to the system, dW is a Wiener noise process which is N(0,dt) and independent for each i, m is the pollutant level where sequestration is half the maximum rate, q determines the slope of f(P) near P=m, r is the renewal rate of the ecosystem service, k is the
impact of the pollutant on renewal of the ecosystem service, and h is a parameter for the effects of extracting or using the ecosystem service.

A rational social planner would consider the net benefits of pollutant-emitting activities (which generate M) and the ecosystem service (which generates hS and is reduced by increasing P), as well as any other economic consequences of P. The expected net benefits, integrated over infinite time with an appropriate discounting factor, would then be considered (Ludwig et al. 2005). Policies for M and h would be chosen to maximize these expected net benefits. We will leave the economic analysis of this model for later work, in order to focus on the ecological dynamics in this paper. Here we will assume that economic considerations lead to some appropriate choice of M and h for each region. These choices may change as events unfold and circumstances change over time. We wish to study the dynamics as M slowly rises past a critical point.

Effects of three levels of M within one region are presented in Figure 2. The blue curve shows s(P) versus P. The red straight lines show the graph of the function of P defined by M+D(Pj-P)-cP for three values of M. Intersections denote equilibria. When M is low, there is one stable equilibrium at a relatively low value of P. When M is high, there is one stable equilibrium at a relatively high level of P. At the intermediate M, there are three equilibria. The intermediate one is unstable, whereas the low-P and high-P equilibria are stable. This paper will focus on the case where M rises slowly and gradually from a value with two stable P equilibria (like the middle red line) past the critical point where the low-P equilibrium disappears, leaving only the high-P equilibrium (like the upper red line).

Model Analysis

To build intuition about the dynamics and variability of P, we plot potential surfaces and stationary distributions for the model, using equations presented in the Appendix. A "ball-and-cup" diagram for the system is given by a plot of the negative potential versus P1 and P2. Here we plot -log(potential + constant) for convenience, where the constant is chosen to yield positive arguments to the log function. Note that maxima and minima of -log(potential + constant) are identical to those of -potential. Over a long period of time, the system approaches a stationary probability distribution (Appendix). The "spread" of the stationary distribution is related to the variability of P near steady state. The stationary probability distribution is the limit distribution which can be shown to be unique for the diffusion systems considered here. It can be found by solving for the steady state density of the Fokker-Planck equation (Horsthemke and Lefever 1984, Berglund and Gentz, 2002b). It is possible to study regime shifts of stochastic systems via bifurcation analysis of the steady-state solutions of the Fokker-Planck equations. We do not follow that path here, because the system has a potential. However, many ecologically-interesting systems of 2 dimensions and higher do not have potentials (Brock et al. 2006). The framework presented in this paper can be generalized to these cases by studying steady-state solutions of the Fokker-Planck equations.
To study the dynamics of variance, we simulated time series of P and S using the Euler method adapted for the Ito solution of stochastic differential equations (Horsthemke and Lefever 1984). We do this by iteratively computing using [2] below. Each time step \( dt \) is of length \( 1/n \). Results presented here use \( 1/n = 36 \). Over each small increment we compute

\[
dP_{i,t} = \left[ M_{i,t} + D(P_{i,t} - P_{i,t-1}) - cP_{i,t} - s f(P_{i,t}) \right] / n + \sigma Z_{i,t} / \sqrt{n}
\]

\[
dS_{i,t} = \left( r / kP_{i,t} - h S_{i,t} \right) / n
\]

where \( Z_{i,t} \) is an independently-drawn random number (normal, mean 0, variance 1) for each small time increment and region. Results presented here use \( n=36 \). For each time step, the \( n \) values of \( P_1, P_2, S_1 \) and \( S_2 \) are used to compute a covariance matrix. The variance indicator for the time step is the largest eigenvalue of that covariance matrix. The Appendix explains why this is a reasonable indicator of variability due to impending regime shift.

RESULTS

Variance Near a Regime Shift: Simple Case

Brock et al. (2006) and Carpenter and Brock (2006) explain the changes in variance that occur in a one-dimensional system, \( dP/dt = V(P,a) \), as it approaches the regime shift where two stable points are replaced by one stable point. For this paper, the analogous case is pollutant dynamics in a single region, as depicted in Fig. 1. Recall that \( dP/dt = V(P,a) \) and let \( \text{var}(da) \) denote the variance of the random disturbances (a constant in our case), and let \( \text{var}(dP) \) represent the variability we observe in \( P \) over a series of small time increments. In more detail assume \( 0 = V(P+dP,a+da) \), i.e. for each shock \( a+da \), we assume the fast dynamics are so fast that the system has relaxed to a new steady state \( P+dP \). Then it can be shown, by expanding \( V(P+dP,a+da) \) in a Taylor series about \( (P,a) \) that

\[
\text{var}(dP) \approx \left( \frac{\partial V/\partial a}{\partial V/\partial P} \right)^2 \text{var}(da)
\]

(Brock et al. 2006, Carpenter and Brock 2006).

To understand the implications of equation 3 for variance near a regime shift, consider Fig. 3 which plots \( V \) versus \( P \) for a case with lower \( M \) (blue line) and higher \( M \) (red line). Suppose the pollutant level is near the lower stable equilibrium. The red line (higher \( M \)) is closer to regime shift. Clearly \( dV/dP \) is lower for the red line, and therefore \( \text{var}(dP) \) must be larger for the case closer to regime shift. Intuitively, the variance of \( da \) gets magnified more and more as \( dV/dP \) (evaluated at steady state \( P \)) gets closer and closer to zero. Earlier papers discuss this point in more detail (Brock et al. 2006, Carpenter and Brock 2006). Now we turn to the more complex spatial case.
Potential Surfaces and Stationary Distributions

To build intuition about changes as the pollutant emission slowly rises in one region, we plotted negative potential surfaces for the pollutants at four levels of emission in region 1 (Figure 4). Valleys in the surface represent attractors, as in a "ball and cup" stability diagram. At the lowest emission rate ($M_1 = 1.25$, upper left), there is a deep attractor at low pollutant levels, and a shallow attractor at moderately high levels of the pollutant in both regions. When the emission rate is increased to 1.5, a third attractor appears, with moderately high pollutant levels in region 1 and low pollutant levels in region 2. The attractor with low pollutant levels becomes more shallow, and the attractor with relatively high levels of both pollutants deepens. When the emission rate increases to 1.75, the attractor with relatively high pollutant levels becomes even deeper, and the attractor with low pollutant levels becomes quite shallow. At an emission rate of 2.0, the high-pollutant valley appears to be the only attractor.

<INSERT FIGURE 4 NEAR HERE>

Stationary probability distributions are roughly the same as the negative potential surfaces turned upside down (Figure 5). At the lowest emission rate ($M_1 = 1.25$, upper left), the probability mass is concentrated at low pollutant levels, and thus the overall variance of $P_i$ is relatively low. When the emission rate is increased to 1.5, there are 3 distinct peaks with the greatest mass of probability at relatively high pollutant levels, suggesting fairly high variance of $P_i$. The dominance of probability peak at high $P$ becomes successively greater when emission is 1.75 and then 2.0, suggesting lower variance of $P_i$.

<INSERT FIGURE 5 NEAR HERE>

Dynamics of Variance

While the potential surfaces and probability distributions are useful for building intuition about the attractors for the system and the dispersion of $P$ over long periods of time, a person who is observing the system and making decisions about it may not have this information. It is more likely that the decision maker will observe time series of pollutant levels, supply of the ecosystem service, or price of the ecosystem service, and draw inferences from these. Observable features of these time series can provide signals of impending regime shift, in advance of the actual regime shift. To illustrate this point, we simulated dynamics over long periods of time while slowly raising $M_1$.

<INSERT FIGURE 6 NEAR HERE>

Time series from such a simulation show two regime shifts (Figure 6). In the first regime shift, region 1 moves to a moderately high pollutant level followed by movement of region 2 to a moderately high pollutant level. In the second regime shift, region 2 moves to a high pollutant level, followed by movement of region 1 to a high pollutant level. Corresponding shifts in ecosystem services are opposite in direction to the shifts in pollutant level.
The variance index spikes before each regime shift (Figure 7). If we focus on a narrower window of time around the regime shift, it appears that the increase in variance occurs roughly 5 to 20 time steps prior to the regime shift (Figure 8). Note that variance is smoothed such that the index at time t = average of index at times t, t-1, t-2. This backward-looking smoothing might degrade the lead time of the indicator. Despite this, there is still a clear rise in the indicator prior to each regime shift.

DISCUSSION

Rising Variance and Regime Shifts

Our findings corroborate earlier work showing that ecosystem behavior becomes more variable prior to a regime shift (Berglund and Gentz 2002a,b, Brock et al. 2006, Carpenter and Brock 2006, Kleinen et al. 2003, Oborny et al. 2003, van Nes and Scheffer 2003). This paper shows that rising variance serves as a leading indicator in a situation in which ecosystems are coupled across space. The detection of rising variance requires no special knowledge of ecosystem dynamics. A simple indicator based on the dominant eigenvalue of the covariance matrix for ecosystem time series can detect the rising variance. This is so because the dominant eigenvalue is strongly influenced by the variable which is closest to a regime shift. Therefore, the dominant eigenvalue can provide early clues of regime shift even in cases where the investigator is unsure about which ecosystem variables are most important. Also, the dominant eigenvalue may be relatively insensitive to junk variables that are highly noisy but carry no signal related to impending regime shift. In the Appendix, we explain general guidelines for constructing variance indicators for regime shifts.

It is important to note that we have studied a "hard" regime shift in which there is a discontinuous change due to the disappearance of an attractor. For other kinds of regime shifts, variance signals may be weak or absent. For example, the pitchfork bifurcation, in which one regime splits continuously into two regimes, may not be preceded by an increase in variance, depending on the standard deviation of the shocks and the relaxation time of the fast variables in the system (Berglund and Gentz 2002b). Ecosystems seem to exhibit a considerable diversity of regime shifts, a range of speed differences between fast and slow variables, and diverse distributions of shocks (Scheffer et al. 2001, Carpenter 2003, Walker and Meyers 2005). More research is needed to ascertain which kinds of ecosystem regime shifts are likely to be signaled in advance by increases in variance.

Ecosystem Services

Our analysis considered changes in an ecosystem service as well as a biogeochemical variable (the pollutant). Clearly a more complete economic analysis is possible for this system, by specifying the form of the expected net present value over future time, although this is not the purpose of the present paper. However, we note that the variance signals discussed here could
readily be transferred to economic signals. For example, a jump in P at some random time in the future will cause V to drop sharply. Furthermore, individuals who have better knowledge of the system may have more accurate expectations of future ecosystem services than those observers who are not as well informed about the details and structure of the system. Hence if there were a stock market for trading of claims to the future flow of ecosystem services, then the equity value (assuming equity shares are the only outstanding claims against the entity) of such an entity would be given by the expression for discounted net present value over future time (Duffie 1988). Grossman (1989) explains how markets aggregate individual specialized information about assets into market values for those assets.

We believe that a market price equation may lead to development of more sensitive indicators of impending regime change than the variance indicators we develop in this paper. Furthermore in the case of corporations that have claims on ecosystem services where there are options markets available on the stocks of such corporations, an even more powerful indicator of impending regime change is available (Bates 1991). Bates (1991) compares the prices of "put" options (options to sell a certain quantity of stock at a specified target price up to a specified date) versus "call" options (options to buy a certain quantity of stock at a specified target price up to a specified date). His result applies to "out-of-the-money" options, that is a call option with target price higher than the market price of the stock, or a put option with target price lower than the market price of the stock. If out-of-the-money put options are selling for a lot more than out-of-the-money call options for a given stock, then this is strong evidence for an impending sharp drop in the value of the stock. In our context, this would be strong evidence for an impending regime change (i.e. a sharp increase in P that harms S). We believe that this approach can be generalized to settings where the pollutant discharge M or harvest function h(S) are chosen to optimize the expectation of future discounted net present value at each point in time (Carpenter et al. 1999, Carpenter and Brock 2004).

However, markets are available for only a small fraction of the world's ecosystem services, and few of these have options markets (M.A. 2006). In most cases, markets are absent or severely distorted by government intervention. Thus in the majority of cases indicators based on direct monitoring of ecological variables or ecosystem services may be the only means of addressing regime shifts.

CONCLUSIONS

While variance provides a signal of impending regime shift, it remains to be seen whether the message can evoke adaptive social responses. It is plausible that variance signals would trigger appropriate responses by some individual investors in ecosystem service markets. However, by the time the variance signal is clear, the regime shift may be "an accident waiting to happen". Momentum in slowly-changing variables may make it difficult to reverse course (Carpenter 2003). Many factors in societies impede adaptive responses to impending environmental breakdowns (Scheffer et al. 2003). Even if ongoing research verifies the reliability of variance indicators of regime shifts, substantial institutional reforms may be needed to employ such indicators in forward-looking management of ecosystem services.
<Literature Cited>


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Figure 1. Box-arrow diagram of the model, showing flows of the pollutant and economic effects in the two regions.

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Figure 3. Net rate of change for a single region versus pollutant level for 2 rates of input to region 1 (blue: M1 = 1.6; red: M1 = 1.9). Circle denotes stable points. Pollutant level in the other region was fixed at 1.6. Other parameters: M2=0.75; D=0.1; c=0.05; s=0.5; m=5; q=6.

Figure 4. Potential surface (negative log potential) versus pollutant in region 1 and region 2 for 4 different input rates to region 1 (M1 = 1.25, 1.5, 1.75 and 2). Other parameters: M2=0.75; c1=0.05; c2=0.1; D=0.1; s=0.5; m=5; q=6).

Figure 5. Stationary distribution versus pollutant in region 1 and region 2 for 4 different input rates to region 1 (M1 = 1.25, 1.5, 1.75 and 2). Other parameters: M2=0.75; c1=0.05; c2=0.1; D=0.1; s=0.5; m=5; q=6).

Figure 6. Time series of ecosystem services (upper panel) and pollutant (lower panel) in each region versus time as M1 increases linearly from 2.0 to 2.1. Other parameters: r=1; h=0.5; k=0.5; M2=0.75; c1=0.05; c2=0.1; D=0.1; s=0.5; m=5; q=6; sigma-0.05.

Figure 7. Pollutant in region 1 and variance index (upper panel) and pollutant in region 2 and the variance index (lower panel) for the simulation shown in Fig. 5.

Figure 8. Results from Fig. 6 with time frame narrowed to focus on the regime shifts. In this case, the variance index is smoothed by averaging the previous 3 observations, i.e. variance index at time t = mean of the variance index at times t, t-1, and t-2.

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![Graph showing variance index over time for different regions.](image)
Appendix

This section of the paper develops theory to suggest how one can construct indexes that do a better job of amplifying the signal of interest and reducing noise that is not of interest. The idea is to construct an index that maximizes a useful notion of "power" to detect an impending bifurcation. The strategy here is analogous to the design of statistical tests that optimize statistical power against certain alternatives to a null hypothesis.

We implement this strategy by choosing weights that sum to one such that the variance of the indicator is maximized as one approaches an impending bifurcation. Intuition suggests that putting nonzero weight on any component except the one that passes through criticality first is just adding unwanted noise to the indicator that lowers its signal to noise ratio. This reasoning suggests that putting all the weight on $P_1$ until $P_1$ shifts, then putting all the weight on $P_2$ after $P_1$ shifts would be best. The dominant eigenvalue used in our computations is one way of accomplishing this.

We wish to consider indices for the system,

\begin{equation}
\dd P_i = [M_i + D(P_j - P_i) - c_i P_i - \frac{P_i^{sm}}{m^q + P_i^q}]dt + \sigma_i dW_i
\end{equation}

where $i=1,2$ and $dW_1$ is independent of $dW_2$. Write (1) as follows

\begin{equation}
\begin{align*}
\dd P_i &= [M_i - c_i P_i - \frac{P_i^{sm}}{m^q + P_i^q}]dt + D(P_j - P_i) + \sigma_i dW_i \\
\end{align*}
\end{equation}

The potential $F$ is given by

\begin{equation}
F = F_1 + F_2 - \frac{D(P_i - P_j)^2}{2}
\end{equation}

Let us simplify the analysis by putting $\sigma_1 = \sigma_2 = \sigma$. Put $x=[P_1 P_2]'$, $a=M_1$, $W=[W_1 W_2]'$, $F(x,a)$ equal to the right side of (3), and $V = \partial F/\partial x$. In this notation, we may write (1) and (2) as the system,

\begin{equation}
\begin{align*}
\dd x &= V dt + \sigma dW \\
\end{align*}
\end{equation}

Note that $E\{dW \cdot dW\}' = \Id$ where $I$ is the $2 \times 2$ identity matrix. Note that in the applications below we may parametrize an arc through the parameter space $(M_1,M_2,D)$ by the parameter $a$ and still work within a context where $a$ is one dimensional.
It is known (Berglund and Gentz (2002a,b)) that the stationary probability density is given by

(5) \( p(x,a) = \exp[bF(x,a)]/N; \quad b = \frac{2}{\sigma^2} \)

where N is a normalization factor. Assume the global maximum, x(a) of F(x,a) is unique and all eigenvalues of the Hessian matrix, \( \partial^2 F/\partial x^2 \) are negative, for all a < a_c, where a_c is a critical value of a where the largest eigenvalue of \( \partial^2 F/\partial x^2 \) first passes through zero. Expand F(x,a) in a Taylor series about x(a) to obtain

(6)

\[
F(x,a) = F(x(a),a) + \frac{\partial F(x(a),a)}{\partial x}(x-x(a)) + \frac{1}{2}(x-x(a))^t \frac{\partial^2 F(x(a),a)}{\partial x^2} (x-x(a)) + o(|x-x(a)|^2)
\]

\[
= F(x(a),a) + 0 + \frac{1}{2}(x-x(a))^t \frac{\partial^2 F(x(a),a)}{\partial x^2} (x-x(a)) + o(|x-x(a)|^2)
\]

= F(x(a),a) + \frac{1}{2}(x-x(a))^t \frac{\partial^2 F(x(a),a)}{\partial x^2} (x-x(a)) + o(|x-x(a)|^2)

where the zero follows from the first order necessary condition for x(a) to be a maximum of F(x,a) and o(|z|^2)/|z|^2 \to 0 as |z| \to 0, where |z| denotes the Euclidean norm of the vector z.

Insert (6) into (5) and cancel the term F(x(a),a) which is common to both numerator and denominator to obtain

(7) \( p(x,a) = \frac{1}{N} \exp\left\{ b \left[ \frac{1}{2}(x-x(a))^t \frac{\partial^2 F(x(a),a)}{\partial x^2} (x-x(a)) + o(|x-x(a)|^2) \right] \right\} \)

where we continue to use N for the normalization factor. Following Berglund and Gentz (2002b), we assume sigma is small enough so the approximation

(8) \( q(x,a) = \frac{1}{N} \exp\left\{ b \left[ \frac{1}{2}(x-x(a))^t \frac{\partial^2 F(x(a),a)}{\partial x^2} (x-x(a)) \right] \right\} \)

is good. But (8) is a Normal distribution for y(a) \equiv x-x(a) which has mean vector zero and variance covariance matrix

(9) \( S(a) = \frac{-1}{b} \left( \frac{\partial^2 F(x(a),a)}{\partial x^2} \right)^{-1} = \frac{-\sigma^2}{2} \left( \frac{\partial^2 F(x(a),a)}{\partial x^2} \right)^{-1} \)
We see right away for the scalar case that $S(a)$ goes to infinity as $a \rightarrow a_c$. This is so because $\partial^2 F / \partial x^2 \rightarrow 0$ as $a \rightarrow a_c$. This observation is the basic foundation for the variance-based indicators which are developed via numerical evidence in this paper. We will sketch the matrix case below.

We believe that a completely rigorous treatment can be developed by adapting Berglund and Gentz (2002a,b) but that task is beyond the scope of the current paper. The basic idea is to move the parameter $a$ slowly enough through time and to assume that $\sigma$ is small enough so that

(i) The system relaxes close enough to the stationary distribution that we can well approximate the distribution at any date by the stationary distribution for $a(t)$ at date $t$.

(ii) $\sigma$ is small enough and $F(x,a)$ is smooth enough so that the potential $F(x(a),a)$ may be approximated by the quadratic

$$(10) \quad F(x(a),a) + (x - x(a)) \frac{\partial^2 F(x(a),a)}{\partial x^2} (x - x(a))$$

where the matrix $\partial^2 F / \partial x^2$ is evaluated at $(x(a),a)$, for each $a=a(t)$. See Berglund and Gentz (2000b) for this type of development.

Returning to the matrix case, we must develop the idea that $S(a)$ becomes "large" as $a \rightarrow a_c$ and we must develop indices that are dominated by the part of $S(a)$ that contains the most information about impending regime change. For example if $S(a)$ is diagonal, the most information is contained in the largest eigenvalue, which is the smallest eigenvalue of $-\partial^2 F / \partial x^2$. In more detail consider first the case $D=0$. In this case $\partial^2 F / \partial x^2$ is the diagonal matrix with the second order partial $\partial^2 F / \partial P_i^2$ on the $i$th diagonal. Thus, letting $y(a) = x - x(a)$ and letting $w \equiv (w_1,w_2)$ be weights whose squares sum to one, we have $w \cdot y(a)$ is normally distributed with mean zero and variance,

$$(11) \quad \text{var}(w, y(a)) = w_1^2 S_{11} + w_2^2 S_{22} = \frac{\sigma^2}{2} \left[ w_1^2 \left( \frac{\partial^2 F}{\partial P_1^2} \right)^2 + w_2^2 \left( \frac{\partial^2 F}{\partial P_2^2} \right)^2 \right]$$

and this convex function (in $w$) takes a maximum at the extreme points of the convex set of $w$'s, i.e. the disk

$$C = \{ w \mid w_1^2 + w_2^2 \leq 1 \}$$

The biggest value is attained by placing all weight on the smallest $|\partial^2 F / \partial P_i^2|$. This result is general in the sense that the biggest value is attained by the largest eigenvector on the boundary of the disk $C$ of an appropriate variance-covariance matrix (Goldberger 1991). When $D>0$, the matrix of cross partial derivatives is no longer diagonal so we have
which is a positive definite quadratic form even if $V$ is not the derivative of a potential and even if the noises $dW_1, dW_2$ are correlated. The maximum is always taken on the boundary of the disk $C$ at the eigenvector associated with the largest eigenvalue of the positive definite matrix in equation 12.

Finally we point out that there are techniques available to develop useful approximations for systems where there is no potential provided the variance of the driving noise $\sigma^2$ is small enough (see Magill (1977) and references therein to the stochastic approximation literature). Indeed Magill (1977) even shows how to develop local spectral analysis for systems that are optimally controlled. This is a much more difficult task than development of approximations for systems under a fixed control as discussed here. We believe that there are fruitful generalizations available for systems that do not have a potential and we believe that the results will end up focusing on the eigenvalue structure of linearization matrices similar to the development of this paper.

LITERATURE CITED


