The Stochastic Lake Game:  
A Numerical Solution

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Abstract

In this paper, we numerically solve a stochastic dynamic programming problem for the solution of a stochastic dynamic game for which there is a potential function. The players select a mean level of control. The state transition dynamics is a function of the current state of the system and a multiplicative noise factor on the control variables of the players. The particular application is to lake water usage. The control variables are the levels of phosphorus discharged (typically by farmers) into the watershed of the lake, and the random shock is the rainfall that washes the phosphorus into the lake. The state of the system is the accumulated level of phosphorus in the lake. The system dynamics are sufficiently nonlinear so that there can be two Nash equilibria. A Skiba–like point can be present in the optimal control solution.

We analyze (numerically) how the dynamics and the Skiba–like point change as the variance of the noise (the rain) increases. The numerical analysis uses a result of Dechert (1978) to construct a potential function for the dynamic game. This greatly reduces the computational burden in finding Nash equilibria solutions for the dynamic game.

Key words:  stochastic optimal control, Markov processes, stochastic dynamic programming, lake water pollution, dynamic games with a potential function

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1 Introduction

The stochastic dynamic nonlinear model describes the economics of a shallow lake. Lakes provide a variety of economic uses: they are sources of drinking water, fishing, recreational sports and pleasant locations for homes. For agriculture, lakes are used for drainage. Runoff from the fields flows into the streams and rivers that feed the lake.

Much of the agricultural runoff includes phosphorus from fertilizers and animal wastes. Phosphorus is the primary nutrient of algae and weeds in water. When it grows excessively from an infusion of phosphorus, algae blooms reduce the oxygen content of the lake and release toxins into the lake. The lake becomes unsafe for certain aquatic species and for recreational purposes.

Recent papers have explored some fundamental issues of the optimal level of pollution in a lake with competing uses (Brock and Starrett (2004), Carpenter, Brock and Hanson (1999), Carpenter, Ludwig and Brock (1999), Dechert and Brock (2000), Wagener (2003), and Mäler, Xepapadeas and de Zeeuw (2003)). Except for a working paper by Ludwig, Carpenter and Brock (2002), the research has used the deterministic version of the model. This paper introduces a stochastic version of the lake game.

In the deterministic model, the dynamics converge toward steady states. In the stochastic model, there is a long-run distribution of the state variable whose support is in a region about the steady states. The optimal policy for a set of deterministic models converges to a safe low level of the state variable. In the stochastic version of these models, a run of bad shocks can lead to convergence to a potentially unsafe high level. The risk of such as run, of course, is at the heart of many environmental debates.

Our primary interest in this paper is to solve the stochastic model numerically and to characterize the similarities and differences between the results from the deterministic and stochastic version of the lake game model. We believe that the research used to characterize systems such as shallow lakes can be extended to games describing larger-scale systems, such as coral reefs (McClanahan, Polunin and Done (2002)) and wetlands (Gunderson and Walters (2002)).

In section 2, we present a stochastic version of the lake model and the dynamics of phosphorus in the lake. In section 3, we show how the model can be interpreted as an open loop dynamic game. We also show how the stochastic
dynamic game can be solved as a single optimal control problem. In section 4, we discuss the computational methods used to solve the stochastic dynamic programming problem and in section 5, we discuss the results of the simulations.

2 The Stochastic Lake Model

Mäler, Xepapadeas and de Zeeuw (2003) presented a deterministic model in continuous time of the dynamics of phosphorus in a shallow lake. Dechert and Brock (2000) adapted their model to a discrete time model and computed numerical solutions for the optimal policy function. In this section, we introduce shocks to the discrete model.

The deterministic form of the dynamic programming model used by Dechert and Brock (2000) was:

$$\max_{\{u_t\}} \sum_{t=0}^{\infty} \beta^t \left( \log(u_t) - k x_t^2 \right) \quad \text{subject to: } x_{t+1} = b x_t + \frac{x_t^q}{1 + x_t^q} + u_t$$  \hspace{1cm} (1)

where $x_t$ is the level of phosphorus in the lake and $u_t$ is the level of loading of phosphorus.

The utility function chosen for the model is one used by ecologists. The term $\log(u_t)$ is a proxy for the profits derived from additional loading ($u_t$), while the term $-k x_t^2$ is a proxy for the disutility from the effects of phosphorus in the lake ($x_t$). The parameter $k$ represents the number of communities using the lake.\(^2\) The utility function is separable. Typically, profits go to farm owners who do not live near the lake, while the disutility is a cost imposed on those who live near the lake and seek clean water services from the lake. Increasing the loading increases the crop yield but reduces the water quality.

In the dynamical system of lake eutrophication, the parameter $b$ is the fraction of phosphorus from date $t$ that remains in the lake at date $t + 1$. For each loading of phosphorus, a portion flows out of the lake, another portion stays suspended in the lake water while the remaining phosphorus becomes a part of the sediment and mud that lines the bottom of the lake. The parameter $q$ is associated with the irreversibility of a lake saturated with phosphorus. On windy days, the mud at the bottom of the lake is stirred up by the action of

\(^2\) The value $k = c/N$ where $c$ is the community value for clean water service and $N$ is the number of communities. When $N = 1$, variations in $k$ describe variations in community values. For this paper, $c = 1$. Any variations in $k$ refers to lakes that differ by the number of communities.
the waves. This causes some phosphorus that had previously sedimented to be reintroduced into the lake water. Notice that the term $x^q/(1 + x^q)$ is negligible when $x \ll 1$ and approaches 1 when $x \gg 1$. This term represents the switching in the ecological behavior of the lake in the unloaded case ($u_t = 0$) from

$$x_{t+1} = bx_t$$

(2)

for low levels of phosphorus to

$$x_{t+1} = 1 + bx_t$$

(3)

for high levels. When the lake is saturated with phosphorus, the limiting level will be $x = 1/(1 - b)$, while a lake with a low level of phosphorus initially will eventually converge to $x = 0$. For shallow lakes, the value of $q$ is relatively small (between 1 to 3) and the transition of the dynamics from equation (2) to equation (3) will be gradual. For deep lakes, the value of $q$ is large and the transition is more abrupt at the critical value of $x = 1$. A typical value for a shallow lake is $q = 2$.

Dechert and Brock (2000) show that there are up to two optimal steady states. The lower steady state represents the oligotrophic steady state and corresponds to a lake with a low level of algae and weeds. The primary economic value at this steady state is clean water service for residents. The upper steady state represents the eutrophic steady state. The high level of algae blooms and weeds at the eutrophic steady state produces a toxin that can render the water unsafe. This state may be irreversible. The primary economic value at the eutrophic state is higher agricultural profit.

Because of the nature of the nonlinearity in the dynamics of phosphorus in a lake, there are aspects of a “switch” that causes the lake to exhibit a phenomenon that is akin to magnetic hysteresis. When there are two locally stable steady states, the nature of the nonlinearity is such that a small displacement from the oligotrophic steady state can lead to a rapid convergence to the eutrophic steady state. Given the parameters in the models, one of two possibilities will happen; 1) the values of the parameters of the model may be such that the lake is irreversible, and even eliminating all of the phosphorus loading in future periods will not allow the lake to return to its oligotrophic steady state; or 2) the lake is reversible, but it requires reducing the loading well below the oligotrophic steady state level to get the lake to return to that steady state level. Hence the comparison with hysteresis.

Although there may be two locally stable steady state solutions of the Euler equation dynamics, they may not be optimal steady states. This depends on global, not local, considerations. When there are two optimal steady states,
there is a Skiba point\(^3\) between the two steady states.

Figures 1 – 3 illustrate the main findings from the deterministic version of the lake game. The lake parameter \(b\) is 0.45. The three figures vary by the number of communities \(N\) in the watershed. In each of these three figures, the loading, \(u\), is plotted on the vertical axis and the level of phosphorus in the lake, \(x\), on the horizontal axis. There is a dashed line (labeled \(x_{t+1} = x_t\)) that corresponds to a level of loading that will keep the lake at the same level of phosphorus from one period to the next. Levels of loading above this line will cause the phosphorus level to increase, while levels of loading below this line will cause the phosphorus level to decrease. The optimal loading is plotted with a solid line, and the steady state is at the point where the optimal loading line crosses the \(x_{t+1} = x_t\) line. In a model with a single community (Figure 1, \(k = 1.0\)), the optimal policy converges to a lower steady state, the state that is the social optimum. The private interest of the community is consistent with the social interest. Increasing the number of communities lead to a divergence between private and social interests, a scenario that is analogous to the tragedy of the commons. When the number of communities is equal to four (Figure 2, \(k = 0.25\)), the optimal policy converges to the upper steady state. With three communities (Figure 3, \(k = 0.33\)), the optimal policy function is discontinuous with a lower and an upper steady state. Optimal policy for this problem depends on the initial value of \(x_0\).

In the deterministic model, the level of loading can be thought of as the per period mean level of loading of phosphorus. In nature, the shocks are random amounts of rainfall that wash the phosphorus into the lake. The model with shocks is:

\[
V(x) = \max_{\{u_t\}} \mathbb{E} \left[ \sum_{t=0}^{\infty} \beta^t \left( \log(u_t) - k x_t^2 \right) \right]
\]

subject to:

\[
x_{t+1} = b x_t + \frac{x_t^q}{1 + x_t^q} + u_t Z_{t+1}
\]

where the shocks \(\{Z_t\}\) are non negative, iid random variables which satisfy \((\forall t) \mathbb{E} \left[ Z_t \right] = 1\).

\(^3\) In a model with a one dimensional state variable, Skiba (1978) (in a continuous time model) and Dechert and Nishimura (1983) (in a discrete time model) showed that when the dynamics are such that there two optimal steady states, there is a point between the two steady states with two optimal paths: one which leads to the lower steady state and one which leads to the upper steady state. (In the lake model in this section, it is also possible for a third, locally unstable, steady state to also be an optimal steady state. This phenomenon occurs for values of \(\beta\) quite a bit less than 1.)
Except when we want to emphasize the functional form of the dynamics we will define

\[ G(x) = bx + \frac{x^q}{1 + x^q} \]  

so that the dynamics are

\[ x_{t+1} = G(x_t) + u_t Z_{t+1} . \]

3 The Stochastic Lake Game

In the stochastic version of the lake game, each community \(i\) that shares the lake and its watershed must make a decision on the per period mean loading of phosphorus, \(u_{i,t}\). Due to the heavy equipment used by farmers, fertilizer is spread before the ground thaws. In each period, a random shock (rain that precedes the thawing of the ground), \(Z_t\), will wash

\[ Z_t \sum_{i=1}^{N} u_{i,t} \]  

units of phosphorus into the lake. The level of phosphorus in the lake follows the stochastic dynamics

\[ x_{t+1} = G(x_t) + Z_t \sum_{i=1}^{N} u_{i,t} \]  

where there are \(N\) communities around the lake. An alternative and equivalent way at looking at the shock is to model each community’s decision as a total loading of \(w_{i,t}\). The rain washes a random fraction, \(R_{t+1}\), of it into the lake. If we denote the mean fraction washed into the lake by \(\mu = E[R_{t+1}]\), then \(u_{i,t} = \mu w_{i,t}\) and \(Z_{t+1} = \mu^{-1}R_{t+1}\). The other aspect of the modeling is that the phosphorus that is applied to the land does not accumulate in such a way that eventually it all washes into the lake. The remaining fraction, \(w_{i,t} (1 - R_{t+1})\), is absorbed into the soil and the plants that are grown on the soil. (It is true that some of the phosphorus leaches into the lake by subsurface drainage, but this can be thought of as included in the value of \(w_{i,t}R_{t+1}\).) Thus, in each time period there is only the fresh application of phosphorus that is available to add to the lake water.
In the open loop dynamic game, each community, $i = 1, \ldots, N$, solves:

$$\max_{\{u_i,t\}} \mathbb{E} \left[ \sum_{t=0}^{\infty} \beta^t \left( v_i(u_{i,t}) - x_{i,t}^2 \right) \right]$$

(8)

subject to: $x_{t+1} = G(x_t) + Z_t (u_{i,t} + U_{i,t})$

where $x_{i,t}^2$ is community $i$’s relative value of clean water services from the lake and community $i$ takes the sequence $\{U_{i,t}\}$ as given. Each community can, in fact, have different relative values, $-k_i x_{i,t}^2$. By dividing the per period objective function of each community by $k_i$, this parameter is incorporated into the utility function, $v_i$.

A Nash equilibrium is a set of solutions, $\{u_{1,t}\}, \ldots, \{u_{N,t}\}$, such that

$$U_{i,t} = \sum_{j \neq i} u_{j,t} \quad i = 1, \ldots, N .$$

(9)

### 3.1 A Potential Function for the Lake Game

A potential function for the dynamic game is a function

$$\Pi(u_1, \ldots, u_N, x)$$

(10)

such that a solution to the control problem

$$\max_{\{u_1,t\cdots u_N,t\}} \mathbb{E} \left[ \sum_{t=0}^{\infty} \beta^t \Pi(u_{1,t}, \ldots, u_{N,t}, x_t) \right]$$

(11)

subject to: $x_{t+1} = G(x_t) + Z_t \sum_{i=1}^{N} u_{i,t}$

is a Nash equilibrium solution to the discrete time dynamic game (8) – (9). The advantage of this approach is that in order to get a solution to the dynamic game, we not only have to solve the dynamic programming problem (8) for each community, but we also have to solve a system of equilibrium equations (9). However, solving the optimal control problem (11) only requires us to solve a single dynamic programming problem. The techniques in Dechert (1978) and Dechert (1997), which are for systems of Euler equations, can be adapted to
the type of problem in this section. Using these techniques lead to

$$\Pi(u_1, \ldots, u_N, x) = \sum_{i=1}^{N} v_i(u_i) - x^2.$$  \hspace{1cm} (12)

Rather than go through the derivation that leads to equation (12), let us show that a solution to the control problem satisfies the best reply property of a Nash equilibrium of the dynamic game. That will demonstrate that (12) is a potential function for the dynamic game. Suppose that

$$\{\hat{u}_{1,t}, \ldots, \hat{u}_{N,t}, \hat{x}_t\}$$ \hspace{1cm} (13)

is a solution to the control problem

$$\max_{\{u_{i,t}\}} E \left[ \sum_{t=0}^{\infty} \beta^t \left( \sum_{i=1}^{N} v_i(u_{i,t}) - x^2_t \right) \right]$$ \hspace{1cm} (14)

subject to: $x_{t+1} = G(x_t) + Z_t \sum_{i=1}^{N} u_{i,t}$

Now let us check the best reply property for community 1. Since (13) is a solution to the optimal control problem, for any sequence, $\{\tilde{u}_{1,t}\}$, and corresponding state sequence

$$\tilde{x}_{t+1} = G(\tilde{x}_t) + Z_t \left( \tilde{u}_{1,t} + \sum_{i=2}^{N} \hat{u}_{i,t} \right)$$

the following inequality holds:

$$E \left[ \sum_{t=0}^{\infty} \beta^t \left( \sum_{i=1}^{N} v_i(\tilde{u}_{i,t}) - \hat{x}^2_t \right) \right] \geq E \left[ \sum_{t=0}^{\infty} \beta^t \left( v_1(\tilde{u}_{1,t}) + \sum_{i=2}^{N} v_i(\hat{u}_{i,t}) - \tilde{x}^2_t \right) \right]$$ \hspace{1cm} (15)

On both side of equation (15) there are common terms of $v_i(\hat{u}_{i,t})$ for $i = 2, \ldots, N$ which can be canceled, leaving

$$E \left[ \sum_{t=0}^{\infty} \beta^t \left[ v_1(\tilde{u}_{1,t}) - \hat{x}^2_t \right] \right] \geq E \left[ \sum_{t=0}^{\infty} \beta^t \left[ v_1(\tilde{u}_{1,t}) - \tilde{x}^2_t \right] \right]$$ \hspace{1cm} (16)

Thus, $\{\hat{u}_{1,t}\}$ is a best reply for community 1 to $\{\tilde{u}_{2,t}, \ldots, \tilde{u}_{N,t}\}$ by the other communities. A similar argument applies for communities 2, \ldots, N as well.
Notice that the solution to the control problem (14) can be recast in the following way. Let

\[ w(u) = \max_{\{u_1, \ldots, u_N\}} \sum_{i=1}^{N} v_i(u_i) \]

subject to: \( \sum_{i=1}^{N} u_i = u \)

and solve:

\[ \max_{\{u_t\}} E \left[ \sum_{t=0}^{\infty} \beta^t \left( w(u_t) - x_t^2 \right) \right] \]

subject to: \( x_{t+1} = G(x_t) + u_t Z_t \)

for the optimal sequence, \( \{\hat{u}_{i,t}\} \). Then solve (17) for the optimal values of \( \{u_{i,t}\} \) given \( \{\hat{u}_t\} \). This problem will have exactly the same solution as the problem in equation (14). This reduces the solution to the dynamic game to solving a single control problem with a single control variable. It is this latter problem that we will solve and discuss in the next section. For the case that

\[ v_i(u_i) = a_i \log(u_i) \]

it can easily be verified that

\[ w(u) = C + A \log(u) \]

where

\[ A = \sum_{i=1}^{N} a_i \]

and the constant \( C \) has no effect on the solution to the dynamic programming problem. By dividing by \( A \) and letting \( k = 1/A \), we come to the formulation in equation (4). By reducing the problem in this way to a single control problem, we can get numerical solutions to the dynamic game very quickly.

It is worth pointing out several observations on the method outlined above. First, there may be more Nash equilibria solutions to the dynamic game than
there are solutions to the control problem (11). However, if there is a unique solution to the dynamic game, then of necessity, there is a unique solution to the optimal control problem. Second, this method only works if the individual objective functions satisfy the restrictions in Dechert (1978) and Dechert (1997). Third, the advantage of this method is twofold. The practical advantage is that we can use dynamic programming to solve the control problem. The theoretical advantage is that for nonconvex problems, the existence of a Nash equilibrium solution to the dynamic game is assured by the existence of a solution to the control problem.

4 Computational Methods

There are two primary methods for solving high dimensional dynamic programming problems. One is to convert the problem to a Discrete Decision Problem and to use the randomization method of Rust (1997). The other is to use soft computing methods such as Keller, Bolker and Bradford (2004) who successfully used a genetic algorithm to solve a climate change model with irreversibilities. The problem with these methods is that they are slow, and do not allow the researcher to explore the space of parameters and models broadly. For some dynamic programming problems, these are the only methods that produce a solution in a reasonable amount of time.

Bertsekas’s modified policy iteration algorithm (Bertsekas (2001)) computes a Bellman operator iteration followed by many policy operator iterations. Policy operator iteration is much faster computationally than Bellman operator iteration because there is no maximization step. The number of such iterations depends on the problem and the discount rate. We used 1000 policy operator iterations for each Bellman operator iteration. Each Bellman operator iteration not only produces the next value function in the iteration scheme, but it also produces the policy function that maximized the Bellman operator at that step. This is the initial policy function used in the next set of policy operator iterations.

The solution of the model in equation (4) was computed primarily by iterating the Bellman operator:

\[
(Tv)(x) = -kx^2 + \max_u \left\{ \log(u) + \beta \sum_{i=1}^n p_i v(G(x) + uz_i) \right\}
\]

(19)

where \( \{z_i\} \) are the distinct values of the shocks and \( \{p_i\} \) the probabilities of their occurrence. The iteration scheme is then:

10
\[ v_{m+1} = T v_m \]

with \( v_0 \) being an arbitrary initial function. It is easy to show that \( T \) is a contraction operator, and that this scheme converges, \( v_m \to V \).

For the policy operator iteration step, we used,

\[ (T_h v)(x) = -k x^2 + \log(h(x)) + \beta \sum_{i=1}^{n} p_i v \left( G(x) + h(x) z_i \right) \]

where \( u = h(x) \) is a specified policy function. If we denote \( V_h \) as the value of using policy \( h \) and with \( v_0 \) arbitrarily chosen, the iteration scheme

\[ v_{m+1} = T_h v_m \]

converges as well, \( v_m \to V_h \).

The results of the numerical computation is the fixed point, \( V \), of the Bellman operator (19) and the policy function, \( h_V \), that satisfy

\[ V(x) = -k x^2 + \log(h_V(x)) + \beta \sum_{i=1}^{n} p_i V \left( G(x) + h_V(x) z_i \right) \]

The shocks \( \{Z_t\} \) used in our simulation were based on log–normal random variables,

\[ Z_t = e^{\sigma W_t - \sigma^2/2} \]

where \( W_t \) is a standard normal and the variance of the shocks is,

\[ \text{Var} [Z_t] = e^{\sigma^2} - 1 \]

One advantage of modeling the shocks in this way is that they are standard and their properties are well understood. The second advantage is that the expectation

\[ \mathbb{E} [f(uZ_t)] \]
can be approximated by the sum
\[
E \left[ f(uZ_t) \right] \approx \sum_{i=1}^{n} w_i f(u e^{\sigma x_i - \sigma^2/2})
\]
where the abscissas, \( \{x_i\} \), and the weights, \( \{w_i\} \), can be found from the Gauss–Hermite quadrature scheme for numerically computing the value of an integral with an integrand of the form \( e^{-x^2} \). An alternative to the Gauss–Hermite quadrature is to choose a finite set of shocks, \( Z_{t,i} = e^{\sigma x_i - \sigma^2/2} \), and assign each with a probability \( w_i \) in such a way to preserve the mean value, \( E[Z_t] = 1 \).

For the simulations, solutions were found more quickly by using the solutions from a prior simulation as initial starting values. Essentially, this method provided a “good initial guess” for the value function.

The programs were written in C++ and the simulations ran on a dual Xeon 2.4GHz PC running Linux 2.6. Simulations produced with programs that did not include the modified policy iteration algorithm took on the order of ten to fifteen hours. Using the modified policy iteration algorithm, simulations took on the order of ten minutes without starting values and one minute with starting values. The mesh size was of the order of \( 10^{-4} \).

5 Simulations

For all simulations, we used a \( b \) parameter equal to 0.45 and a \( \beta = 0.997 \). Although this seems unusually close to no discounting at all, the \( \beta \) value corresponds to an actual annual interest rate (for Lake Mendota in Madison, Wisconsin) of 2.1\%. The value of \( \beta \) is the result of changes in physical and temporal units that comes about from the transformation of the continuous time model to the discrete time model. See Dechert and Brock (2000) for the details. As a baseline for comparison, Figures 1 – 3 are solutions to problems without shocks (\( \sigma^2 = 0 \)).

The \( y \) axes for the stochastic models (Figures 4 – 12) are a transformation of the \( y \) axes used in the deterministic problems (Figures 1 – 3). It is customary in dynamic phase portraits to draw the line that corresponds to the level of control for which \( x_{t+1} = x_t \). Denote this line by \( a(x) = x - G(x) \), Then, the optimal policy function, \( \hat{h}(x) \), is plotted. From such a graph we can see how the system will behave when the optimal control is applied to it. In our figures, we plot the difference, \( h(x) = \hat{h}(x) - a(x) \) multiplied by each shock, \( Z_t \). Then, when \( Z_t h(x) \) lies above the horizontal axis, we know that the level of the state variable will increase, while when it is below the axis the level of the state
variable will decrease. In effect, we are plotting \( G(x) - x + uZ \) with \( u = h(x) \) on the vertical axis and have labeled them as such.

Figures 4 – 12 show the policy function, \( h(x) \), multiplied by the value of each of the possible shocks, \( Z_i \). We have chosen the shocks and their probabilities so that: \( Z_{\text{low}} < Z_{\text{mid}} < Z_{\text{high}} \), and \( Z_{\text{mid}} = 1 \). The values of \( Z_{\text{low}} \) and \( Z_{\text{high}} \) are chosen so that \( \mathbb{E}[Z] = 1 \) and \( \text{Var}[Z] = \sigma^2 \).

Long-run distributions are included in Figures 4 – 12. The simulations are based on running the model for 500,000 periods and then plotting the next 500,000 observations. For each problem, we generated a large number of simulations by varying the value of \( x_0 \). In three cases (represented in Figures 4 – 6), we indicated that the long-run distribution was sensitive to the value of \( x_0 \).

Let us now focus on the Skiba–like case with shocks. Setting \( k = 0.3152 \), we increase the value of the variance of the shock, \( \sigma^2 \).

In Figures 4 – 6 are two long-run distributions based on simulating the time series. In Figures 4–5, there are two basins of attraction for the stochastic dynamics. In Figure 4, for values of \( x \) between 0.45 and 0.95, \( Z_i h(x) \) lines lie below the \( x_{t+1} = x_t \) line, and, in this range, \( x_{t+1} < x_t \) with probability one. Similarly, for values of \( x \) between the 0.95 and 1.7, \( Z_i h(x) \) lines lie above the \( x_{t+1} = x_t \) line and \( x_{t+1} > x_t \) with probability one. Thus, depending on the initial value of \( x_0 \), the process will converge to one of the two distributions.

In Figures 4 – 6, we see the effects of increasing the variance on the long-run distribution. Not only is the increased variance of the long-run distribution of the state variable, \( x \), apparent, the transition from two stable distributions to a single distribution and a transient distribution is also evident.

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4. The stochastic equivalent of a Skiba point is a transient set between two basins of attraction. The basins of attraction are recurrent sets for the Markov dynamics. The two basins of attraction are separated by a region where there is a positive probability that the dynamics will end up in the lower basin of attraction, and a positive probability they will end up in the upper basin of attraction. In the sequel we will refer to this as the stochastic Skiba point (SSP for short), even though it is not a point but a set.

5. Values of \( k \) that are near the value of 0.33 will generate two optimal steady states. The policy functions represented in all figures are individual solutions over a grid. The coarseness of the grid often produced poorer quality graphs. This was true for \( k = 0.33 \). Our choice of \( k \) for the stochastic problems was based on solving a SSP case that produced more accurate numerical solutions which resulted in higher quality graphs.

6. The distributions are not in scale vertically so that they can be superimposed on the same graph. The areas under the two distributions should be equal when drawn on the same scale.
In Figure 6, the long-run distribution is centered at $x = 1.95$. There is a second distribution centered at $x = 0.3$ that is actually a transient distribution. The way one can determine that this distribution is a transient one is to note that the policy function times the high shock, $Z_{high}h(x)$, is always above the line where $x_{t+1} = x_t$. Thus, a run of bad luck, i.e., a long sequence of predominately high shocks, will eventually lead the state to a value such that $x_t > 0.89$. Between $x = 0.89$ and $x = 1.2$, $Z_t h(x)$ are above the $x_{t+1} = x_t$ line, and so the state increases into the region where $1.2 < x < 4.0$. This latter is the support of the asymptotic distribution for this model with these parameters. Although this is the support of the true long run distribution, numerically it appears to be $1.6 < x < 2.3$. Essentially, the probability of being outside this range is so small that even with 1,000,000 simulations there are no points observed in the extremes of the long-run distribution.

In Figures 4 – 5, the value of $Z_{high}h(x)$ drops below the $x_{t+1} = x_t$ line, and so both distributions are long-run distributions. In this sense, the features of the deterministic Skiba diagram are preserved (at least at lower variances) in the stochastic case. If the initial value of the state, $x_0$, is low enough, the optimal policy will lead to a long run in the oligotrophic region. On the other hand, if the level of $x_0$ is high, then the optimal policy will keep the state in the eutrophic region.

Another feature that can be seen in Figures 4 – 6 is there is no longer a single Skiba point as in the deterministic case. (See the discussion in footnote 4.) In the one dimensional case, the value function is not differentiable at the Skiba point, except when the locally unstable steady state is also an optimal steady state. In the deterministic case depicted in Figure 3 the policy function has a discontinuity at $x = 0.95$, and consequently the value function is not differentiable at that point. Uncertainty “smooths out” the value function so that it is differentiable. In the graphs of the numerical results it is hard to see that the policy function is indeed continuous since the slope is so steep in this range of $x$. In Figure 6 one can see some dots at $x = 0.89$. The other feature of these graphs is as $\sigma^2$ increases (from 0.13 to 0.72 to 0.99), the value of $x$ at the SSP decreases (from 0.94 to 0.92 to 0.89 respectively).

Figures 7 - 12 show the impact of increasing the variance for the cases that there is not an SSP. It is evident from the graphs that an increase in the variance of the shocks causes the state to move away from the oligotrophic region. This is because the high shocks become less likely as the variance of the shocks increases.

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7 Although it is a transient distribution, it is based on 1,000,000 observations of the time series.
8 In this figure the probability of a high shock is 0.18. Therefore, the probability of a run of 20 high shocks in a row is approximately $1.27 \times 10^{-15}$ which is a very low probability event. This is the reason why the initial segment of $10^6$ simulations remains in the oligotrophic region of the state space.
9 Florian Wagener, private correspondence. See Wagener (2003) for details on this and other cases in a deterministic, continuous time model of the lake.
variance of the shocks, $\sigma^2$, leads to an increase in the variance of the long-run distribution of the state variable. It also can have a dramatic effect on the support of that distribution. Compare Figures 8 and 9. In the former the $Z_{\text{high}} h(x)$ line falls below the $x_{t+1} = x_t$ line at $x = 0.7$, and so the support is the interval $[0.2, 0.7]$. In Figure 9 the $Z_{\text{high}} h(x)$ line is above the $x_{t+1} = x_t$ line for $x$ up to 2.8, and so the support of the distribution is the interval $[0.2, 2.8]$, which is a dramatic increase in length. Admittedly, there is essentially a zero probability of observing a value of the state variable in the upper part of the distribution.

6 Conclusion

The real application of this type of research is of course to help us understand the economic forces in the model so that we can better regulate our use of the environment. If indeed the net benefits, $B(u, x) = \log(u) - kx^2$, represent the private per period revenues minus costs, then the system will follow the policy function that we have solved for in this paper.\textsuperscript{10} This would tell us the potential outcomes of an unregulated system. Similarly, if $B(u, x)$ is the net social benefit per period, then the policy function shows us how the system should be regulated.

There is another application of this type of research that reinforces the application of the precautionary principle of Holling (1978). Consider Figures 6 and 9 and suppose that the system is initially at a low value of the state variable, $x$. We may understand the benefits and costs at these low levels (i.e., the function $B(u, x)$ is known in that region), however, we may not know all of the costs that might be associated with high levels of $x$. In that case one might argue that even though the optimal policy, $h(x)$, is to take the risk that we might end up in the high state region, we should be cautious and use a lower value of control so that the state will remain in the low region with probability one. Figure 9 shows a variant of this theme. In this case the support of the long-run distribution of the process $x_t$ goes up to $x = 2.8$ where $Z_{\text{high}} h(x)$ cuts the $x_{t+1} = x_t$ line. So even though there is not a “flip” that occurs that leaves us permanently in the high range of the state variable, nevertheless the optimal policy is to take the risk that the system may at times be in the eutrophic region. If indeed the costs of having high levels of the state variable are poorly understood, it may well be that the best policy would be to reduce the control so that the probability of reaching such levels of the state variable is zero.

This actually raises a number of issues that deserve much more discussion and

\textsuperscript{10} Naturally, other functions, $B(u, x)$, can be used in solving the model.
analysis than is presented here. For example, suppose that the lake is eutrophic for \( x > 1.7 \) and let \( T \) be the first date such that \( x_T > 1.7 \) in Figure 9. Notice that \( T \) is finite with probability one. If \( E[T] = 100,000 \) years say, then one might argue that other phenomena that operate on a geological time scale are going to have a much larger impact on the lake than our current decisions about phosphorus loading. Therefore the optimal policy might indeed be the best policy, even if it runs the risk of a catastrophic event.

References


URL: http://algol.ssc.wisc.edu/research/research/dgames.pdf


Fig. 1. Oligotrophic Steady State

Fig. 2. Eutrophic Steady State
Optimal policy \( x_{t+1} = x_t \)

Fig. 3. Skiba Point

Policy function convergence
for \( x_0 > x_{SSP} \)

\( Z_{HIGH} = 1.50, \ p = 0.32 \)
\( Z_{MID} = 1.00, \ p = 0.20 \)

Policy function convergence
for \( x_0 < x_{SSP} \)

\( Z_{LOW} = 0.67, \ p = 0.48 \)
\( x_{t+1} = x_t \)

\( \sigma^2 = 0.13 \quad k = 0.3152 \)
Policy function convergence
for $x_0 > x_{SSP}$
for $x_0 < x_{SSP}$

$Z_{HIGH} = 2.50, p = 0.23$
$Z_{MID} = 1.00, p = 0.20$
$Z_{LOW} = 0.40, p = 0.57$

$x_{t+1} = x_t$

Fig. 5. $\sigma^2 = 0.72 \; k = 0.3152$

Policy function convergence
for $x_0 < x_{SSP}$
for $x_0 > x_{SSP}$

$Z_{HIGH} = 3.50, p = 0.18$
$Z_{MID} = 1.00, p = 0.20$
$Z_{LOW} = 0.29, p = 0.62$

$x_{t+1} = x_t$

Fig. 6. $\sigma^2 = 0.99 \; k = 0.3152$
Fig. 7. \( \sigma^2 = 0.13 \) \( k = 0.45 \)

Fig. 8. \( \sigma^2 = 0.72 \) \( k = 0.45 \)
Policy function convergence

\[ G(x) = -x + uZ \]

\[ x_{t+1} = x_t \]

\[ Z_{HIGH} = 3.50, p = 0.18 \]
\[ Z_{MID} = 1.00, p = 0.20 \]
\[ Z_{LOW} = 0.29, p = 0.62 \]

Fig. 9. \( \sigma^2 = 0.99 \) \( k = 0.45 \)

Policy function convergence

\[ G(x) = -x + uZ \]

\[ x_{t+1} = x_t \]

\[ Z_{HIGH} = 1.50, p = 0.32 \]
\[ Z_{MID} = 1.00, p = 0.20 \]
\[ Z_{LOW} = 0.67, p = 0.48 \]

Fig. 10. \( \sigma^2 = 0.13 \) \( k = 0.25 \)
Policy function convergence

$Z_{\text{HIGH}} = 2.50, p = 0.23$
$Z_{\text{MID}} = 1.00, p = 0.20$
$Z_{\text{LOW}} = 0.40, p = 0.57$

$x_{t+1} = x_t$

Fig. 11. $\sigma^2 = 0.72 \ k = 0.25$

Policy function convergence

$Z_{\text{HIGH}} = 3.50, p = 0.18$
$Z_{\text{MID}} = 1.00, p = 0.20$
$Z_{\text{LOW}} = 0.29, p = 0.62$

$x_{t+1} = x_t$

Fig. 12. $\sigma^2 = 0.99 \ k = 0.25$