PROFILING PROBLEMS WITH PARTIALLY IDENTIFIED STRUCTURE

by

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ABSTRACT

We study two main classes of profiling problems where the relevant structural inputs to the problems may be only partially identified: (i) Class I is where profilees are not "strategic" in altering their characteristics; (ii) Class II is where profilees strategically alter their characteristics at a cost. These problems are divided into two main subclasses. Subclass A (B) are problems where the objective of the profitor is the same (different, maybe even opposite) as the profilees. An example of a Subclass A type of problem is program evaluation where people are admitted (i.e. "profiled") into the program based upon observable characteristics and the objective function of the program designer (i.e. the profitor) is the same as the profilees. But the profilees may have an incentive to alter their observable characteristics in order to get into the program. Examples of Subclass B types of problems are airport screening, lending, employing, etc.

1. INTRODUCTION

We study two main classes of profiling problems where the relevant structural inputs to the problems may be only partially identified: (i) Class I is where profilees are not "strategic" in altering their characteristics; (ii) Class II is where profilees strategically alter their characteristics at a cost. These problems are divided into two main subclasses. Subclass A (B) are problems where the objective of the profitor is the same (different, maybe even opposite) as the profilees. An example of a Subclass A type of problem is program evaluation where people are admitted (i.e. "profiled") into the program based upon observable characteristics and the objective function of the program designer (i.e. the profitor) is the same as the profilees. But the profilees may have an incentive to alter their observable characteristics in order to get into the program. Examples of Subclass B types of problems are airport screening, lending, employing, etc.

In section 2, we lay out a basic framework and develop the problem of a lender who profiles across observable type strata as an expository vehicle. We draw connections between the maximin and minimax regret problems and zero sum game theory. This suggests tools of analysis as well as potential extensions by exploiting the large literature on zero sum stochastic games, and adaptive learning in such contexts. We sketch a simple profiling-adaptive learning scheme "in population" and compare maximin and minimax regret learners. In the context of our simple scheme the minimax regret learner learns everything it wants to learn in just one period.

Section three continues the analysis by considering problems with unobservable "sub-type" distributions associated with each observable type. This raises issues of "selection bias" and we show a connection between the size of the bias and tools from monotone comparative statics such as assumptions about increasing differences and assumptions about first order stochastic dominance. We barely scratch the surface of potential applications of this connection.

Section 4 conducts what we call a "top down" approach in contrast to the
earlier sections which conducted what we call a "bottoms up" approach. The
"top down" approach replaces the "large" "ambiguity sets" associated with
planning under unknown structures with "small" ambiguity sets centered around
a "baseline." This approach enables us to draw a connection between the
rapidly growing area of research on robust control and our concerns here. We
find that there are consistent patterns in the directions where the planner
needs to robustify treatment choice more intensely as a function of the
structure of the ambiguity set of partially identified structures.
Furthermore in continuous settings the formulae reveal an interesting
connection with hazard functions. Again, we have barely scratched the
surface.

Section 5 briefly applies the tools developed in previous sections to
Class II profiling problems where profilees have a strong incentive to falsely
report their type to the planner. We restricted ourselves to a subclass of
these problems where the cost function to false reporting is "L-shaped" in the
sense that a profilee can report a type up to "e" greater than its true type
for a cost of zero, but the cost becomes infinite if it reports a type that
deviates above its true type by more than "e". We sketch how the tools can be
applied to this class of problems and sketch what the maximin and minimax
regret solutions look like.

Finally, Section 6 contains summary, conclusions and suggestions for
future research. In particular, it contains a brief sketch of a problem of
detecting discrimination by employers and employer learning about unfamiliar
types. This example is included not only because it is of current policy
importance but also to suggest potential applications of the tools developed
in earlier sections.

2. BASIC FRAMEWORK

A "Treatment Rules" Framework

The basic framework of the paper builds on Manski's article, (2004a), on
structures, Brock and Durlauf (2004) on local robustification, and Brock,
Durlauf, and West (2003), on model uncertainty. We use Manski's (2004a)
notation as much as possible. There is a planner who wishes to choose a
policy \( t \) in \( T \) (i.e. a "treatment rule") to maximize population mean welfare.
Manski assumes that the welfare from assigning treatment \( t \) to person \( j \) is
\( u_j(y_j(t), t, x_j) \) where \( u(\ldots) : Y \times T \times X \rightarrow R \) is the welfare function which is
"individualistic", i.e. \( u \) does not depend upon the outcomes of persons other
than \( j \) and \( x_j \) denotes observable characteristics of person \( j \). In this paper
the welfare function \( u \) is known only to lie in a set \( U \) which may contain
welfare functions where person \( j \)'s welfare depend upon outcomes of others,
e.g. person \( j \)'s welfare may depend upon the welfare of people in \( j \)'s
"reference group". As in Manski's book (2003) observed data samples
\( \{y_j(t), x_j, t \in T, x \in X, j = 1, 2, \ldots, N(t, x)\} \) may be corrupted by unknown
selection on unobservables, partially identifiable Data Generating Processes
(DGP's) and the like. Furthermore, we deal with the extra complication that
individuals may alter their observable characteristics in order to get what
they think is a more desirable "treatment", i.e. policy \( t \) assigned to them,
and so on. Since the area of investigation we wish to deal with is vast, we
write this paper as a series of examples that are unified by a common style of
attack.

EXAMPLE 1: LENDING TO A PARTIALLY IDENTIFIABLE POPULATION
In this section of the paper we will develop a leading example of a lender who profiles on an observable characteristic \( x \) in \( X \). This problem will set the stage for the other examples to follow.

Consider a lender who is screening, i.e., "profiling", loan applications from members of a population \( J \) and wants to know \( P(1|x) \), the probability of paying off the loan by a member of \( J \) with one dimensional observable characteristic \( x \) in \( X \). Think of \( x \) as an observable index of payback potential. As in Manski (2004a), we will assume that \( X \) is finite, when convenient, in order to keep the exposition simple. In some of the analysis below we will assume \( X \) is the non-negative orthant of the \( n \)-dimensional reals.

The lender observes performance only by those members of \( J \) with characteristics \( x \) who applied for a loan before, were accepted by the bank, and actually took out their loan from the bank. Denote this event by \( z=1 \). Suppose the bank knows \( P(x) \), the fraction of people in \( J \) with characteristics \( x \) in \( X \). Assume \( P(1|x) \) is increasing in \( x \). Let profits, \( R[P(1|x)] \), be increasing in \( P(1|x) \), negative for small values of \( P(1|x) \), positive for large values of \( P(1|x) \). If the bank knows \( P(1|x) \) the problem is simple: Find a cutoff level, \( x^* \), and lend to members of \( J \) with \( x \geq x^* \). Note that this way of posing the problem gives an optimal rule that is independent of \( P(x) \). Formulations where the rule depends upon \( P(x) \) are studied also.

Following, Manski (2003, p. 6), observe that

\[
(2.1) \quad P(1|x) = P(1|x,z=1)P(z=1|x) + P(1|x,z=0)P(z=0,x),
\]

and note that the bank observes \( P(1|x,z=1) \), \( P(z=1|x) \), \( P(z=0,x) \), but knows nothing about \( P(1|x,z=0) \) except that it lies in \([0,1]\). Think of \( x \) as an "observable type" and think of \( P(z=1|x) \) as the fraction of observable type \( x \)'s where the outcome of the loan has been observed by the lender, and think of \( P(z=0,x) \) as the fraction of observable type \( x \)'s where the outcome of the loan has not been observed by the lender. Hence, \( P(1|x,z=0) \) can take any value in \([0,1]\). We will emphasize the special case where \( P(z=1,x)=1 \) for a subset of \( x \)'s and \( P(z=1,x)=0 \) for the complement.

Alternatively, but analytically similar, is to think of a program evaluator profiling people into a program whose "success" probability for an \( x \) is \( P(1|x) \) but whose "cost" (which includes welfare from the alternative "treatment zero" program) per profilee if admitted into the program is \( A \) so that expected net welfare per person of type \( x \) from the program is proportional to \( P(1|x)-A \) as in the lender's problem.

This paper uses the lender's problem as an expository vehicle to attempt to formalize "good" approaches to profiling on \( x \) in \( X \) under various types of partial identification by building on Manski (2003) for both Case I and Case II profiling situations. We will attempt to develop a "bottoms up" approach (where one increases assumptions about unknown objects by paying the price of loss of credibility) like that of Manski (2003) where we assume nothing about \( P(1|x,z=0) \) and where we add the same kind of considerations to other missing objects such as \( P(x|z=0) \) when the structure of payoffs is such that \( P(x) \) itself effects the set of "good" strategies.

At the same time we will investigate a "top down" approach where we start out with identified objects and relax such exact knowledge by increasing a cloud of ignorance around one or more "baseline" cases (Brock, Durlauf, and West (2003)). This latter exercise is in the spirit of recent approaches to control theory that go beyond "robust" control by concentrating on "good" control designs.

At the risk of repeating what was said above, we study two main classes of profiling problems where the relevant structural inputs to the problems may be only partially identified:
(i) Class I will refer to problems where profilees are not "strategic" in altering their characteristics. These problems will have an analytical structure quite close to that of Manski (2004a,b,c,2003).

(ii) Class II will refer to problems where profilees strategically alter their characteristics at a cost.

We tackle a class I problem first and use it as an expository example. First, we shall do a "bottoms up" approach first that adapts Manski (2004a,b,c,2003). Second, we shall do a "top down" approach that adapts Brock, Durlauf, and West (2003), and Brock and Durlauf (2004). We will separate issues that can be treated assuming infinitely sized samples from those issues that arise because of finite samples. For the most part we will be assuming that samples are infinitely sized here.

The expository style of this article will be rather informal in an attempt to reach a wide audience. Yet we will try to esposit ideas with enough precision that formalists can see how matters can be made precise.

A "BOTTOMS UP" APPROACH

We adapt the spirit of Manski (2004a,b,c,2003) where we do an analysis that we shall call a "bottoms up" approach. The idea here is to impose minimal assumptions on the structure of the problem in order to purchase maximal scientific credibility, i.e. to purchase as wide an agreement about assumptions amongst fellow scientists as possible. After this minimal assumptions analysis is done, then add the most credible assumptions and gradually tighten the conclusions obtained. The "top down" approach is to use economic theory or some such device to make assumptions that give precise conclusions at the price of credibility loss. Then relax these incredible assumptions and study what conclusions remain "robust" to this relaxation. The "top down" approach studies various ways of "robustifying" conclusions.

Consider the lender who is screening loan applications from members of a population \((J,D)\) where the notation focuses attention not only on the set of people, \(J\), but also the date, \(D\). The lender wants to know \(P(1|x)\), the probability of paying off the loan by a member of \(J\) with characteristics \(x\) for population \((J,D)\). It also may want to extrapolate what it learns to some other population \((J',D')\), for example, the population under scrutiny, but at a later date \(D'\), where the population may have changed. For expositional simplicity, we assume the support of \(x\) is bounded (and finite for the most part) and normalize it to lie in \([0,1]\) when convenient. As in Manski (2004a,b,c) we assume infinite sized samples in order to abstract from small sample issues.

Assume loan size is fixed at \(L\), the borrowing rate for the lender is \(r\), the lending rate is \(R>r\), then the expected profit, \(R(x)\), per loan to a member of "stratum" \(x\), is given by (assuming all of the lender's loan, \(L\), is lost when borrower fails to repay),

\[
R(x) = L[(1+R)P(1|x)-(1+r)] = L(1+R)[P(1|x)-P^*], \quad \text{where} \quad P^* = (1+r)/(1+R).
\]

Recall (2.1),

\[
P(1|x) = P(1|x,z=1)P(z=1) + P(1|x,z=0)P(z=0),
\]

and recall that the lender observes \(P(1|x,z=1)\), \(P(z=1)\), \(P(z=0)\), but knows nothing about \(P(1|x,z=0)\) except that it lies in \([0,1]\).

The goal function of the lender lies in what I'll call a goal
Identification region, since $P(1|x)$ lies in an identification region (Manski (2003, Equation (2.2)). Given the identification region for $P(1|x)$, we can compute the identification region for $R(x)$ using (2.1) and (2.2). Provided that the results of this analysis are not extrapolated to another population ($J', D'$), this computation should achieve consensus on upper and lower bounds to returns, $R(x)$.

Suppose the lender is deciding whether to lend to members of some other population ($J', D'$) that it thinks is the same as ($J, D$). I.e. it is willing to make an "invariance assumption" and extrapolate its findings for ($J, D$) to ($J', D'$) by assuming the observed objects it estimated are the same. What should the lender do? The best case is when it knows $P(1|x)$. In this case the optimal action is to lend to $x=x^*$ where $x^*$ satisfies,

$$r(x)=\{P(1|x)-P^*\}=0.$$\

Assume

**Assumption 2.1:** $P(1|x, z=1)$ is continuous in $x$ and increases in $x$.

In order to compute the worst case scenario, assume that $P(1|x, z=0)=0$ and lend accordingly. We compute the worst case cutoff, $x=c_1$, by computing the worst case bound for $P(1|x)$.

This computation shows that $c_1$ satisfies the equation,

$$P(1|x, z=1)P(x, z=1)+0.P(x, z=0)=P(1|x, z=1)P(x, z=1)=P^*.$$\

Call a lender who behaves this way a pessimistic lender.

We call a lender who profiles by assuming the best case, $P(1|x, z=0)=1$, instead of zero to compute a cutoff $c_2$, an optimistic lender. Note that

$$c_2<x^*<c_1.$$\

It would seem that one would not want either type as manager because the pessimist would lose a lot of potentially good loans to the competition and the optimist would draw in too many nonperforming loans. Hence, we investigate another behavioral option, e.g. minimax regret.

Let us follow Manski (2004b) and compute minimax regret for each $x$. We start by defining the following objects,

$$H(x):=P(1|x, z=1)P(x, z=1)+gP(x, z=0), \text{ g in } [0, 1],$$

$$H_1(x):=P(1|x, z=1)P(x, z=1),$$

$$H_2(x):=P(1|x, z=1)P(x, z=1)+P(x, z=0),$$

$$U(t, g, x):=tAL[P(1|x, z=1)P(x, z=1)+gP(x, z=0)-P^*]:=tAL[H(x)-P^*],$$

$$U^*(g, x):=\max\{U(t, g, x)\}=AL\{1[H(x)-P^*\geq 0]\}\{H(x)-P^*\}\}$$

$$R(d, g, x):=U^*(g, x)-W(d, g, x)=AL\{t^*(H(x)-P^*)-d(H(x)-P^*)\},$$

$$t^*=1[H(x)-P^*\geq 0],$$

$$W(d, g, x)=dAL[P(1|x, z=1)P(x, z=1)+gP(x, z=0)-P^*]=dAL[H(x)-P^*],$$

where the unknown $g:=P(1|x, z=0)$ lies in $[0, 1]$, and $t$ is the "treatment choice"
(t=1 means "lend", t=0 means "do not lend").

Since \( g \) is in \([0,1]\) it is trivial to find the minimal and maximal values of \( H(x) \) that we called \( H_1(x) \) and \( H_2(x) \) in (2.6a). Obviously, \( H_1(x) < H_2(x) \). In the case \( H_2(x) - P^* < 0 \), we have \( t^* = 1 \) if \( H_2(x) - P^* > 0 \), i.e. the lender never lends in this case because there is no \( H(x) \) large enough to support lending. Hence, the set of \( x \)'s where \( H_2(x) - P^* < 0 \) is "profiles out" of the set of potential ledees. For \( x \)'s, such that \( H_1(x) - P^* = 0 \), there is an \( H(x) \) such that \( H - P^* = 0 \), for "state of the world" \( H(x) \), it is optimal to lend, i.e. set \( t^* = 1 \) if \( H - P^* > 0 \). For \( d > 0 \), regret is largest when \( H(x) = H_2(x) \) in this case provided there is a feasible \( H(x) > P^* \). If \( H_1(x) - P^* < 0 \), there is an interval of \( H(x) \) where it is optimal to set \( t^* = 0 \). In this case regret is largest when \( H(x) \) is smallest. These computations lead to an expression for maximum regret for the case, \( H_1(x) - P^* < 0 \), \( H_2(x) - P^* > 0 \). We have,

\[
(2.7a) \quad R^*(d, x) = AL\{\min((H_2(x) - P^*)(1-d), (H_1(x) - P^*)(-d))\}.
\]

Maximum regret is minimized at \( d^*(x) \) which solves the equation,

\[
(2.7b) \quad (H_2(x) - P^*)(1-d) = (H_1(x) - P^*)(-d), \text{ i.e.}
\]

\[
(2.7c) \quad 0 \leq d(x) = (H_2(x) - P^*)/[H_2(x) - H_1(x)] \leq 1,
\]

for \( x \) in \([x_*, x^*] \) where \( H_2(x_*) - P^* = 0 \), \( H_1(x^*) - P^* = 0 \). Note that \( x_* < x^* \), so the minimax regret lender lends to a positive fraction, \( d(x) \), of \( x \)'s in the set \([x_*, x^*] \) that the maximin lender does not lend to, and the positive fraction \( d(x) \) falls from one to zero as \( x \) falls from \( x^* \) to \( x_* \). To put it another way, the minimax regret lender conducts "experiments" where it draws at random a positive fraction \( d(x) \) for \( x \) in \([x_*, x^*] \).

To put it yet another way, in plain English, a maximin lender will not lend to a stratum, \( x \), if the minimum possible net return on lending to that stratum is negative while the minimax regret lender will lend when the minimum net return is negative provided that the ratio of the maximum net return to the sum of the absolute values of the minimum and maximum net returns is positive. Both measures can be criticized for being too sensitive to "models" e.g. values in the set of possible net returns, that have extremely small probabilities under a "reasonable" or "data determined" set of priors, e.g. in the context of data informed exercises like Bayesian Model Averaging, or approaches to dealing with ambiguity by using sets of priors (Brock, Durlauf, and West (2003) and references). This suggests that future research may want to concentrate on "trimmed" versions of maximin or minimax regret as in Brock, Durlauf, and West (2004). More will be said about this below. Here we wish to concentrate on the interaction of choice of criterion with "adaptive" learning.

Hence, at the population level, the minimax regret lender has learned \( P(1|x) \) in one period for all \( x \) in \([x_*, x^*] \)! Since, \( P(1|x) \) is assumed to be continuous, the minimax regret lender has also learned \( P(1|x_*) \). Thus if \( N(x) \), the number of \( x \)'s in the population is large, \( d(x)N(x) \) is an approximation to the size of sample of \( x \)'s in \([x_*, x^*] \) that the minimax regret lender takes.

While the minimax regret lender takes a smaller fraction of \( x \)'s as \( x \) falls, the maximin lender takes no samples at all. Hence, it would appear that the minimax regret objective could end up playing an important role in designing workable learning schemes that do a good job of protecting the decision maker against misspecification but not be overly conservative as is maximin.
LEARNING: BOTTOMS UP

Assume the density $P(x)$ is known and assume $P(1|x)$ is increasing and continuous in $x$. Let us write down a simple learning scheme. Let the lender start at the beginning of period $s$ with a belief set

(2.8) $Q(1|x,s) = [L(x,s), U(x,s)]$, $0 \leq L(x,s) \leq U(x,s) \leq 1$, on $P(1|x)$,

which is a closed interval of possible values for the function $P(1|x)$. Assume that the lower bound, $L(x,s)$, and the upper bound $U(x,s)$ are continuous and increasing functions of $x$. Suppose the lender maximizes the same expectation of the profit expression as above and that the lender is pessimistic, i.e. it is a maximin lender. Hence, it chooses $c(s)$ such that $L(x,s) - P^* = 0$ at $x = c(s)$ and lends for sure to all $x'$'s with $x > c(s)$. At the end of period $s$, it observes

(2.9) $P(1|x)$ for all $x > c(s)$.

Thus, it now knows $P(1|x)$ for all $x > c(s)$ and replaces the set $Q(1|x,s)$ with the unique function $P(1|x)$ for $x > c(s)$. Since $P(1|x)$ is continuous, the lender also knows $P(1|c(s))$. In the discussion below, we assume $P(1|c(s)) > P^*$.

What is a sensible strategy for the lender for updating its belief set $Q(1|x,s)$ for $x < c(s)$ into a new belief set $Q(1|x,s+1)$? Presumably it should be some kind of fan shaped set where the point of the fan starts at the point $(c(s), P(1|c(s)))$. But there is no justification for this type of assumption based upon the data alone. If the lender is pessimistic, i.e., a maximin lender, it continues to assume that $P(1|x) = L(x,s)$, for $x < c(s)$ because it has not seen performance for any of these $x$'s,

(2.10) $L(x,s+1) = L(x,s)$,

and chooses a cutoff, $c(s+1)$ that satisfies,

(2.11) $L(x,s+1) - P^* = 0$,

which is the same as $c(s)$ because lender still continues to believe that the worst case value of $P(1|x)$ for $x < c(s)$ is $L(x,s)$ because it has no information to prove otherwise.

LIPCHITZ LEARNING

But a "reasonable" person might argue that $P(1|x)$ has enough "smoothness" that one should assume that it is, for example Lipchitz with constant $K$, which is bounded above by a known bound $B$, e.g.

(2.12) $|P(x|1) - P(x'|1)| \leq K|x - x'| \leq B|x - x'|$.

Under this assumption one may compute $L(x,s+1)$ using

(2.13) $|P(x|1) - P(c(s)|1)| \leq K|x - c(s)| \leq B|x - c(s)|$,

giving the lower bound,

(2.14) $L(x,s+1) = \max \{B|x + P(1|c(s)) - Bc(s), 0\}$, $x < c(s)$,

and the cutoff $c(s+1)$ level given by,
(2.15) \( B(c(s+1)-c(s)) + P(1|c(s)) = \* \).

Equation (2.15) defines a simple one dimensional dynamical system,

\[(2.16) \quad c(t+1) = c(t) + [P(\* - P(1|c(t)))/B], \quad t = s, s+1, \ldots, \quad c(s) \text{ given.} \]

If we assume \( P(1|0) = 0 \), since \( P(1|x) \) is assumed Lipchitz with constant \( K < B \), since the fixed point \( x(\*|) \) which is given by \( \* - P(1|x(\*)) = 0 \) is unique (recall that \( P(1|x) \) is assumed to be increasing in \( x \) and continuous in \( x \), therefore the RHS of (2.16) is nondecreasing with "slope" bounded above by one, it must converge to \( x(\*) \). Thus the "Lipchitz" maximin learner converges to the true lending cutoff.

Similarly we may write down an analog of this process for the optimistic lender who cuts off using the upper bound of the belief set at each date \( s \) rather than the lower bound. In this case, if the upper bound is increasing in \( x \), the learner learns all it needs to know in one period. A "Hurwicz-\( \alpha \)" learner might set Hurwicz's \( \alpha \) index (which weights optimistic maximax and pessimistic minimin) by looking at the goal of improving the learning rate over time as well as a personal preference. Thinking about the learning objective gives new insight into the classic Luce and Raiffa (1957) review of various decision criteria including maximin, minimax regret, Hurwicz-\( \alpha \), etc.

For example while they agree that minimax regret is an improvement over the extreme pessimism of maximax, they still stress Chernoff's classic critique of minimax regret (e.g. it fails to satisfy a version of Independence of Irrelevant Alternatives). But the interaction of the learning goal with the goal of "reasonableness" of a decision criterion gives a different perspective on the evaluation of that decision criterion. For example, we shall see that minimax regret has nice learning properties. Turn now to a parallel discussion for the minimax regret lender.

MINIMAX REGRET LEARNING

As we saw above, at date \( s \), the minimax regret lender lends to a fraction \( d(x) \) for \( x \) in the set \([x_1, x_2] \) where,

\[(2.17) \quad U(x_1, s) = \*, \quad L(x_2, s) = \*, \quad \]

where the positive fraction \( d(x) \) decreases from unity at \( x = x_2 \) to zero at \( x = x_1 \). Thus the minimax regret learner observes \( P(1|x) \) on \([x_1, x_2] \) in just one period. Since \( x(\*) \) will typically be greater than \( x_1 \), all learning is completed in just one period by a minimax regret learner.

Hence, we see that in both cases, for one dimensional characteristic \( x \), for maximin and minimax regret, one can construct a class of "workable" "adaptive learning by lending" processes that would enable the lender to uncover the true \( P(1|x) \) eventually, and, in the case of minimax regret with one dimensional characteristic, all learning is completed in just one period. Turn now to a short sketch of how one could generalize these results to multidimensional \( x \).

Assume \( L(x), U(x), \) and \( P(1|x) \) are defined on the non-negative orthant of the \( n \)-dimensional reals and are continuous, increasing in \( x \) and are bounded between zero and one as above. Furthermore assume both \( VL(P) = \{x | L(x) \geq P \}, \quad VP(P) = \{x | P(1|x) \geq P \} \) are convex sets for each \( P \) in \([0,1] \). I.e. assume \( L(x) \) and \( P(1|x) \) are quasi-concave, continuous, increasing functions of \( x \). As above, the lender lends to all \( x \)'s in \( VL(P) \) and learns \( P(1|x) \) for \( x \)'s in \( VL(P) \) after one period. Consider \( x \) in the boundary of \( VL(P) \) and \( t \) in \([0,1] \). It is natural to apply the above scalar analysis to the functions \( L(tx') \),
P(1|tx'). In order to do this we need to assume the functions L(tx'), P(1|tx') are Lipchitz with constant K(t) bounded above by a known B(t). A sufficient condition for this property is the assumption that L(x) and P(1|x) are Lipchitz with constant K bounded above by known constant B. Hence, just as above, the Lipchitz learner learns P(1|tx') for t in [t(x'),1] where 
\[ (2.18) \quad P(1|tx'x') = P^*. \]

Since it can learn P(1|x) each ray <x'>:={tx'|t is in (t(x'),1)}, it can learn P(1|x) on VP(P*). Indeed, with more thought, one should be able to design a learning scheme that is more efficient at exploiting the convexity of the sets VL(.) and VP(.) than simply learning "ray by ray." We leave this interesting topic to future research.

Our discussion of minimax regret learning can be generalized to rays as above to show that the minimax regret learner can learn P(1|x) on the set VP(P*) in just one period.

It is not clear how to formulate a notion of "optimal" learning in the partially identifiable structural context here. Indeed it might be argued that the economics profession should back off from its fixation on optimal learning schemes (which may be lacking in "robustness" to specification of the supporting assumptions which few may agree to) and concentrate on finding "good" learning schemes which work under a wide set of assumptions (which many can agree to).

But it seems plausible that any notion of "workable" learning will involve some form of updating a set of beliefs. There is a rapidly developing literature on modeling "ambiguity" and learning under ambiguity that replaces the usual prior in Bayesian learning with a set of priors, each of which is updated that should be useful in this regard (e.g., Epstein and Schneider (2002), (2004)). Note that we maintained an assumption of stationarity here as learning progresses. If the population of profiles (i.e. "lendees") is changing as time progresses (which is likely) then learning must deal with this new "injection" of ambiguity. Epstein and Schneider (2004) model injection of "new" ambiguity with a family of likelihoods that is used to update each member of a set of priors.

Another possible approach to developing learning schemes is to adapt ideas reviewed in Marimon (1997) as well as ideas in Mannor and Shimkin (2001, 2002) and Schlag (2003). Adaptation of the Empirical Bayes Envelope used by Mannor and Shimkin (2002) for repeated problems under time average objectives and the methods of Schlag (2003) for repeated problems under discounted sum objectives appears especially promising. More will be said about this below.

GENERAL STRUCTURE OF MAXIMIN AND MINIMAX REGRET PROBLEMS

Let us formulate a general approach that not only encompasses all of the above for the binary case but also extends to multi-period problems. We write out an abstract general structure for one period problems first. This structure will include Manski's (2004b) minimax regret problem. Fix a particular covariate x in X and suppress it in the notation for now. We change Manski's (2004b) notation so that it is compatible with the general framework to be described below. Let T be the treatment set, T={1, 2, ..., n}, let π=(π(1), ..., π(n)) denote the vector of π(t):=E{u(y(t),t)}. Manski (2004b) treats a case where
\[ (2.19) \quad \pi(t) = \pi(t\{0\})P(t\{0\}) + \pi(t\{not 0\})P(t\{not 0\}) \]

where "0" ("not 0") stands for "observed" ("not observed") and the rest of the notation is self explanatory. The problem is that the scientist can not
observe \( g(t) = \pi(t) \) (not \( 0 \)) so we assume it takes values in the closed interval \([a(t), b(t)]\). Thus \( \pi(t) \) can be any value in the closed interval \([\pi_*(t), \pi^*(t)]\) where the lower and upper limits are defined by (2.19) and \([a(t), b(t)]\). With this notation the minimax regret problem for each covariate becomes,

\[
(2.20) \quad \min_d \max_u \{\max_{\pi} \Sigma u(t)\pi(t) - \Sigma d(t)\pi(t)\} \text{ where } u, d
\]

range over the \( n \)-dimensional simplex \( S_n \).

As in Manski (2004b) note that the \( \max_u \) takes place at extreme points of the simplex \( S_n \), thus, the simplex \( S_n \) can be written as a union over \( t \) of subsets \( S_{tn} \) where \( t \) is optimal. On these subsets, \( S_{tn} \) we observe that the largest value of

\[
(2.21) \quad (1-d(t)) \pi(t) - \Sigma'd(s)\pi(s) \text{ is } (1-d(t))\pi^*(t) - \Sigma'd(s)\pi_*(s),
\]

where \( \Sigma' \) denotes summation over all \( s \) not equal to \( t \). Thus minimization of maximum regret collapses to

\[
(2.22) \quad \min_d \max_t \{ (1-d(t))\pi^*(t) - \Sigma'd(s)\pi_*(s) \},
\]

which is the minimum of the upper envelope described by a family of \( n \) linear functions of the vector \( d \) in \( S_n \). The upper envelope is convex and this reduces the complexity of the problem. The same argument applies for each value of the covariate \( x \) in \( X \).

A special case is the case \( T=\{1,2\} \). Since \( d(1)+d(2)=1 \), (2.22) boils down to

\[
(2.23) \quad \min_d \max\{(1-d(1))[\pi^*(1)-\pi_*(2)], d(1)[\pi^*(2)-\pi_*(1)]\}.
\]

The four basic cases are

\[
(2.24) \quad [\pi^*(1)-\pi_*(2)]>0 \quad (\leq 0), \quad [\pi^*(2)-\pi_*(1)]>0 \quad (\leq 0).
\]

For each of these four cases we observe that the "max" operator (2.23) is over a pair of linear functions. The maximum over a set of linear functions is a convex function. Therefore (2.23) is a minimization over \( d \) of a convex function in \( d \). Hence the problem breaks up into two basic cases: (i) [(ii)] the minimization takes place on the boundary [interior] of the domain of \( d \).

It is easy to see that for

\[
(2.25a) \quad [\pi^*(1)-\pi_*(2)]>0 \quad \text{and} \quad [\pi^*(2)-\pi_*(1)]\leq 0; \quad d(1)=1,
\]

\[
(2.25b) \quad [\pi^*(1)-\pi_*(2)]>0 \quad \text{and} \quad [\pi^*(2)-\pi_*(1)]>0;
\]

\[
(2.25b.1) \quad 0<d(1)=[\pi^*(1)-\pi_*(2)]/[\pi^*(1)-\pi_*(2)]+[\pi^*(2)-\pi_*(1)]<1
\]

\[
(2.25c) \quad [\pi^*(1)-\pi_*(2)]\leq 0 \quad \text{and} \quad [\pi^*(2)-\pi_*(1)]>0; \quad d(1)=0,
\]

\[
(2.25d) \quad [\pi^*(1)-\pi_*(2)]\leq 0 \quad \text{and} \quad [\pi^*(2)-\pi_*(1)]\leq 0;
\]

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These results extend to covariates $x$ in $X$ by inserting "x" into the $\pi$'s in (2.25).

Notice that the results follow from the listing of the possibilities for the minimum of two lines in (2.23), i.e. the listing of the possibilities for the upper envelope generated by the two lines each of which can have positive or non-positive slope.

If we have three treatments, the two one dimensional lines in (2.23) get replaced by three two dimensional planes and the slopes of the lines get replaced by the direction numbers of the planes. While it is possible to list the possibilities for these direction numbers and, hence, list the possibilities for the upper envelope, we content ourselves with pointing out that this geometry shows why one should expect "mixed strategies" to be quite common. Karlin (1959) provides techniques for listing out the possibilities and explicitly constructing mixed and pure strategies in closely related structures.

At the risk of repeating, if we now insert $x$ in $X$ into the above notation and results we have Manski’s (2004b) result. Notice Manski’s emphasis on the possibility of a mixed treatment strategy. Mixing will play an important role in adaptive learning and in the other formulations of dealing with different forms of the planner’s ignorance below.

Let us apply this work to the lender’s problem. Let the "treatments" be denoted 1=“do not lend”, 2=“lend”. Notice that net return expressions of the form $\pi:=P(1|\pi,g)-P^*$ play the key role in the above analysis where $g$ is unknown and lies in some set. Let $\pi:=\pi(2)$ denote unknown net return which is assumed to lie in the set $\Pi$, let $\pi(1)=0$ because return is zero if “do not lend”, and let $t$, $d$ denote "treatments" in $T={1,2}$. Then the maximin problem is given by

\[
\max_d \min_{\pi} (dm) = \max_d \min_{\pi} \{ (d(1)\pi(1)+d(2)\pi(2)) \} = \max_d \{d(2)\pi_*(2)\}.
\]

Hence the RHS is zero with $d(2)=0$, if $\pi_*(2)=0$ and the RHS is $\pi_*(2)$ with $d(2)=1$, if $\pi_*(2)>0$. Turn now to minimax regret.

The minimax regret problem is given by

\[
\min_d \{ \max_{\pi} [\max_t (\pi t)-dm] \} = \min_d \{ \max_{\pi} [\max_t (\pi t)-dm] \}.
\]

But by (2.23) above, we have,

\[
\max_{\pi} [\max_t (\pi t)-dm] = \min_d \{ (d(2)(-\pi_*(2))+(1-d(2))\pi^*(2)) \},
\]

since $\pi^*(1)=\pi_*(1)=0$.

It is easy to see that $\pi^*(2)=0$ implies $d(2)=0$ (we take the $d(2)=0$ in the boundary case $\pi^*(2)=0$), $\pi_*(2)>0$ implies $d(2)=1$, and $\pi_*(2)<0$, $\pi^*(2)>0$ implies

\[
0<d(2) = \pi^*(2)/[(\pi^*(2)+(-\pi_*(2))] < 1.
\]

All of the above generalizes to lending problems with covariates $x$ in $X$ by writing $\pi_*(2,x)$ and $\pi^*(2,x)$ and repeating the above analysis "x by x".
3. PLANNER’S PROBLEM WITH UNKNOWN TYPE DISTRIBUTIONS

Fix covariate x in X and suppress it in the notation for the moment. But let there be unobserved types k=1,2,...,K and consider the general treatment effect problem, (Manski (2004a, p. 1225)),

(3.1) \( \max_t \{ \Sigma_k E(u(y(t),t,k)\mid k)p(k) \} \), \( \Sigma \) is over k=1,2,...,K types,

where the types probabilities are not known but the payoffs

(3.2) \( a(t,k) := E(u(y(t),t\mid k) \)

are known.

If a decision maker faces a problem of this form, how should it play? If it plays maximin, this is a zero sum matrix game problem of the form analyzed by Karlin (1959), i.e. let the primary player, the maximizing player, be denoted Player I and the minimizing player be denoted Player II. Then Karlin shows that the \(|T| \times K\) finite matrix game

(3.3) \( \max_u \min_p \{ u'p \}, \) where,

u is in the compact convex \(|T|\) dimensional simplex, \( p \) is in the K dimensional compact convex simplex, has a value \( v \), i.e. a saddle point where the maximin=minmax theorem of zero sum game theory holds, the presence of equilibrium mixed treatment strategies \( u^* \) is general, and explicit solution techniques are given for \( 2 \times K \) and \(|T| \times 2\) games which are still somewhat tractible for \( 3 \times K \) and \(|T| \times 3\) games. He also gives solution techniques for \(|T| \times K\) games where \( n=|T|=K\). We develop this below as well as some theory for minimax regret solutions. But first we wish to present a possibly more realistic problem.

Consider the problem,

(3.4) \( \max_t \{ \Sigma_k E(u(y(t),t,k)\mid k)p(k) \} \), \( \Sigma \) is over k=1,2,...,K types, where,

(3.5) \( p(k\mid t) = p(t,k) / q(t) \),

(3.6) \( \Sigma_k E(u(y(t),t,k)\mid k)p(k\mid t) = \Sigma_k E(u(y(t),t,k)\mid k)p(t,k) / q(t) = v(t) \),

(3.7) \( \Sigma_k p(t,k) = q(t) \), t=1,2,...,T; \( \Sigma_t p(t,k) = p(k) \), k=1,2,...,K.

This problem represents Player I’s attempt to optimize the same goal as in (3.1), but selection of unknown form of types, which is induced by the presence of the treatment \( t \) presents an obstacle. It is assumed that Player I knows \( a(t,k) := E(u(y(t),t,k)\mid k) \) and observes \( p(t), v(t) \), t in \( T \) from observational data. Given this knowledge and knowledge of the constraints on the unknown joint distribution \( p(\ldots) \), what should Player I do?

This problem is an attempt to stylize the real problem in settings where one is trying to use information from observational data in order to select an "optimal" treatment, but where selection on unobservable "type" characteristics takes place and this selection is induced by the presence of the observed treatment in the data. To put the question another way, if Player I can not observe \( p(t,k) \), but it knows \( a(t,k) \) and it can observe \( v(t) \)
and \( q(t) \), how should it choose \( t \) in \( T \) to do "well" by exploiting the constraint information contained in (3.4)-(3.7) as best as it can?

One "conventional" approach is to use economic theory to formulate a "model" for the selection probabilities

\[
(3.8) \quad p(t|k, \theta),
\]

or the joint probabilities

\[
(3.9) \quad p(t, k, \theta),
\]

formulate a model for the outcome equation, \( Y(t, \theta) \), e.g. (3.6), and use the data to estimate \( \theta \), and equipped with estimates of (3.8) or (3.9) construct an optimal solution to the original problem. For example one could insert (3.9) into

\[
(3.10) \quad \sum_k a(t, k, \theta) p(t, k, \theta) = Y(t, \theta), \quad \sum_k p(t, k, \theta) = q(t), \quad t = 1, 2, \ldots, T
\]

and provided that (3.10) identifies a unique value of \( \theta \) (after putting the covariates \( x \) in \( X \) back into the notation and writing out the counterparts of the outcome equation and the treatment selection equation for this setup), these equations identify \( \theta \) from population data. But this approach assumes a lot of structure on \( Y(t, \theta) \), and \( p(t, k, \theta) \) which may be controversial.

However, see Newey, Powell, and Walker (1990) for an example of a comparison of the standard parametric approach to estimation of a selection model to a semi-nonparametric estimation where the results of variation of assumptions on the joint distribution of the errors in the outcome and selection equations made little difference in the estimates. But changes in the specification of regressor sets made a bigger difference. Their example suggests robustness with respect to specification of regressor sets may be more important than robustness with respect to joint distributions of the errors.

In any event, here, in this part of the paper we pursue a nonparametric approach (in this very stylized setting) in an attempt to purchase more agreement at the cost of less precise conclusions. Later on in this paper we will pursue a "local" approach which allows variations constrained to a neighborhood of a "baseline." Our main goal is to attempt to unify different approaches to "robustification" against uncertainties including sensitivity analysis, robust control, and bounding.

Let us put this problem into the notation and formal setup of zero sum matrix game theory. For the Maximin problem, we have

\[
(3.11) \quad \max_{u} \min_{p} \{u'Ap\} \text{ s.t.}
\]

\[
(3.12) \quad \text{diag}(AP') = b, \quad Pe = q, \quad P'f = p,
\]

where \( b(t) = v(t)q(t) \), \( t = 1, 2, \ldots, T \), \( P \) is \( |T| \times K \) matrix with \( t, k \) element, \( p(t, k) \), \( e \) is a \( K \) dimensional vector of ones, \( f \) is a \( |T| \) dimensional vector of ones, and \( \text{diag}(AP') \) selects the \( |T| \) diagonal elements of the \( |T| \times |T| \) matrix \( AP' \), i.e. \( [AP'](t) = \sum_k a(t, k)p(k, t) \). Since \( u \) is in the compact convex \( |T| \) dimensional simplex, and \( p \) is in the compact convex set described by (3.12), by standard game theory, the game (3.11) has a value and has a saddle point. We also must be sure that the constraint set (3.12) is non empty. For example, if there are too few types relative to treatments the first two equations of (3.12), \( \text{diag}(AP') = b, \quad Pe = q \), may impose such strong restrictions on
that \( P'f = p \) can not hold. The most plausible case would seem to be the case of more types than treatments. Hence the constraint set (3.12) is likely to be nonempty, but this must be checked for each application. In any event it is compact and convex so we may adapt received zero sum matrix game theory to this problem.

**PLANNER FACES AN UNKNOWN, BUT TREATMENT INDEPENDENT, TYPE DISTRIBUTION**

Recall the problem (3.3) for the special case \(|T|=2\),

\[
(3.13) \quad \max_u \min_p \{u'Ap\} = \max_u \min_j \{L_j(u)\} \quad \text{where,}
\]

following Karlin (1959, pp. 41-42), the line \( L_j(u) \) is given by

\[
(3.14) \quad L_j(u) = u(1)A(1,j) + (1-u(1))A(2,j),
\]

the lower envelope of the set of \( K \) lines \( \{L_j(u)\} \) is concave (cf. Karlin (1959, Figure 2.1)), and the maximizer \( u \) maximizes the concave function, \( \min_j \{L_j(u)\} \).

Much as in the above analysis of minimax regret, we can divide the analysis into Case (i): The maximum is at the boundary of \([0,1]\), Case (ii) The maximum is in the interior of \([0,1]\). The interior maximum is a mixed strategy. The reader may now follow Karlin (1959) to complete the analysis and extend it to cases of \( 2 \times K \) and \( 3 \times K \) games and \(|T| \times 2 \) and \(|T| \times 3 \) games. We wish to concentrate on minimax regret solutions.

The minimax regret problem is given by

\[
(3.15) \quad \min_d \max_u \{\max_p (u'Ap - d'Ap) \}
\]

\[
= \min_d \max_u \max_p \{\sum_j A(k,j)p(j) - \sum_k \delta(i)(\sum_l A(i,l)p(l))\} := \text{RHS},
\]

where \( \sum'_l \) is the sum over all \( l \) not equal to \( k \). The last equality follows from recognizing that maximization over \( u \) in the \(|T| \) dimensional simplex takes place at the extreme points of that simplex. We gain insight into the structure of the problem by considering the special case \(|T| = 2, K = 2\). In this case, the expression in \( \{.\} \) becomes, for \( k = 1 \),

\[
(3.16) \quad (1-d(1))[A(1,1)p(1)+A(1,2)(1-p(1))]-d(2)[A(2,1)p(1)+A(2,2)(1-p(1))]
\]

\[
= (1-d(1))[Dp(1)+b], \quad \text{where}
\]

\[
(3.17) \quad D := A(1,1) - A(1,2) - (A(2,1) - A(2,2)), \quad b := A(1,2) - A(2,2).
\]

Similarly, using again that \( 1 = d(1)+d(2) \) and \( 1 = p(1)+p(2) \), we may write the expression in \( \{.\} \) in (3.15) for \( k = 2 \) as,

\[
(3.18) \quad (1-d(2))[A(2,1)p(1)+A(2,2)(1-p(1))]-d(1)[A(1,1)p(1)+A(1,2)(1-p(1))]
\]

\[
= d(1)[-Dp(1) - b].
\]

Hence, \( \text{RHS}(3.15) \) becomes,

\[
(3.19) \quad \text{RHS} = \min_d \max_p \max_u \{\sum_k (1-d(1))[Dp(1)+b], d(1)[-Dp(1) - b]\}
\]

Put \( \pi := Dp(1)+b \), and rewrite (3.19) as

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(3.20) \( \min_d \max_{\pi} \max_{\pi} \{(1-d(1))\pi, -d(1)\pi\} \).

The inner maximization over \( \pi \) takes place at the extreme points of the set \( \Pi=\{\pi_*, \pi^*\} \), so we now choose \( d(1) \) to solve the problem,

(3.21) \( \min_d \{(1-d(1))\pi^*, -d(1)\pi_*\} \).

For the "main" case, \( \pi_* < 0, \pi^* > 0 \), this gives the solution,

(3.22) \( d(1) = \pi^*/(\pi^* + (-\pi_*)) \).

We apply this solution below.

Consider a stylized innovative government program \( t=2 \), to give training to disadvantaged people, at a cost of \( c(2) \) each. Let \( t=1 \) represent doing nothing, e.g. "placebo" or using a received status quo program. Put \( A(1,1)=A(1,2)=0, c(1)=0 \). The numbers here could represent values for the two types under "doing nothing" or they could represent differences between the more expensive innovative (but experimental) program and the existing status quo program.

Suppose there are two types, type 1 has low treatment gain potential, i.e. \( A(2,1)=0 \), and type 2 has high treatment gain potential, i.e. \( A(2,2)-c(2)>0 \). The planner does not know \( p(1) \) and \( p(2) \), but it does know \( \{A(t,j), c(t)\} \). What should it do? If it does maximin, it solves

(3.23) \( \max_{u} \min_{p} \{u'A'p-u'c\} = \max_{u} \min_{p} \{u(2)[A(2,2)p(2)-c(2)]\} \)

which gives the solution \( u(2)=0 \) for every \( c(2)>0 \), no matter how high \( A(2,2) \) is.

This is a version of the recurring theme of extreme pessimism of maximin. Of course the extreme pessimism of maximin is "toned down" in the presence of prior information such as knowledge of a lower bound on \( p(2) \) that is positive. Turn now to minimax regret.

The minimax regret problem is given by (omitting familiar steps),

(3.24) \( \min_d \max_{\pi} \max_{\pi} \{(1-d(2))[A(2,2)p(2)-c(2)], -d(2)[A(2,2)p(2)-c(2)]\} \).

Application of the above solution technique gives

(3.25) \( d(2) = [A(2,2)-c(2)]/A(2,2) \),

which is a "net return" measure on treatment 2 measured relative to "gross return" \( A(2,2) \). This seems more sensible than the maximin solution of zero for any \( c(2)>0 \), no matter how small, and any \( A(2,2)>0 \), no matter how large. This point is related to Manski's (2004a) point that maximin "ignores" the data whereas here the point is that maximin "ignores the return".

BOUNDING "SELECTION BIAS" UNDER INCREASING DIFFERENCES

There is another important issue that is raised by the unobservable types that are present in this problem. In a stylized version of reality, using observational data, one observes \( \Sigma A(t,k)p(k|t) = v_0(t) \), chooses
\( t^* = \arg\max \{ v_o(t) - c(t) \} \), where \( c(t) \) is the cost of treatment \( t \), but actually obtains \( \Sigma A(t^*, k)p(k) - c(t^*) = v(t^*) - c(t^*) \), if \( t^* \) is assigned to the whole population. It is of interest to uncover sufficient conditions for "overoptimism" due to \( p(.|t) \) instead of \( p(.) \), i.e. \( v_o(t) \geq v(t) \), all \( t \). Label the types with increasing type indices corresponding to "better" types and write \( \Sigma n(t) = \pi(1) + \Sigma u(t) = \pi(t) - \pi(1) \), using \( \Sigma u(t) = 1 \), so we can interpret \( A(2,k) \) below as the difference between treatment 2 and treatment 1 for type \( k \). Thus, W.L.O.G., we consider a, not so special case, \( A(1,k) = 0 \), all \( k \), but we assume that \( A(2,.k) \) monotone increasing in \( k \) and nonnegative. This assumption is a version of the increasing differences assumption that is common in monotone comparative statics exercises.

It is easy to see that \( v_o(2) \geq v(2) \) for all monotone increasing functions \( A(2,.k) \) (i.e. first order stochastic dominance) implies

\[
(3.26) \quad p(K|t=2) \geq p(K), \quad p(K-1|t=2) + p(K|t=2) \geq p(K-1) + p(K), ...
\]

Thus, it is reasonable to conjecture that there is a quite tight relationship between notions of stochastic dominance and measures of "over optimism" due to the presence of \( p(.|t) \) in observational data rather than the relevant \( p(.) \) when treatments (e.g. policies) are chosen based on estimates from historical data when there are unobserved covariates present. Of course there is a huge literature on methods for correction of such bias. We simply point out the potential connection of this literature with notions of stochastic dominance here and refer study of this potentially very interesting connection in future research.

Before turning abstracting out the general structure of these problems and attempting to fit them into received analytical approaches available in the general literature let us discuss one more example.

LENDING UNDER RISK AVERSION: BOTTOMS UP

Return to the lender's problem and suppose the lender is risk averse. Let the support of \( x \) have two points \( x_1 < x_2 \) with probabilities \( P(x_1), P(x_2) \). Let \( l_1 := 1[+1, x_1] \) \( 1[0, x_1] \) denote the event: An individual with characteristic \( x \) pays off (repays zero of the loan) the loan. For simplicity we assume the individual pays off all of the loan or none of it. Hence, \( 1[+1, x_1] + 1[0, x_1] = 1 \) for all types \( x \). Of course

\[
(3.27) \quad P(1|x_1) = E(1[+1, x_1]), \quad i=1,2.
\]

Profit from lending to type \( x \) is given by the random variable,

\[
(3.28) \quad R(x) = L((R-r)(1[+1, x] - 1[0, x])) = LA(1[+1, x]^* - P^*).
\]

If \( P(1|x) \) is known the mean and variance of profit is given by expected profit

\[
(3.29) \quad E(R(x)) = L((R-r)P(1|x)) - LP(0|x) = L((R-r+1)P(1|x)) - 1) = LA[P(1|x) - P^*].
\]

If there is only one type, \( x \), the variance of profit is given by

\[
(3.30) \quad V(R(x)) = LA^2 P(1|x)(1-P(1|x)).
\]

At the risk of repeating, we can expose several key issues when there is
only one type, \( x \). First, the equation

\[(3.31) \quad P(1|x) = P(1|x, z=1)P(z=1) + gP(z=0),\]

where the lower case quantity \( g = P(1|x, z=0) \) in \([0,1]\) is not revealed by the sampling process, creates ambiguity in the variance (3.31) as well as in the mean, (3.29).

For example if profits are "risk adjusted", as for example in mean variance objectives, one can do both maximin analysis and minimax regret analysis w.r.t. \( g = P(1|x, z=0) \) in this setting and compare the outcomes.

Second, one can consider a finite number of types and extend the analysis when covariance terms are present. The presence of covariance terms raises issues of how to model dependence of quantities such as \( P(1|x) \) as a function of the "distance between types," \(|x_i - x_j|\) as well as the dependence of the sampling process \( z \) on the distance between types. It might seem at first blush that since one only observes \( P(1|x_1, x_2, \ldots, x_n, z_1 = 1, \ldots, z_n = 1) \) that "bounds" would widen drastically with \( n \). But closer types should be more likely to be observed as a pair because of increasing dependence as types get closer. We formulate the problem at an abstract level here.

Let \( d_i \) in \([0,1]\) be the probability that the lender lends to type \( i \), define the following objects,

\[(3.32) \quad J = E(R) - (a/2) \text{Var}(R),\]

\[(3.33) \quad M_i = L\{+1, x_i\} - P\}, \quad m_i := EM_1, \quad b_{ij} : = \text{Covariance}(M_i, M_j),\]

and obtain, using the formulas above,

\[(3.34) \quad J = E(R) - (a/2) \text{Var}(R) = d^\prime m - (a/2)d^\prime B d : = J(d, \theta), \quad \theta : = (m, B),\]

in obvious notation. Unlike the usual mean variance portfolio problem, the lender operates here under ambiguity about \( \theta \). The ultimate source of ambiguity is the joint distribution of \( \{1, 1_2, \ldots, 1_n\} \). If we let \( F \) denote the convex compact set of joint probability distributions of \( \{1, 1_2, \ldots, 1_n\} \), the construction of \( \theta = (m, B) \) in (3.32) induces a mapping from \( F \) to \( \Theta \) whose image can be "convexified" into a convex and compact subset of \( \Theta \), by taking the closed convex hull of the image, call it \( K \). Hence we can construct an "approximate" zero sum game to represent an "approximate" maximin problem for the lender,

\[(3.35) \quad \max_d \min_{\theta}(J(d, \theta)),\]

where the \( \max \) is taken over the convex compact \( n \)-simplex and the \( \min \) is taken over the convex compact set \( K \). Notice also that \( J \) is concave in the maximizing player's choice, \( d \), and \( J \) is convex (e.g. linear) in the minimizing player's choice, \( \theta \). The linearity (e.g. convexity) of \( J \) will lead to the minimum being taken on the extreme points of \( K \), which will be, in some cases, the extreme points of the original image before the closed convex hull is taken.

It is well known in general zero sum game theory that if the objective \( J \) is quasi-concave in \( d \) and quasi-convex in \( \theta \) and the strategy sets are convex compact, then a value \( v \), and a saddlepoint, \((d^*, \theta^*)\) exists for this game in general locally convex spaces (Hofbauer and Sorin (2002) with references to Ky
Fan, M. Sion, et al.). Furthermore Hofbauer and Sorin (2002) show that best response dynamics converge to a saddle point under modest additional regularity conditions. The minimax regret problem becomes,

\[(3.36) \quad \min_{d} \max_{\theta} \{ \max_{u} (J(u, \theta)) - J(d, \theta) \} = \min_{d} \max_{\theta} \{ (J(u^*(\theta), \theta)) - J(d, \theta) \} = \min_{d} \max_{\theta} \{ R(d, \theta) \}, \]

which is another zero sum game problem where the d-player is now the minimizing player and the \(\theta\)-player is now the maximizing player. Both are still operating over convex compact strategy sets, but we must deal with possible non quasi-concavity for the maximizing player and non quasi-convexity for the minimizing player. Most of the analytical difficulty is caused by the "theta" player because of the presence of \(u^*(\theta)\) in \(J\).

It is important to recognize that in contrast to all of the above problems which have planner's objective functions additively separable in covariates, as in Manski (2004a,b,c), we now have a planner's objective that is not additively separable in covariates. We now have ensemble dependence through the covariance matrix, \(B\). Of course if we considered problems across communities, so long as the communities were independent, we would restore a form of additive separability. Game theory, however, points the way to potential tractability of a class of problems that are not additively separable across covariates. We will say more about this later, especially when we turn to analysis of problems where the lendses can modify their covariates in an attempt to entice loans out of the lender.

Once one sees that the general structure of the MaxMin analysis is similar to the general structure of received zero-sum game theory in general spaces, we have access to a large literature which not only assists us in analyzing these problems but also assists us in formulating approaches to learning from data flows (e.g. "on line" learning) by making use of the huge literature on zero sum stochastic games (e.g. Mannor and Shimkin (2002), Schlag (2003)), especially the literature on possible attainment of "lower envelopes" (e.g. the Bayes Envelope of Mannor and Shimkin (2002) which are convex hulls of nonconvex objects which stand in the way of tractability. This appears to be a fruitful avenue for future research in this area.

4. A "TOP DOWN" APPROACH

We take the same lender's problem treated above and use it to illustrate a top down approach. Here we assume a "baseline" function \(p(1|x)\) which is "close" to the unknown function \(P(1|x)\) in the spirit of Hansen and Sargent (2003) as modified by Brock, Durlauf, and West (2003) and Brock and Durlauf (2004). This "top down" approach may be useful for situations where the lender has reason to believe that the function \(p(1|x)\) that it obtained from one population should serve as a good baseline for the current population that is under scrutiny. For example the lender may have learned \(p(1|x)\) for one population using the procedure in Section 3 and now wishes to lend to a new population that it believes is "close" using a relevant metric.

The analytical structure of what follows is close to that of zero sum game theory except the minimizing player is constrained to a small neighborhood of a "baseline." The smallest of the neighborhood enables useful approximations by Taylor expansion in the size of the neighborhood of the zero sum game equilibrium around the baseline (Brock and Durlauf (2004)).

For ease of exposition we assume \(P(x)\) has finite support

\[(4.1) \quad x_1 < x_2 < \ldots < x_n \]
with probabilities $P(x_i) = q_i$, $i = 1, 2, \ldots, n$. Let $u_i = 1$ if the lender lends to $x_i$, zero if not. First consider the risk neutral case where we assume the lender maximizes expected profits. Fix the size of each loan at $L$ dollars and let

$$J(u, a) = \mathbb{E}[\{u_i \cdot AL\cdot [(1+x_i)^{-1} - P^*]\}] = \mathbb{E}[\{u_i \cdot (a_i - P^*)\}]q_i,$$

where the baseline value of $a$ is given by,

$$(4.3) \quad a_i = p(1|x_i), \quad i = 1, 2, \ldots, n.$$ 

As we showed above the solution to this problem is to find a cutoff $c^*$ (i.e. $c^*$ solves $p(1|x) = P^*$) that sets mean profits to zero and lend to all who have $x > c^*$. Hence optimal $u_i$ is unity for all $x_i > c^*$, zero otherwise.

In order to represent a local form of "ambiguity aversion" introduce an Adversarial Agent, AA, who tries to hurt $J$ within a constraint set

$$(4.4) \quad \{a | \sum |a_i - p(1|x_i)|^2 \leq \epsilon^2\},$$

for $\epsilon$ small and positive. Consider Nash equilibria $(u^*, a^*)$ that solve,

$$(4.5) \quad J(u^*, a^*) \geq J(u, a^*), \quad \text{for all } u = (u_1, \ldots, u_n), \quad 0 \leq u_i \leq 1, \quad i = 1, \ldots, n,$$

$$(4.6) \quad J(u^*, a^*) \leq J(u^*, a), \quad \text{for all } a \quad \text{that satisfy } (4.4).$$

It is easy to write out the first order necessary conditions (Brock and Durlauf 2004)) and show that, for $\epsilon$ small enough, we have

$$(4.7) \quad a_i^* = p(1|x_i) - eu_i^* q_i / D^*, \quad D^* = (\sum u_j^* q_j)^{1/2}, \quad u_i^* = 1[a_i^* \geq 0],$$

where $\sum'$ denotes the summation over all $j$ with $u_j^* = 1$.

The second order necessary conditions for AA's problem determines whether the perturbation to $p(1x)$ is positive or negative. It is negative for a minimizer. Here $D^*$ is the number of $x_j$'s (weighted by the type probability, $q_j$) such that $x_j > c^{**}$ where $c^{**} = c^*$ is a 'robustified' cutoff which is determined by the requirement that mean profits be non-negative when $p(1|x_i)$ is replaced by $a_i^*$, $i = 1, 2, \ldots, n$. Notice that $u_i^*$ in (4.7) is $u_i^* = 1[a_i^* \geq 0]$.

Hence (4.7) is a rather awkward fixed point equation.

It is helpful to solve the analogous problem when $X$ is a continuum, e.g. an interval in the real line. The objective, (4.2) is now

$$(4.8) \quad \mathbb{E}[u(x)\cdot a(x) - P^*]dx,$$

and Nash equilibrium becomes

$$(4.9a) \quad a^*(x) = p(1|x) - eu^*(x)q(x)/D^*,$$

$$(4.9b) \quad D^* = (\int_x u^*(x)q(x)dx)^{1/2} = (1-F(x^*))^{1/2},$$

$$(4.9c) \quad u^*(x) = 1[a^*(x) \geq 0],$$

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where \( F(x) : = \text{Prob}(X \leq x) \). Note that \( q(x) = F'(x) = f(x) \). Hence, \( x^* \) must solve the equation

\[
(4.10a) \quad P^* = p(1|x) - ef(x)/(1-F(x^*))^{1/2}, \quad \text{i.e.,}
\]

\[
(4.10b) \quad P^* = p(1|x^*) - h(x^*)/(1-F(x^*))^{1/2},
\]

where \( h(x) = f(x)/(1-F(x)) \) is the hazard function \((\text{Miller (1981)})\). Differentiation of both sides of \((4.10b)\) w.r.t. e and evaluation at \( e = 0 \) yields

\[
(4.11) \quad x^*(0) = h(x^*)(1-F(x^*))^{1/2}/p'(1|x^*) > 0,
\]

which shows that the cutoff, \( x^*(e) \), increases (at least for \( e \) near zero) which is what one would expect.

Turn now to a variation where lender robustifies only against lending to those who carry covariates with which it has less "experience." Suppose the true \( P(1|x) \) satisfies \((2.2)\) above,

\[
(4.12) \quad P(1|x) = P(1|x,z=1)P(x,z=1)+p(1|x)P(x,z=0),
\]

with the unknown part \( P(1|x,z=0) = p(1|x) \). This could represent a situation where lender has experienced lending to a fraction \( P(x,z=1) \) of the \( x \)'s (or it has gathered data from other lenders to reveal this quantity) in the "new" population. Now lender only needs to locally robustify against its ignorance of \( P(1|x,z=0) \). The analog of \((4.7)\) is now,

\[
(4.13) \quad a_i^* = p(1|x_i) - eu_i^*q_i(P(x_i,z_i=0))/D^*, \quad D^* = (\sum u_j^*q_j(P(x_j,z_j=0)))^{1/2}.
\]

Recall that \((4.7)\) (and \((4.10a)\)) implies that the AA worsens \( p(1|x_i) \) relatively more, the more probable \( x_i \) is, i.e. the higher is \( q_i \) for all \( i \)'s that are "active", i.e. the ones that lender lends to. Under \((4.13)\) the AA now "slants" the worsening even if all the \( q_i \)'s are the same. It hits the "in-active" \( x_i \)'s relatively harder the larger is \( q_i P(x_i,z_i=0) \), i.e. the larger the chances they did not receive past loans weighted by the prevalence of that type.

This finding is related to what we will call, a "power principle" (See Hansen and Sargent's (2003) Chapter on frequency domain methods). E.g. Brock and Durlauf (2004) showed that, in the context of design of variance minimizing feedback control, that the AA hits the frequency distribution of "outside" shocks to a linear dynamics which is being optimally controlled relatively softer where the spectral density of the shocks has more power. The mechanism behind BD's finding was this. The optimal control design at \( e = 0 \) tended to "flatten" the areas of high power so AA attacks the parts where the optimizer is relatively "careless." These areas are where a marginal change in a parameter has the biggest impact on the optimized value of the objective at \( e = 0 \). A related thing happens here. I.e. AA does more damage for i's where \( u_i^*q_i P(y_i,z_i=0) \) is relatively largest. To put it another way the derivative of \( J \) w.r.t. \( a_i \) measures the marginal change of the planner's goal w.r.t. parameter \( a_i \) and the AA changes the parameter \( a_i \) in the objective function relative more the bigger the marginal change w.r.t. \( a_i \). This implies, using the envelope theorem, that \( dJ(u^*,a^*)/de = -\lambda J |_a \) at \( e = 0 \). I.e. the AA finds the
local direction where it can do the most damage for a given allotment. This
direction represents the direction of maximal vulnerability that a planner who
is uncertain about specification of problem structure should robustify itself
against.

It is useful to pause at this point and reflect upon the differences
between the lender’s problem and Manski’s (2003,2004a,b) formulations. The
analogy with his "treatment rule" formulation is very close. In fact a
version of the lender’s problem where the lender draws random samples from the
population as does Manski (2004) in his treatment rules problem can be
directly mapped into Manski’s binary case of two "treatments" (do not lend is
treatment t=0 and lend is treatment t=1, outcome y=0 is default, outcome y=1
is pay off the loan, utility is lending profit). This formulation would be
appropriate for a lender whose objective is the long run average payoff as in
Manski’s "Waldean" formulation (2004a). Indeed, we believe, that Manski’s use
of Hoefding’s large deviations theorem can be adapted to produce bounds for
maximin regret lenders.

One may view our local robustification ("top down") analysis as a form of
"local" robustification of Manski’s (2004a,b) "ideal" problem. The same
analysis could be applied to obtain local robustification in Manski’s
(2004c) context of profiling in crime control.

Our "bottoms up" formulation may be viewed as a form of "adaptive
learning" in Manski’s (2004a,b) world. I.e. we focus here on "adaptive
learning" situations where there is an interaction of the sampling process
z=1,z=0 and the underlying economics of the problem because z=1,z=0 for this
period is determined by last period’s decisions by the lender. This
interaction between the sampling process z=1,z=0 and the economics of the
problem suggests not only novel learning processes but also raises issues
about how to optimally learn as well as optimizing tradeoffs between losses
today and useful information on P(1|y) for better optimizations in the future.
We set these interesting issues aside for future research.

Turn now to a mean variance version of the lender’s problem. Let \(C_{ij}\)
denote the covariance of the random variable \(1[+1,x_i]\) with \(1[+1,x_j]\). Let
\(P(1|1):=E\{1[+1,x_i]\}\), \(u_i:=d_i L_i\). Here \(d_i\) is one if the lender lends to i and is
zero otherwise. The mean and the variance of profits for loans of size \(L_i\) to
\(x_i\), \(i=1,2,...,n\) is given by

\[
(4.14) \quad EJ:=\sum_{i} u_i (AP(1|1)-1) q_i \\
(4.15) \quad VJ:=A^2 \sum_{i,j} C_{ij} u_i u_j q_i q_j.
\]

We continue to abstract from determination of the optimal size of the loans
for simplicity. Future research should endogenize the size of \(L\) in the
obvious way.

Suppose the lender maximizes \(EJ-(a/2)VJ\) and the AA strikes the \(C_{ij}\). The
partial derivative of the objective w.r.t. \(C_{ij}\) is \(-aA^2 u_i u_j q_i q_j\) so we have

\[
(4.16) \quad C_{ij}^*=C_{ij} + e(u_i u_j q_i q_j / D^*), \quad D^*:=\left[\sum u_i u_j q_i q_j\right]^{1/2}.
\]

Hence, if, for example, all types are equally likely, \(q_i=1/n\), the lender acts
as if it faces a "larger" \(C_{ij}\) matrix whose elements increase relatively more
the larger the "exposure" as measured by

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Notice that the sizes of the loans as well as the actions (whether to lend or not) appear, so the label, "exposure" is appropriate. We leave further exploration along these lines to future research now that the path is clear.

LOCAL ROBUSTIFICATION OF "IDEALIZED" TREATMENT RESPONSE

The lender's problem is a useful introduction to the more general problem we wish to address now.

Consider Manski's (2004a, Section 2.1) "idealized" treatment response problem, where there is a planner whose goal is to maximize population mean welfare which is achieved by

\[
U^*(P) := \sum_p(x) \{\max \{E[u(y(t),t,x) | x] \} = \sum p(x) \{ \max \{ m(t,x) \} \},
\]

where \( \sum \) is over \( x \) in \( X \), \( m(t,x) := E[u(y(t),t,x) | x] \), \( \max \) is over \( t \) in \( T \), and I change the notation slightly from Manski's to ease typing. It is easy to map the lender's problem as a special case of this framework. I.e. put \( T = \{ 0 = \text{do not lend}, 1 = \text{lend} \} \), \( y(0) = 0 \), \( y(1) = \{ 0 = \text{default}, 1 = \text{pay off loan} \} \), \( u(y(0), 0, x) = 0 \), \( u(y(1), 1, x) = AL[y(1) - P^*] \). Suppose now that the planner faces a new problem whose structure is "close" to the above problem in some metric. For example, in the binary case \( E[u(y(1), 1, x) | x] = AL[E(y(1) | x) - P^*] \) above, we locally robustified against

\[
a_o(x_1^i) := E\{u(y(1), 1, x_1^i) | x_1^i\} - E\{u(y(0), 0, x_1^i) | x_1^i\}, i = 1, 2, \ldots, n,
\]

for \( X := \{ x_1^1, x_2^1, \ldots, x_n^1 \} \).

Profiling appeared here because lender profiled on \( x \) and chose the optimal cutoff \( x^* \) for \( t = 1 \) over \( t = 0 \). The cutoff, \( x^* \), was determined by the rule

\[
a(x) := E\{u(y(1), 1, x) | x\} - E\{u(y(0), 0, x) | x\} > 0, \text{ for } x > x^*
\]

where the planner's (i.e. the lender's) objective in this very special case of (4.18) can be written as

\[
J(u, a) := \sum_{x_1^i} a(x_1^i) p(x_1^i).
\]

Here \( u_1 = 1 \) if the lender lends (i.e. gives treatment 1) to \( x_1^i \) and zero otherwise.

In many binary treatment applications it is natural to take \( t = 0 \) as "status quo" and (4.21) as the planner's objective. For these cases, when the planner knows that the population it faces is structurally near the baseline population where objective function (4.21) is appropriate and it has no reason to suspect that the new population structurally deviates in any particular direction, then it can locate the directions of structural deviation against which it is most vulnerable by introducing a fictional Adversarial Agent and finding Nash equilibria, as follows,

\[
J(u^*, a^*) \geq J(u, a^*), u := \{u(x_1^1), \ldots, u(x_n^1)\}, u(x_i^1) \in \{0, 1\},
\]

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\[(4.23) \ J(u^*, a) \leq J(u^*, a^*), \text{ for all } a \text{ such that } \sum (a(x_i) - a_o(x_i))^2 \leq \varepsilon^2,\]

where \(\varepsilon > 0\) is appropriately small. It is natural to take \(\varepsilon\) smaller the closer the planner believes the new population is to the baseline population in a metric across the set of structural descriptions. Assume \(a(x)\) in (4.15) is increasing and continuous in \(x\) and that there is \(x^*\) such that \(a(x^*) = 0\). Compute Nash equilibrium \((u^*, a^*)\) to obtain,

\[(4.24) \ u^*(x_i) = 1[a^*(x_i) \geq 0], \quad a^*(x_i) = a(x_i) - e(p(x_i)1[a_i^* \geq 0]/D^*),\]

\[(4.25) \ D^* = \sum_{k=1}^{n} [a_k^* \geq 0], \text{ where } \sum \text{ is over } k = 1, 2, \ldots, n.\]

When \(\varepsilon = 0\), Nash equilibrium is \(u_i = 1[a_i(x_i) \geq 0]\) which is the optimum for the original problem. When \(\varepsilon\) is not zero the solution (if it exists) to (4.24) and (4.25) involves a rather taxonomic approach.

But we can shed light on the nature of the solution by going to a continuous state space analog. Let \(X\) be the closed interval, \([a, b]\), which we replace by \(X = [0, 1]\) via a rescaling. Replace the "baseline" \(a(x_i)\) by \(a_o(x)\), replace \(u(x_i)\) by \(u(x)\), put \(J(u, a) = \int u(x)a(x)dx\), replace \(u(x)\) in \([0, 1]\) by \(u(x)\) in \([0, 1]\), assume \(a_o(x)\) is strictly increasing in \(x\) and is continuous in \(x\) and there is unique \(x^*\) in \((0, 1)\) such that \(a_o(x) < 0\) for \(x < x^*\), \(a_o(x) > 0\) for \(x > x^*\). Then for \(e = 0\) it is easy to show that the optimal \(u^*(x) = 1[a_o(x) \geq 0]\). Now let \(e > 0\).

Let us solve for Nash equilibrium where the planner, which we call the Primary Agent, hereafter, "PA", solves the problem

\[(4.26) \ \text{maximize} \{J(u, a) = \int u(x)a(x)dx\} \text{ over } u,\]

and the Adversarial Agent, hereafter "AA", solves the problem,

\[(4.27) \ \text{minimize} \ \{J(u, a) \text{ s.t. } \int (a(x) - a_o(x))^2dx \leq \varepsilon^2\}.\]

It is easy to see that PA's best reply to AA is \(u^*(x) = 1[a(x) \geq 0]\) and AA's best reply to PA is

\[(4.28) \ a^*(x) = a_o(x) - ep(x)1[a^*(x) \geq 0]/\{\int p(z)1[a^*(z) \geq 0]\}^2dz\}^{1/2}.\]

Since \(a_o(x)\) is assume to be strictly increasing and continuous in \(x\) it is natural to look for a threshold \(x^*(e)\) that solves the equation,

\[(4.29) \ a^*(x) = a_o(x) - ep(x)/\{\int_x p(z)^2dz\}^{1/2}.\]

For \(e\) small enough (4.29) will have a solution \(x^*(e) > x^* = x^*(0)\) that is nearest to \(x^*\) as well as one closest to the upper bound "1" of \(X\) if the function,

\[(4.30) \ g(x) = p(x)/\{\int_x p(z)^2dz\}^{1/2},\]

becomes infinite as \(x\) tends to 1. This creates a bit of an artifact since it seems sensible for PA to protect itself against misspecification of \(a(x)\) around the baseline \(a_o(x)\) by increasing the threshold \(x^*\) to \(x^*(e)\) but it does
not make sense to refuse to assign $t=1$ to even higher $x$'s near 1.

A useful way to fix this problem is to introduce a sensible lower bound to $a(x)$ in formulating the AA's problem (2.47) as well as the requirement that $a(x)$ be close to $a_0(x)$ in some useful metric. Application of the Kuhn Tucker Theorem to the reformulated AA's problem will deliver a new $a^*(x)$ that produces a unique threshold $x^*(e)$ if $a_0(.)$ produces a unique threshold provided that $p(.)$ is continuous.

LOCAL ROBUSTIFICATION FOR "IDEAL" TREATMENT RESPONSE PROBLEMS

We have now developed enough examples that it is appropriate to briefly sketch what a general local robustification analysis of (4.18) would look like. Recall (4.18) and rewrite it as follows,

(4.31) $U^*(P) = \sum_p(x) \{ \max \{ E[u(y(t),t,x)|x] \} : \} = \sum_p(x) \{ \max \{ \sum d(t,x)m(t,x) \} \}$

$= \sum_p(x) \{ \max \{ \sum d(t,x)m(t,x) \} \}$

$= \sum_p(x) \{ m(0,x) + \max \{ \sum d(t,x)[m(t,x) - m(0,x)] \} \}$

$= \sum_p(x) \{ m(0,x) + \max \{ \sum d(t,x)[a(t,x)] \} \}$,

where "max" is over $\{ d(0,x), d(1,x), \ldots, d(K,x) \}$ such that $\sum d(t,x) = 1$, $d(t,x) \geq 0$, for all $t \in T = \{ 0, 1, \ldots, K \}$, for all $x \in X$, and $\sum' \text{ is over } t = 1, 2, \ldots, K$. It is easy to see that maximizing over $K+1$ choices is the same as maximizing the above problem over the $K+1$ simplex in the nonnegative orthant of $K+1$ dimensional Euclidean space. This formulation is convenient for the analysis that follows which exploits the $J(u,a)$ framework used earlier.

We take the position here that the main interest in robustification is to robustify the treatment decision $d(t,x)$ against misspecification of the structure of the problem. We see from (4.31) that the structure that matters from this position is the collection of functions $\{a(t,x), t \in T, x \in X\}$. Hence we map this problem into the $J(u,a)$ framework by putting $u=d$. The Lagrangian for AA's problem is given by

(4.32) $L = \sum_p(x) \{ m(0,x) + \{ \sum d^*(t,x)[a(t,x)] \} \}$

$\quad + \sum' \lambda(k)[e(k)^2 - \Sigma[a(k,x) - a_0(k,x)]^2]$, 

where $d^*(t,x)$ is the indicator function of the event that treatment $t$ in $\{1,2,\ldots,K\}$ is the best treatment for $x$.

We ignore the issue of imposing a lower bound to $a(\ldots)$ to prevent "artifacts" in order to focus on the essential structure of the problem. It will be clear in a moment how to deal with such artifacts. The first and second order necessary conditions for a minimum yield

(4.33) $a^*(k,x) = a_0(k,x) - e(k)p(x)d^*(k,x)/[\sum(p(x)d^*(k,x))^2]^{1/2}, k=1,2,\ldots,K,$

for all $x \in X$. But $d^*(k,x)$ is the indicator function of the event that $k$ is the best treatment for $x$ under $a^*(k,x)$ which creates a rather awkward fixedpoint problem to deal with.

We can shed light on this problem by assuming $X$ is the closed interval [0,1] and that all functions $a_0(\ldots)$, $p(.)$ are continuous in $x$ as we did in the binary case above. Let us get started by looking at the general binary case. Treatment 1 is chosen for $S(1) = \{ x|a(1,x) > 0 \}$ which is an open set in
[0, 1] (i.e. a "relative to [0, 1]" open set) by continuity of \( a(1, x) \) in \( x \) with boundary set \( dS(1) = \{ x \mid a(1, x) = 0 \} \). In the case where \( a(1, x) \) was strictly increasing in \( x \) that we analyzed above there was at most one point \( x^* \) in \( dS(1) \). Equation (4.33) suggests that, since the function \( a^*(1, x) \) falls, that the cutoff, \( x^* \), will rise, i.e. a higher threshold will be imposed to accept treatment 1 over treatment 0. In general the "acceptance set" for treatment one will shrink. It is relatively straightforward generalize this type of analysis to more treatments than two. We leave this to future research. Turn now to other types of robustification.

In many situations a planner may feel that a population is "nearby" a baseline population so robustification against misspecification of the covariate probabilities is also needed. A natural way of doing this is to add more terms to the lagrangian (4.32) to obtain

\[
(4.32') \quad L = \sum p(x)\{m(0, x) + \{ \sum d^*(t, x)[a(t, x)] \} \} + \sum \lambda(k)[e(k)^2 - \sum(a(k, x) - ao(k, x))^2] + \lambda[e^2 - \sum(p(x) - po(x))^2] + \mu[1 - \sum p(x)],
\]

and obtain the following formula for \( p^*(x) \)

\[
(4.34) \quad p^*(x) = po(x) / e\delta D(x) / (\sum e\delta D^2(z)dz)^{1/2},
\]

\[
(4.35) \quad \delta D(\text{x}) = D(\text{x}) - E(D(\text{x})), \quad D(\text{x}) = m(0, x) + \{ \sum d^*(t, x)[a(t, x)] \},
\]

and \( E(D(\text{x})) \) is the expectation of \( D \) over the maximum entropy measure on \( X \). For example for \( X = \{ x_1, x_2, \ldots, x_n \} \), \( E(D(\text{x})) = (1/n)\sum D(x_1) \), for \( X = [a, b] \), \( E(D(\text{x})) = fD(z)(dz/(b-a)) \).

A closely related type of robustification is useful when the results need to be useful to a variety of planners who do not necessarily agree on the utility functions \( u[y(t), t, x] | x \). In cases like this the objects of interest are the joint outcome densities \( f(y(t), t, x) \) and the conditional outcome densities, \( f(y(t), t|x) \). Suppose a random sample of data for each treatment \( t \) is available for a covariate \( x \), denote it by \( \{ y(t, j), j = 1, 2, \ldots, N(t, x) \} \). Form an estimator of the density \( f(y(t), t, x) \) and denote it by \( \hat{f}(y(t), t, x) \). See Pagan and Ullah (1999) for a concise treatment of various estimators, including kernel density estimators. Given a utility \( u[y(t), t, x] | x \), a natural estimator of \( m(t, x) = Eu[y(t), t, x] | x \) is

\[
(4.36) \quad Eu[y(t), t, x] | x = \int dyu[y, t, x]fo(y, t, x) / p(x).
\]

It is natural to locally robustify against uncertainty in \( f \) for each \( t, x \) when \( p(x) \) is known. Taylor expansion in the size of the neighborhood can now be done for whatever objective is of interest to the analyst, e.g. localized uncertainty about the utility function. We leave this to future research.

5. CLASS II PROFILING PROBLEMS

In class II profiling problems, profiteers can alter their characteristic \( x \) to \( x' \) at a cost \( C(x, x') \). If we assume the distribution of types and \( C(x, x') \) is common knowledge to both lenders and lendees, we can study equilibrium concepts. There is a sub class of these problems where our treatment may be brief by mapping the analysis into a special case of the above.

In order to get started let us treat a simplest example first. Suppose \( C(x, x') = 0 \) for \( x' = x + e, \) and is infinite for \( x' > x + e, \) for some \( e > 0. \) Thus an \( x \) will always pose as \( x' = x + e \) if it is worthwhile to pose at all. Suppose there are only two possible types \( x_1 < x_2 \) with probabilities \( q_1, 1-q_1 \) and this
information is common knowledge to both sides. Suppose \( P(1|x) \) increases in \( x \) and is known to the lender.

Suppose, at first blush, the lender uses the rule: Lend \( L \) if \( x=x^* \) where \( x^* \) solves \( P(1|x)-P^*<0 \). Assume \( x_1<x^* \leq x_2 \), so that lender plans to reject \( x_1 \), but lend to \( x_2 \). If the "bad" type \( x_1 \) differs from the "good" type \( x_2 \) by no more than \( e \), then \( x_1 \) will pose as \( x_2 \). But lender knows this and knows that the cost to \( x_1 \) of such posing is zero. Hence lender must protect itself by assuming it faces a "pool" of \( q_1 \) \( x_1 \)'s and \( 1-q_1 \) \( x_2 \)'s. Mean profits for lender facing this population of lendees is

\[
(5.1) \quad \text{LAD} \{d[q_1P(1|x_1)-P^*]+q_2P(1|x_2)-P^*]\}
\]

where \( d=1 \) if lender lends to the group, \( d=0 \) otherwise. Thus, lender acts as if it faces a "worst case" scenario where all the bad types pose as good types. I.e. it lends to the "atom" \((x_1,x_2)\) if the pooled probability of payoff of that atom satisfies

\[
(5.2) \quad q_1P(1|x_1)+q_2P(1|x_2) \geq P^*
\]

Turn now to a more general version of this problem. If the cost function to lendees of posing as \( x'=x \) is zero for \( x'-x \leq e \geq 0 \), and is infinite for \( x'-x>a \), and it is common knowledge that \( P(x) \) is a continuous distribution with bounded support, for example, \([0,1]\), and \( e<1 \), then lender knows that if it announces a cutoff lending strategy \( x^* \), then lendee of type \( x \) will pose as type \( x+a \) to maximize the chance of getting a loan. Thus there is an "atom" at \( x=1 \) which consists of all types \( x \) such that \( x+e \geq 1 \), because all types \( x \) in \([1-e,1]\) can pose as \( 1 \)'s for free. Furthermore lender should expect that all types \( x \) in \([0,1-e]\) will pose as \( x'=x+e \). Therefore if lender knows \( P(1|x) \) and \( P(x) \), it should lend to \( x' \geq x^* \), for \( x^* \) such that,

\[
(5.3) \quad P(1|x^*-e) \geq P^*
\]

if such \( x^* \) exists. If \( e=1 \), all \( x \)'s pose as \( 1 \)'s so it lends to the "atom" \([0,1]\) if

\[
(5.4) \quad \int P(1|x)dP(x) \geq P^*.
\]

Hence, we see that if "ambiguity" is present on the size of \( e \) in the interval \([0,e]\), the lender's problem can be mapped into the general framework developed in this paper. I.e. the maximin lender assumes, for each type \( x_1 \) in the set, \( A \), of admissible types, \( A:=\{x_1, x_2', \ldots, x_n\} \), it observes the worst case pool, call it \( P_*(x_1') \), that can be formed from the set of admissible types, \( A:=\{x_1, x_2', \ldots, x_n\} \) with their probabilities \( \{q_1, q_2', \ldots, q_n\} \), and lends provided that the net return on that pool, denote it by \( \pi_*(x_1') \), is positive. Notice that if any type \( x_1 \) "lies" i.e. reports that it is a higher type \( x_j>x_1 \) in \( A \), that it always "lies to the max" and reports the largest "reachable" type \( x_j \) in \( A \) that is in the "reachable" \([x_1, x_1+e]\). Meanwhile the best possible pool \( P^*(x_1') \) is just the truth. I.e. the reported \( x_1' \) that lender
observes is just the true type $x_1'$ and the lender gets the true net return, $\pi^*(x_1')$, on this type. Now we are ready to apply our machinery above to this particular subclass of Class II, profiling problems. As we saw from our above work, the maximin lender does not lend to reported type stratum, $x_1'$ if $\pi^*(x_1')<0$. The minimax regret lender, on the other hand, lends to the positive fraction $d(x_1')$ defined by (5.5) below of reported type stratum $x_1'$ provided that, $\pi^*(x_1')>0$.

(5.5) $0<d(x_1')=\pi^*(x_1')/(-\pi^*(x_1')+\pi^*(x_1'))$.

Of course there are details to work out, but the sketch above is suggestive of a start on dealing with Class II profiling problems using the tools developed in earlier sections.

This formulation can be generalized in many ways.

(i) One can study problems where the cost of posing as $x'>x$ is a convex increasing function, $C(x'-x)$, with $C(0)=0$. The formulation above is enough to suggest that lender will not want to constrain itself to designing mechanisms where each type is induced to report its true type. Furthermore, suppose the support of $x$ is bounded, W.L.O.G., assume it is the closed interval, $[0,1]$ and this is common knowledge to both sides. We have learned enough from the above work to expect an "atom" to appear at $x=x^*$ where $x^*$ is a cutoff level for lending. This raises issues of existence or nonexistence of "pooling" and "separating" equilibria, analogous to the "screening" literature of the 1970's. Some of these nonexistence problems could be "fixed" by introducing "noisy discrete choice" lenders, producing existence of equilibria indexed by the level of the noise, and taking the noise level to zero to study potential convergence of the equilibria to a "natural" limit. This issue merits study.

(ii) We can study problems where lender has a wider strategy set. For example let it have access to "atom splitting" strategies where it can randomly audit members of a suspected atom at a cost of $S$ ("S" stands for "search" or "screen") for each. Let the audit detect the true type with probability one if it is conducted and let the punishment cost to the poser if caught be $F$. Suppose $F$ is fixed by law or custom. If lender could choose $F$ the problem would become trivial because lender should choose an infinite $F$ and audit with infinitesimal probability.

(iii) Potentially more interesting would be to study problems where the possibility of posing interacts with the lack of knowledge of $P(1|x,z=0)$ for various sampling processes. For example, the lender could open a trial "loan window" where it accepts $N$ applications from a self reported $x'$-stratum and attempts to learn $P(1|x)$ for the population but only gets to observe $P(1|x,z=1)$ where $(z=1,x)$ denotes acceptance of an application by a type who reports as type $x$. A "structural" econometrician might use economic theory to build a model of this joint process and attempt to use data to estimate that model as in the literature on selection bias, for example. One could, in the standard way, look for sufficient conditions for that model to be identified. However, it may be difficult to get any kind of consensus in the research community that this model is the "right" model. But one might be able to get more consensus on a "cloud" of models surrounding a "baseline" model. If so, then one could adapt the Bayesian Model Averaging approach of Brock, Durlauf, and West (2003).
6. SUMMARY, CONCLUSIONS, SUGGESTIONS FOR FUTURE RESEARCH AND APPLICATIONS

This paper has attempted to connect several areas of research that appear unconnected at first thought: (i) Treatment effects and profiling analysis under partially identified structures; (ii) Zero sum game theory, deterministic and stochastic; and (iii) Robustification analysis in control theory. We used a lender's problem as a main expository vehicle. We could have used an employer's problem and attempting to detect discrimination as an example, which we explain in some detail here because it suggests some potentially interesting future research.

For example, suppose there are two types of people, blue and green. An experiment is conducted where fake CV's are constructed with qualifications x for each type where x is varied. These CV's are sent out in response to prospective employers and a response variable is measured, e.g. the number of callbacks received. It is observed that blues receive more callbacks than greens for the same x and it is even observed that the relative callback rate for greens falls as qualifications x increases. What is going on here? Is this evidence of employer discrimination against greens in favor of blues?

Let us use the lender's problem above to explore this issue. Let P(1|x,w) be the probability that type w with characteristics x will be a "successful" employee and let L(x,w) denote the level at which this employee is employed. For example L(x,w) could represent the size of the employment contract or commitment given to (x,w). Suppose profits per employee are given by

\[ \pi(x,w) = L(x,w)\{A(1+\alpha|x,w)|-1\}, \]

where \(1+\alpha|x,w\) denotes the event that (x,w) turns out to be a good employee which makes revenue A per unit time employed and -1 is the normalized wage. We assume here that if the event \(1+\alpha|x,w\) does not occur, i.e. the employee turns out to be not good, then nothing is produced. Thus the employer's problem has the same analytical structure as the lender's problem.

We can make the following observations now. First, in order to conclude that the experiment reveals discrimination we need an "invariance assumption". i.e. the hypothetical population constructed by the experimenter must be the same as the actual population faced by the employers. The experimenter constructed a hypothetical population where the qualifications of blues and greens were exactly the same. If the employer knew this the callback rate should have been exactly the same conditional upon qualifications x. However, invariance is not likely to hold in the field. This lack of invariance is perhaps one reason why the legal notion of "intent" plays such a big role in actual litigation. What can we conclude from our hypothetical experiment?.

We can use such hypothetical experiments plus our work on the lender's problem above to suggest possible alternative reasons for the observed behavior and, possibly design tests to rule out these alternative reasons.

Case 1: Suppose \(P(1|x,\text{green}) < P(1|x,\text{blue})\).

Under Case 1, if employers are maxi-miners then they will be reluctant to hire greens at any x because they do not have as much experience with greens. This outcome will be the case if we assume, as is plausible, that the "ambiguity set" (cf. (6.4) below) is larger with a worst, worst case scenario, the less experience the employer has with a particular type. Our discussion of learning above suggests that if this lack of experience combined with maximin behavior is the cause, that a subsidy for employment of greens will induce experience, which raises \(P(1|x,\text{green})\) which will eliminate the
resistence of employers.

Notice that improving greens' access to litigation against employers may induce the very maximin behavior that is causing the problem in the first place. This is so because it may increase the size of the "ambiguity set" that employers would face, thus inducing a more negative worst case scenario. More negative worst case scenarios, even if highly improbable, also make minimax regret employers less likely to hire greens.

Notice also that under Case 1, if employers have less experience at higher levels of \( x \), i.e. the

\[
(6.2) \quad \text{"experience gap" } G(x) := P(z=1|x, \text{blue}) - P(z=1|x, \text{green}),
\]

increases with \( x \) then the callback rate would fall as \( x \) increases if employers are maxi-miners.

This possibility could be "tested" by modifying the experiment. Suppose CV's are sent out for a new type, violet, where everyone knows that \( P(z=1|x, \text{violet}) = 0 \). Hence the experience gap \( G(x) \) is maximal for violets. If the callback rate for violets is higher than greens, the case for discrimination against greens would be strengthened. Furthermore, in the field, if one observed a company with lots of green employees who had been there for quite some time and who were fired after a new president took over, one would also be suspicious.

Manski (2004a) has focused attention on the objective of minimax regret in contrast to maximin because, among other things, maximin is too rigid in its reponse to additional data. We can ask at least two questions, for example, how might we use hypothetical experiments like those above to reveal evidence of minimax regret behavior on the part of employer in contrast to maximin behavior? Will minimax regret "look like" discriminatory behavior as does maximin?

We can directly apply our work above to compute minimax regret. The minimax regret employer will employ a positive fraction, \( d(x, w) \) if

\[
(6.3) \quad 0 < d(x, w) = \pi^* (x, w) / \left( \left( -\pi^*_* (x, w) \right) + \pi^* (x, w) \right),
\]
even though the worst case, \( \pi^*_* (x, w) < 0 \), where the maximin employer hires no one from stratum, \( (x, w) \). Of course the minimax regret employer is still vulnerable to being unduly influenced by a perceived-to-be very large, worst case belief, \( \left( -\pi^*_* (x, w) \right) \), for stratum \( (x, w) \) even though the probability of this value is tiny under any plausible set of priors (Epstein and Schneider (2002)) that the employer may have.

The idea of using knowledge gleaned from data and other sources to "trim" the set of "admissible" values of \( \pi \) which was used by Brock, Durlauf, and West (2004) in the context of monetary policy under model uncertainties might be used here to improve the performance of maximin and minimax regret or any criterion that is "unduly" impacted by "extreme" and "implausible" values of \( \pi \).

In any event, when coupled with our discussion of adaptive learning above, we see that the "experiments" that the minimax regret employer conducts under (6.3) causes it to learn much faster than the adaptive maximin employer. E.g. in population, the ambiguity set,

\[
(6.4) \quad A = \left\{ \pi^*_* (x, w), \pi^* (x, w) \right\},
\]
shrinks to a point for any stratum where \( \pi^* (x, w) > 0 \), even though \( \pi^*_* (x, w) \) may be
very negative, after only one period for the minimax regret employer using our
adaptive learning scheme discussed above. In contrast the maximin learner
gingerly steps into the employee pool, \((x,w)\) and creeps towards reducing the
ambiguity set, \(\pi^*(x,w)\), to a point at a rate depending upon the
assumed bound on the Lipchitz constant in Equation (2.12) above.

Turn now to discussion of future research and potential applications and
extensions.

First, we have said nothing about issues raised by the reality of finite
samples. Since our prototypical problem, e.g. the lender's problem, is
essentially a special case of Manski (2004a), the natural extension would be
to adapt his discussion of "empirical success strategies" and the "analogy
principle" (which replaces any population object with its sample counterpart)
to produce a discussion of sampling issues for any population discussion in
our work above. Coupling this extension with adaptive learning would be a
good topic for future research.

Second, Manski (2004c) has written a recent paper that studies the
problem of a planner who wishes to minimize cost of crime control under
partially identified offense functions the set of which is constrained by data
on observed policing search rates and realized offense rates, and a
monotonicity assumption on the response of the offense rate to increased
police searches. It would be interesting to study adaptive learning by police
in his framework.

Third, the lender's problem is close to the classical literature on
markets, screening, adverse selection, etc. which was plagued by nonexistence
problems. It would be interesting to extend the work above to equilibrium
contexts and to discuss empirical issues raised by the interaction of
partially identified structures with equilibration. Nonexistence problems
could be potentially dealt with by using noisy discrete choice and taking the
limit as the "intensity of choice" goes to zero. This idea is closely related
to ideas in game theory under the label of "quantal response" games as well as
ideas in dynamical systems theory under the label of "natural" limiting
measures.

Fourth, we barely scratched the surface above in our very brief treatment
of connections between different forms of selection bias in treatment choice
and ideas from monotone comparative statics such as increasing differences
assumptions and assumptions on stochastic dominance. These issues arise when
one must use observational data to inform treatment choice and a choice must
be made. For example such cases are important when one does not have access
to randomized experiments, or the "randomized experiments" are "contaminated"
by compliance, attrition, subjects "jockeying" for what they perceive is the
"best" treatment, and other problems in reality.

Fifth, we said nothing about issues raised by the possible dependence of
treatment effect of treatment \(t\) on stratum \(x\) on "endogenous social
interactions."

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