The Lake Game\textsuperscript{1}

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1 The Lake Model

We use a simplified model for the level of phosphorus in a lake. Phosphorus is one of the nutrients that are used by algae, and high levels of algae are associated with poor water quality. The model and a discussion about its implications and origins can be found in Carpenter, Brock, and Hanson (1999), Carpenter, Ludwig, and Brock (1999) and Mäler, Xepapadeas, and de Zeeuw (1999), as well as in the introduction to this volume and in Scheffer (forthcoming).

The dynamics of phosphorus sequestered in algae is modeled by

\[ \dot{P}(t) = L(t) - \delta P(t) + \frac{rP(t)^q}{m^q + P(t)^q} \quad (1.1) \]

where \( P \) is the level of phosphorus sequestered in algae, \( L \) is the loading of phosphorus per unit time, and \( \delta, r, m \) and \( q \) are parameters of a given lake. The loading typically comes from the runoff of excess fertilizer and manure from agriculture and home use.

The preferences of a representative agent are given by

\[ \int_0^\infty e^{-\rho t} U(L(t), P(t))dt \quad (1.2) \]

where \( U \) is the utility of the loading and the phosphorus level and \( \rho \) is the rate of discounting the future. Increases in the loading are assumed to bring increased crop yields which in turn implies higher consumption levels. Increases in the level of phosphorus are assumed to reduce the water quality. Thus the loading is an economic "good" and the level of phosphorus is an economic "bad" in utility terms. In this paper we will be analyzing solutions of the differential equation (1.1) that maximize the objective (1.2).

The game that we analyze is one in which there are \( N \) communities in the drainage basin of the lake that have representative consumers with preferences as in (1.2)

\[ \int_0^\infty e^{-\rho t} U_i(L_i(t), P(t))dt \quad (1.3) \]

where the preferences of the representative consumer from the \( i^{th} \) community depend only on the loading from that community, \( L_i \), and the level of phosphorus sequestered in algae in the lake, \( P \). Our analysis will focus on the Nash equilibria solutions to this dynamic game.

By using the following transformations,
\[
\tau = \frac{rt}{m} \\
x(\tau) = \frac{P(m\tau/r)}{m} \\
a(\tau) = \frac{L(m\tau/r)}{r} \\
b = \frac{m\delta}{r} \\
\bar{\rho} = \frac{m\rho}{r} \\
u_i(a, x) = \frac{r}{m} U_i(ra, mx)
\]

we get the following dimensionless\textsuperscript{1} optimization problem,

\[
\max \int_0^\infty e^{-\bar{\rho} \tau} u_i(a_i(\tau), x(\tau))d\tau \\
\text{subject to: } \dot{x}(\tau) = a_i(\tau) - bx(\tau) + \frac{x(\tau)^q}{1 + x(\tau)^q} + \sum_{j \neq i} a_j(\tau)
\]

where, for the Nash equilibrium, the \(i^{th}\) community takes the loading by the other communities

\[
\sum_{j \neq i} a_j(\tau)
\]

as given.

There are several approaches that one can take to solve for the Nash equilibria numerically. The one we take in this chapter is to first convert the continuous time dynamic game into a discrete time version, and then to use discrete dynamic programming to get a numerical solution for the game. Naturally, we will have to use a specific form of the utility function as well as specific values of the parameters in the differential equation.

To change this problem into a discrete time one, first let \(\tau = nh\) for \(n = 0, 1, \ldots\), and \(h\) is a small increment of time. The integral can then be written as:

\textsuperscript{1}Not only does this transformation convert the problem into a standardized form, but the parameter space has been reduced from a 5 dimensional one, \((\rho, r, m, q, \delta)\), to a 3 dimensional one, \((\bar{\rho}, b, q)\). In essence, the set of parameters that give rise to different solutions of the optimal control problem is a 3 dimensional manifold in a 5 dimensional space.
\[
\int_0^\infty e^{-\hat{\beta}r} u_i(a_i(\tau), x(\tau))d\tau = \sum_{n=0}^{\infty} \int_{nh}^{(n+1)h} e^{-\hat{\beta}r} u_i(a_i(\tau), x(\tau))d\tau
\approx \sum_{n=0}^{\infty} e^{-\hat{\beta}nh} u_i(a_i(nh), x(nh))h
\]

If we let \( x_n = x(nh) \), \( a_{i,n} = a_i(nh) \) and \( \beta = e^{-\hat{\beta}h} \), then a discrete time version of the continuous time optimization is:

\[
\max_{\{a_{i,n}\}} \sum_{n=0}^{\infty} \beta^n u_i(a_{i,n}, x_n)
\]

subject to:
\[
x_{n+1} = G(x_n) + ha_{i,n} + hA_{i,n}
\]

where player \( i \) takes the sequence \( \{A_{i,n}\} \) as given, and the function \( G \) is:

\[
G(x) = x + h \frac{x^q}{1 + x^q} - hbx.
\]

A Nash equilibrium is a solution, \( \{a_{i,n}\} \), such that

\[
A_{i,n} = \sum_{j \neq i} a_{j,n} \quad j = 1, \ldots, N.
\]  

2 \hspace{1cm} \textbf{A Potential Function for the Dynamic Game}

A potential function for the dynamic game is a function

\[
\Pi(a_1, \ldots, a_N, x)
\]

such that a solution to the control problem

\[
\max_{\{a_{i,n}\}} \sum_{n=0}^{\infty} \beta^n \Pi(a_{1,n}, \ldots, a_{N,n}, x_n)
\]

subject to:
\[
x_{n+1} = G(x_n) + h \sum_{i=1}^{N} a_{i,n}
\]

is a Nash equilibrium solution to the discrete time dynamic game (1.5) – (1.6).

The advantage of this approach is that in order to get a solution to the dynamic
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game, we not only have to solve the dynamic programming problem for each player, but we also have to solve a system of equilibrium equations. However, solving the optimal control problem (2.2) only requires us to solve a single dynamic programming problem. The techniques in Dechert (1978, 1997), which are for systems using Euler equations, can be adapted to the type of problem in this chapter.

Next, we need to show under what conditions a potential function for the dynamic game exists. First let us consider a Nash equilibrium solution for the dynamic game in Hamiltonian form. Player i’s Hamiltonian for the N player game is

\[ H_{i,n}(a_i, x, p_i) = u_i(a_i, x) + p_i \left( G(x) + ha_i + hA_{i,n} \right) \]

and the first order condition and dynamics are:

\[ 0 = \frac{\partial u_i}{\partial a_i}(a_{i,n}, x_n) + p_{i,n+1} h \quad (2.3) \]

\[ x_{n+1} = G(x_n) + ha_i + hA_{i,n} \quad (2.4) \]

\[ \beta^{-1}p_{i,n} = \frac{\partial u_i}{\partial x}(a_{i,n}, x_n) + p_{i,n+1}G'(x_n) \quad (2.5) \]

The Nash equilibrium conditions are

\[ A_{i,n} = \sum_{j \neq i} a_{i,n} \quad (2.6) \]

Next, consider the Hamiltonian for the optimal control problem (2.2)

\[ H(a_1, \ldots, a_N, x) = \Pi(a_1, \ldots, a_N, x) + p \left( G(x) + h \sum_{i=1}^{N} a_i \right) \]

The first order conditions and dynamics are:

\[ 0 = \frac{\partial \Pi}{\partial a_i}(a_{1,n}, \ldots, a_{N,n}, x_n) + p_{n+1} h \quad (2.7) \]

\[ x_{n+1} = G(x_n) + h \sum_{i=1}^{N} a_{i,n} \quad (2.8) \]

\[ \beta^{-1}p_{n} = \frac{\partial \Pi}{\partial x}(a_1, \ldots, a_N, x) + p_{n+1}G'(x_n) \quad (2.9) \]
Now by comparing equations (2.3) – (2.5) with equations (2.7) – (2.9) we can see that for $\Pi$ to be a potential function for the dynamic game the following must hold:

$$\frac{\partial \Pi}{\partial a_i}(a_1, \ldots, a_N, x) = \frac{\partial u_i}{\partial a_i}(a_i, x) \quad i = 1, \ldots, N \quad (2.10)$$

$$\frac{\partial \Pi}{\partial x}(a_1, \ldots, a_N, x) = \frac{\partial u_i}{\partial x}(a_i, x) \quad i = 1, \ldots, N \quad (2.11)$$

From equation (2.11) we get that

$$\frac{\partial^2 \Pi}{\partial x \partial a_j} = 0 \quad j = 1, \ldots, N$$

which implies that

$$\frac{\partial^2 u_i}{\partial x \partial a_i} = 0 \quad i = 1, \ldots, N$$

which in turn implies that there are functions $v_i$ and $c_i$ such that

$$u_i(a_i, x) = v_i(a_i) + c_i(x) \quad i = 1, \ldots, N.$$

Equation (2.11) also implies that there must be a common function, $c$, such that

$$c_i(x) = c(x) \quad i = 1, \ldots, N.$$

Therefore, assuming that the utility function of the representative consumer from the $i$th community is

$$u_i(a_i, x) = v_i(a_i) + c(x) \quad (2.12)$$

the potential function would have to be

$$\Pi(a_1, \ldots, a_N, x) = \sum_{i=1}^{N} v_i(a_i) + c(x). \quad (2.13)$$

With the potential function (2.13) the solution to the Hamiltonian system is:
\begin{align}
0 &= v'_i(a_{i,n}) + h p_{n+1} \quad i = 1, \ldots, N \\
x_{n+1} &= G(x_n) + h \sum_{i=1}^{N} a_{i,n} \\
\beta^{-1}p_n &= c'(x_n) + p_{n+1}G'(x_n)
\end{align}
\hspace{1cm} (2.14) (2.15) (2.16)

and with the individual utility functions given by (2.12) the Nash equilibrium solution to the dynamic game Hamiltonian systems for each \( i = 1, \ldots, N \) are:

\begin{align}
0 &= v'_i(a_{i,n}) + h p_{i,n+1} \\
x_{n+1} &= G(x_n) + h \sum_{i=1}^{N} a_{i,n} \\
\beta^{-1}p_{i,n} &= c'(x_n) + p_{i,n+1}G'(x_n)
\end{align}
\hspace{1cm} (2.17) (2.18) (2.19)

By inspection, a solution to equations (2.14) – (2.16) is also a solution to equations (2.17) – (2.19).

Let us show that indeed a solution to the control problem satisfies the best reply property of a Nash equilibrium of the dynamic game. Suppose that

\[ \{\hat{a}_{1,n}, \ldots, \hat{a}_{N,n}, x_n\} \]
\hspace{1cm} (2.20)

is a solution to the control problem

\[ \max_{\{a_{i,n}\}} \sum_{n=0}^{\infty} \beta^n \left[ \sum_{i=1}^{N} v_i(a_{i,n}) + c(x_n) \right] \]

subject to:

\[ x_{n+1} = G(x_n) + h \sum_{i=1}^{N} a_{i,n} \]

Now let us check the best reply property for community 1. Since (2.20) is a solution to the optimal control problem, for any sequence, \( \{\hat{a}_{i,n}\} \), and corresponding state sequence

\[ \tilde{x}_{n+1} = G(\tilde{x}_n) + h\hat{a}_{i,n} + h \sum_{i=2}^{N} \hat{a}_{i,n} \]
the following inequality holds:

$$\sum_{n=0}^{\infty} \beta^n \left[ \sum_{i=1}^{N} v_i(\hat{a}_{i,n}) + c(\hat{x}_n) \right] \geq \sum_{n=0}^{\infty} \beta^n \left[ \hat{a}_{1,n} + \sum_{i=2}^{N} v_i(\hat{a}_{i,n}) + c(\hat{x}_n) \right]$$

(2.21)

On both side of equation (2.21) there are common terms of $v_i(\hat{a}_{i,n})$ for $i = 2, \ldots, N$ which can be canceled, leaving

$$\sum_{n=0}^{\infty} \beta^n \left[ v_1(\hat{a}_{i,n}) + c(\hat{x}_n) \right] \geq \sum_{n=0}^{\infty} \beta^n \left[ \hat{a}_{1,n} + c(\hat{x}_n) \right]$$

(2.22)

Thus, $\{\hat{a}_{1,n}\}$ is a best reply for community 1 to $\{\hat{a}_{2,n}, \ldots, \hat{a}_{N,n}\}$ by the other communities. A similar argument applies for the other communities as well.

It is worth pointing out several observations on the method outlined above. First, there may be solutions to equations (2.17) – (2.19) with distinct sequences of the costate variables, $\{p_{i,n}\}$, which would not be solutions to equations (2.14) – (2.16), and therefore there may be more Nash equilibria solutions to the dynamic game than there are solutions to the control problem (2.2). However, if there is a unique solution to the dynamic game, then of necessity there is a unique solution to the optimal control problem. Second, clearly this method does not work with all dynamic games. As equation (2.12) shows, this method only works if the individual objective functions satisfy the restrictions imposed by equations (2.10) and (2.11). Third, the advantage of this method is twofold. The practical advantage is that we can use dynamic programming to solve the control problem. The theoretical advantage is that for non convex problems, the existence of a Nash equilibrium solution to the dynamic game is assured by the existence of a solution to the control problem.

3 A Dynamic Programming Solution

For the remainder of the paper, we are going to focus on a solution to a symmetric dynamic game, where the communities are homogeneous with preferences represented by

$$u_i(a_i, x) = \ln(a_i) - cx^2$$

(3.1)

The value function for the $N$ community game is
\[ V_N(x) = \max_{\{a_{i,n}\}} \sum_{n=0}^{\infty} \beta^n \left[ \sum_{i=1}^{N} v_i(a_{i,n}) + c(x_n) \right] \]

subject to: \[ x_{n+1} = G(x_n) + h \sum_{i=1}^{N} a_{i,n} \]

and the Bellman equation is

\[ V_N(x) = -cx^2 + \max_{\{a_i\}} \left\{ \sum_{i=1}^{N} \ln a_i + \beta V_N \left( G(x) + h \sum_{i=1}^{N} a_i \right) \right\} \quad (3.2) \]

One approach to solving this equation is to represent \( x \) as a discrete variable, and to solve the problem by iterating the value function. Since \( \beta < 1 \), the contraction mapping theorem assures that the function \( V_N \) is unique, and that iteration converges.

From a computational point of view, the maximization in (3.2) is over \( N \) variables and, because of symmetry, can be simplified by considering the following control problem:

\[ \tilde{V}_N(x) = -cx^2 + \max_{a \geq 0} \left\{ N \ln a + \beta \tilde{V}_N \left( G(x) + hNa \right) \right\} \quad (3.3) \]

By using the concavity of the logarithm function it can be shown that \( \tilde{V} = V \), and so solving (3.3) is equivalent to solving (3.2). To put the problem in standard form, introduce

\[ y = G(x) + hNa \]

and the dynamic programming problem can be rewritten as:

\[ \tilde{V}_N(x) = -cx^2 + \max_{y \geq G(x)} \left\{ N \ln \left( \frac{y - G(x)}{hN} \right) + \beta \tilde{V}_N(y) \right\} \quad (3.4) \]

\[ = -cx^2 - N \ln(hN) + \max_{y \geq G(x)} \left\{ N \ln(y - G(x)) + \beta \tilde{V}_N(y) \right\} \]

Now let us simplify this a little further. We can rewrite (3.4) as:

\[ \frac{\tilde{V}_N(x)}{N} + \frac{\ln(Nh)}{1-\beta} = -kx^2 + \max_{y \geq G(x)} \left\{ \ln(y - G(x)) + \beta \left( \frac{\tilde{V}_N(y)}{N} \right) + \frac{\ln(Nh)}{1-\beta} \right\} \]
where \( k = c/N \). Finally, let \( W_k \) be the fixed point of

\[
W_k(x) = -kx^2 + \max_{y \geq G(x)} \{ \ln(y - G(x)) + \beta W_k(y) \} \quad (3.5)
\]

With this notation, the solution for the \( N \) player game can be found from solving (3.5) for \( W_k \) and then

\[
\tilde{V}_N(x) = N \left( W_k(x) - \frac{\ln(Nh)}{1 - \beta} \right). \quad (3.6)
\]

It is in the form (3.5) that we implement the discrete state space dynamic programming problem.

3.1 Properties of Solutions

Before looking further at the computational issues, let us consider some of the abstract properties of the solution to the symmetric equilibrium dynamic game. First, the dynamic optimization corresponding to (3.5) can be written as:

\[
W_k(x_0) = \max_{\{x_t\}} \sum_{t=0}^{\infty} \beta^t \left( \ln(x_{t+1} - G(x_t)) - kx_t^2 \right). \quad (3.7)
\]

Now let \( F(x, y) = \ln(y - G(x)) - kx^2 \). Then

\[
\frac{\partial^2 F}{\partial x \partial y} = \frac{G'(x)}{(y - G(x))^2} > 0 \quad (3.8)
\]

and so by the result in Benhabib and Nishimura (1985), the optimal solution is monotone.\(^2\) The Euler equation for (3.7) is

\[
\frac{1}{x_t - G(x_{t-1})} - \beta \left( \frac{G'(x_t)}{x_{t+1} - G(x_t)} + 2kx_t \right) = 0 \quad (3.9)
\]

which can be written as a system (the Hamiltonian dynamics)

\[
x_{t+1} = G(x_t) + u_t \quad (3.10)
\]

\[
\frac{\beta^{-1}}{u_{t-1}} = \frac{G'(x_t)}{u_t} + 2kx_t
\]

\(^2\)i.e., the optimal solution satisfies: either \( x_{t+1} \geq x_t \) for all \( t \), or \( x_{t+1} \leq x_t \) for all \( t \).
where \( u = Nha_t \). Thus the steady states for this system are defined by the solution(s) to

\[
\begin{align*}
    u &= x - G(x) \\
    2kux &= \beta^{-1} - G'(x)
\end{align*}
\]  

(3.11)

It can be shown that there can be three solutions to these equations. When there are three solutions, the middle one is locally unstable and the other two are locally stable. It is also possible for the latter two solutions both to be \textit{optimal steady states}. In that case the argument in Dechert and Nishimura (1983) shows that there is a unique point, \( x_S \), which we call a \textit{Skiha point},\(^3\) with the property that if \( x_0 < x_S \), then the optimal path converges towards the lower steady state, and if \( x_0 > x_S \), the optimal path converges towards the upper steady state. In the case that \( x_0 = x_S \), there are two optimal solutions: one converging to the lower steady state and one converging to the upper steady state. At points of differentiability, the value function satisfies

\[
W'_k(x) = -2kx - \frac{G'(x)}{y - G(x)}
\]  

(3.12)

and since there are two solutions at \( x_S \), the value function is not differentiable at \( x_S \). The left and right derivatives are

\[
\begin{align*}
    W'_k(x_S^+) &= -2kx - \frac{G'(x_S)}{x_U - G(x_S)} \\
    W'_k(x_S^-) &= -2kx - \frac{G'(x_S)}{x_L - G(x_S)}
\end{align*}
\]

where \( x_L \) is the optimal solution (from \( x_S \)) that converges to the lower steady state, and \( x_U \) is the optimal solution (also from \( x_S \)) that converges to the upper steady state. Since \( x_L < x_U \)

\[
W'_k(x_S^+) > W'_k(x_S^-)
\]  

(3.13)

By the same argument in Dechert and Nishimura (1983), it can also be shown that the solution is unique except at \( x_S \) and so \( W_k \) is differentiable everywhere except at \( x_S \). From equation (3.12) it is the case that \( W'_k < 0 \) and so in (3.13) the left hand slope is steeper than the right hand slope. This is one of the characteristics that we will look for in the numerical solution of the dynamic programming problem.

Now let’s turn to the optimality of the lower steady state. If \( \beta \) is close to 0 (i.e., \( \rho \) is very large) then the optimal solution is to load the lake with a large

\(^3\)Skiha (1978) analyzed a continuous time growth model with a non-convex technology where this phenomenon occurred.
amount of phosphorus (i.e., the optimal value of \( u_0 \) is very large). For small \( \beta \) there will be only one steady state, and the lake will be oligotrophic.⁴

Consider the case that there are two (a non-generic case) or three solutions to equations (3.11). Let the one with the smallest \( x \) value be \((u_O, x_O)\) and the one with the largest, \((u_E, x_E)\).⁵ In order for this to happen (in the hysteretic region) the parameter \( b \) must satisfy

\[
\frac{1}{2} < b < \frac{3}{8}\sqrt{3}
\]

and \( \beta \) must not be too small. Since \( W_k \) is monotonically decreasing, if

\[
\log(u_E) - kx_E^2 \geq \log(u_O) - kx_O^2
\]

then the lower steady state is not optimal. For the parameter values of \( b = 0.52, c = 1, N = 4 \) and \( \rho = 0.003 \) it turns out \( x_E = 2.30 \) is an optimal steady state, and \( x_O = 0.31 \) is not.

### 3.2 Social Optimum

The social optimum for this problem is the solution to

\[
S(x_0) = \max_{\{a_{i,n}\}} \sum_{n=0}^{\infty} \beta^n \sum_{i=1}^{N} [\ln a_{i,n} - cx_n^2]
\]

subject to: \( x_{n+1} = x_n + hg(x_n) + h \sum_{i=1}^{N} a_{i,n} \) \hspace{1cm} (3.14)

The solution will exhibit symmetry, so we can write the corresponding dynamic programming problem as:

\[
S(x) = \max_a \{ N[\log(a) - cx^2] + \beta S(G(x) + Nh a) \}
\]

and by using the same method as before, let \( k = c \) in (3.5), and then

\[
S(x) = N \left( W_c(x) - \frac{\log(Nh)}{1 - \beta} \right)
\]

Notice that in equations (3.6) and (3.15) the only difference is the value of the parameter \( k \). Thus we can use the same program to calculate the social optimum as well as the solution to the dynamic game.

⁴See Brock and Starrett (April 1999) for a detailed analysis of the continuous time phase diagram and steady state properties. They also develop a new style of support argument for the zero discounting case.
⁵These correspond to the oligotrophic steady state and the eutrophic steady state, respectively.
⁶While this may appear to be an excessively small (annual) interest rate, due the rescaling of time in the original model it represents an actual annual interest rate of 0.021.
With the parameter values in the previous section, it can be shown that for the social optimum the upper steady state is \( x_E = 1.45 \) and that \( x_O = 0.28 \) is to the left of \( x_T \). So, for the social problem, \( x_O \) is an optimal steady state. This is one of the critical situations where if the initial value of \( x_0 \) is small, private and social interests differ dramatically. The dynamic game converges to a steady state with nine times the level of phosphorus as the social optimum. Numerical results on this case are presented in Section 4.1.

### 3.3 A Dynamic Game with Two Lakes

Our treatment has, so far, focused on analysis of the dynamics of optimal management of one lake ecosystem. This section sketches an analysis of coupled lake ecosystems for the special case of two lakes, one “downstream” from the other. Even this simple case generates some analytical challenge due to the coupling. We believe this sketch is enough to suggest that the analysis of optimally managed “spatially” ecosystems may not only present plenty of analytical challenges, but also some potentially surprising results. This seems to be a rich field for applications of bifurcation analysis.

We present here a sketch of the problem of optimally managing two lakes which are connected so that some of the nutrients loaded into the first lake can travel to the second lake by a connecting pathway. Our theoretical analysis is be in continuous time.\(^7\)

The dynamics of the lakes are similar to those of equation (1.1), except for the flow of \( \gamma P_1 \) from lake one to lake two:

\[
\dot{P}_1 = L_1 - \delta_1 P_1 + r_1 F_1(P_1) - \gamma P_1
\]

\[
\dot{P}_2 = L_2 - \delta_2 P_2 + r_2 F_2(P_2) + \gamma P_1
\]

where

\[
F_i(P) = \frac{P_{\text{crit}}}{m_{\text{crit}} + P_{\text{crit}}}
\]

Consider the social optimization problem

\[
\max \int_0^\infty e^{-\rho t} \left[ u_1(L_1) - c_1(P_1) + u_2(L_2) - c_2(P_2) \right] dt,
\]

subject to (3.16) and (3.17). Let \( H_i(P_i, q_i) \) for \( i = 1, 2 \) denote the current value Hamiltonian when \( \gamma = 0 \),

\(^7\)For analytical clarity, we will not convert the model to non-dimensional units in this section.
\[
H_i(P_i, q_i) = \max_L \left\{ u_i(L) - c_i(P_i) + q_i \left( L - \delta_i P_i + r_i F_i(P_i) \right) \right\}.
\]

Then, in general, when \( \gamma \neq 0 \), the first order necessary conditions of optimal control are

\[
\begin{align*}
\dot{q}_1 &= \rho q_1 - D_P H_1(P_1, q_1) + \gamma q_1 - \gamma q_2 \quad (3.19) \\
\dot{P}_1 &= D_q H_1(P_1, q_1) - \gamma P_1 \quad (3.20) \\
\dot{q}_2 &= \rho q_2 - D_P H_2(P_2, q_2) \quad (3.21) \\
\dot{P}_2 &= D_q H_2(P_2, q_2) + \gamma P_1 \quad (3.22)
\end{align*}
\]

The Jacobian matrix of equations (3.19) – (3.22) is

\[
J = \begin{bmatrix}
\rho + \gamma - D_{Pq} H_1 & -D_{PP} H_1 & -\gamma & 0 \\
D_{qq} H_1 & D_{qP} H_1 - \gamma & 0 & 0 \\
0 & 0 & \rho - D_{Pq} H_2 & -D_{PP} H_2 \\
0 & \gamma & D_{qq} H_2 & D_{qP} H_2
\end{bmatrix} \quad (3.23)
\]

Observe that when \( \gamma = 0 \) the Jacobian decomposes into two \( 2 \times 2 \) blocks, one block for each lake. Our analysis above applies to each of these blocks when \( \gamma = 0 \). We have a complete characterization of the optimal solution for each lake when \( \gamma = 0 \) under the same assumptions on the \( F_i(P) \) as in the previous section. That analysis suggests that we undertake a bifurcation analysis with respect to the parameter \( \gamma \), starting at \( \gamma = 0 \). Let \( q_i(P_i, \gamma) \) be the optimal costate values as functions\(^8\) of \( P_i \) and \( \gamma \) so that

\[
q(P, \gamma) = D_P V(P, \gamma)
\]

where \( V \) is the value function, and define \( H \) as

\[
H(P, q(P, \gamma), \gamma) = \begin{bmatrix}
H_1(P_1, q_1(P_1, \gamma)) - \gamma P_1 \\
H_2(P_2, q_2(P_2, \gamma)) + \gamma P_1
\end{bmatrix}
\]

\(^8\)Throughout this section we are sketching the outline of a mathematical argument rather than being precise about all of the details. The perceptive reader will have noted that for certain values of \( P_i \) and \( \gamma \) there may be more than one optimal value of the costate variable, \( q_i \).
Now consider the bifurcation of

$$\dot{P} = D_{q}H(P, q(P), \gamma) \quad . \quad (3.24)$$

At $\gamma = 0$, equations (3.24) decompose into two independent lakes analyzed in the previous sections. For $\gamma = 0$, locally stable steady states for (3.24) correspond to saddle points for the co-state, state equations for the individual lakes.

Here we sketch an analysis how to locate sufficient conditions for a Hopf bifurcation in (3.24) or, equivalently, in the $4 \times 4$ co-state - state system (3.19) and (3.22). This will be done by writing out the 4th degree characteristic polynomial, $p_{4}(\lambda)$, and exploiting the fact that if $\lambda$ is a root, so also is $\rho - \lambda$.

Let $J_{1}$ and $J_{2}$ denote the $2 \times 2$ diagonal blocks of $J$ and let $J_{ij}$ denote the individual elements of $J$. The characteristic polynomial for $J$ is

$$p(\lambda) = \lambda^{4} - 2\rho\lambda^{3} + (\rho^{2} + |J_{1}| + |J_{2}|)^{2} - \rho(|J_{1}| + |J_{2}|)\lambda + |J| \quad (3.25)$$

where $|\cdot|$ denotes the matrix determinant,

$$|J| = |J_{1}| |J_{2}| + \gamma J_{21}J_{34}$$

and

$$|J_{1}(\gamma)| = (\rho + \gamma - D_{p_{q}H_{1}})(D_{q_{p}H_{1}} - \gamma) + D_{p_{q}H_{1}}D_{q_{q}H_{1}}$$

$$|J_{2}| = (\rho - D_{p_{q}H_{2}})D_{q_{p}H_{1}}H_{2} + D_{p_{p}H_{2}}D_{q_{q}H_{2}}$$

$$J_{21} = D_{q_{q}H_{1}}$$

$$J_{34} = -D_{p_{p}H_{2}}$$

Define $\mu$ by

$$\mu = \lambda - \frac{\rho}{2} \quad (3.26)$$

and recall that roots of $p(\lambda)$ come in “Poincaré Pairs” $(\lambda, \rho - \lambda)$. What this implies in terms of the transformation (3.26) is that if $\mu$ is a root of the polynomial, then $-\mu$ is also a root of the polynomial. It turns out that it is easy to factor the polynomial $p(\rho/2 + \mu)$ so that it can be written in the form of a

Note that $J_{1}$ depends on $\gamma$, while $J_{2}$ does not. When we wish to emphasize this fact, we will write $J_{3}(\gamma)$.
polynomial in \( \mu^2 \). First, let \( T = |J_1| + |J_2| \). Then the polynomial (3.25) can be written as:

\[
p(\lambda) = \lambda^4 - 2\rho\lambda^3 + (T + \rho^2)\lambda^2 - \rho T \lambda + |J|
\]

or in terms of \( \mu \),

\[
p(\mu + \rho/2) = (\mu + \rho/2)^4 - 2\rho(\mu + \rho/2)^3 + (T + \rho^2)(\mu + \rho/2)^2 - \rho T (\mu + \rho/2) + |J|
\]

\[
= (\mu + \rho/2)^4 - 2\rho(\mu + \rho/2)^3 + T(\mu + \rho/2)^2 + \rho^2(\mu + \rho/2)^2 + \rho T (\mu + \rho/2) + |J|
\]

\[
= (\mu + \rho/2)^2 [(\mu + \rho/2)^2 - 2\rho(\mu + \rho/2) + \rho]\n
+ T(\mu^2 - \rho^2/4) + |J|
\]

\[
= (\mu + \rho/2)^2 [\mu^2 - \mu \rho + \rho^2/4] + T(\mu^2 - \rho^2/4) + |J|
\]

\[
= (\mu + \rho/2)^2 (\mu - \rho/2)^2 + T(\mu^2 - \rho^2/4) + |J|
\]

\[
= (\mu^2 - \rho^2/4) + T(\mu^2 - \rho^2/4) + |J|
\]

By the quadratic formula,

\[
\mu = \mu \pm \sqrt{\mu^2 - \rho^2/4}
\]

or

\[
\mu = \pm \sqrt{\frac{\rho^2}{4} - \frac{T \pm \sqrt{T^2 - 4|J|}}{2}}
\]

and in terms of \( \lambda \),

\[
\lambda = \frac{\rho}{2} \pm \sqrt{\frac{\rho^2}{4} - \frac{T \pm \sqrt{T^2 - 4|J|}}{2}}
\]  

(3.27)

This is known as Dockner’s Formula.\(^{10}\) For \( \gamma = 0 \), at any locally stable optimal steady state for lake \( i \), we have \( |J_i| < 0 \) because both roots are real and one

\(^{10}\)Dockner (1985). We have shown the derivation here for our case since the Jacobian matrix has a special structure.
is negative, one is positive. Since this corresponds to two uncoupled lakes, equation (3.27) simplifies to

\[ \lambda = \frac{\rho}{2} \pm \sqrt{\frac{\rho^2}{4} - |J_1(0)|}, \quad \frac{\rho}{2} \pm \sqrt{\frac{\rho^2}{4} - |J_2|} \]

For any value of \( \gamma \geq 0 \), the Poincaré–Bendixon theorem tells us that the \( 2 \times 2 \) reduced form optimal dynamical system (3.24) has severely limited dynamic behavior: generically, starting from any initial condition \( P(0), P(t) \) must (i) converge to a steady state, (ii) converge to a limit cycle, or (iii) become unbounded. Since (iii) is ruled out by the ecological dynamics, only (i) or (ii) is possible. In order for a limit cycle to appear as \( \gamma \) increases, system (3.24) must pass through a Hopf bifurcation. But this requires that a pair of roots of \( J \) pass from the left half of the complex plane to the right half, i.e. the real parts of that pair must pass through zero.

In the case of linear lakes \( (r_1, r_2 = 0) \) or where the dynamics are concave, we may obtain convergence theorems for quite general cases for “small \( \rho \)”. Suppose there are \( N \) lakes which are connected by \( P \)-flows amongst them. Let the dynamics be represented by the system

\[ \dot{P} = F(P) + L \]

where \( P \) is an \( N \)-vector with \( P_i \) denoting \( P \) in lake \( i \), \( F \) maps \( N \) dimensional real space into itself, and \( L \) is an \( N \)-vector of loadings. Let \( U(L, P) \) denote the joint utility function of the community. The “Q” condition of Brock and Scheinkman (1976) can be applied to show that for \( \rho \) small enough we have global asymptotic stability of the socially optimal management path independently of the ecosystem dynamics \( \dot{P} = F(P) \), provided that \( F(P) \) is linear in \( P \). A structural stability argument could then be used for dynamics of the form \( \dot{P} = F(P) + \epsilon G(P) + L \) where \( G \) is convex and \( \epsilon \) is sufficiently small. See also (Brock and Scheinkman 1977).

This gives us a neat separation of the “economics” and the “ecodynamics”. The sufficient conditions are especially transparent in the case where \( U \) is linear–quadratic and separable across lakes and separable across the loading interests and the recreational interests.

4 Computational Solutions

In this section we discuss the numerical methods used to solve the dynamic programming problem (3.5) for the one lake dynamic game. Let \( \Delta \) be the largest value of \( x \) for which we solve the dynamic programming problem. Pick \( \{\xi_i\} \) so that

\[ 0 = \xi_0 < \xi_1 < \cdots < \xi_m = \Delta . \]
Introduce the notation \( w(i) = W_k(\xi_i) \) and the dynamic programming problem is

\[
w(i) = -k\xi_i^2 + \max_{\xi_j \geq G(\xi_i)} \left\{ \ln(\xi_j - G(\xi_i)) + \beta w(j) \right\}
\]

or if we let

\[
R_{i,j} = \ln(\xi_j - G(\xi_i)) - k\xi_i^2
\]

then we can rewrite the discrete state space dynamic programming problem as:

\[
w(i) = \max_{j \leq j \leq m} \left\{ R_{i,j} + \beta w(j) \right\}
\]

where \( j_0(i) \) is the smallest integer, \( j \), such that \( \xi_j > G(\xi_i) \). There are several algorithms\(^{11}\) to choose from to solve (4.1). Each of the main methods (value function iteration, policy function iteration and linear programming) has substantial drawbacks for this problem. However, there's a very neat way to solve for the value function when the problem has one or more fixed points and the solution exhibits monotonicity. Here's a synopsis of this method: let \( \xi_{i_0} \) be an optimal steady state. Then

\[
w(i_0) = \frac{\log(\xi_{i_0} - G(\xi_{i_0}))}{1 - \beta}
\]

\[
h(i_0) = i_0
\]

where \( h() \) is the optimal (discrete state space) policy function. So we have solved the problem for \( i = i_0 \). Now consider \( i = i_0 + 1 \). By the monotonicity of optimal solutions, the optimality of the steady state, and the fact that this is a discrete dynamic programming problem, \( h(i) = i \) or \( h(i) = i_0 \) are the only two possibilities. So we do not have to search the entire set of states in (4.1). In general, if we have solved the problem for \( i_0, i_0 + 1, \ldots, i \), then for the solution at \( i + 1 \) we need only search the set of states \( j \) such that \( h(i) \leq j \leq i + 1 \). The monotonicity property of optimal paths implies that it would never be optimal to pick a state \( j < h(i) \). With this technique, we can solve the problem for \( i = i_0, i_1, \ldots, m \). Following a similar scheme, we also can solve the problem for \( i = i_0 - 1, i_0 - 2, \ldots, 0 \). This algorithm is implemented in section 4.1.

How do we handle multiple steady states? Take the case of two steady states, as in this paper, and say that they are \( i_0 \) and \( i_1 \). Solve for the value (as above) as though \( i_0 \) is an optimal steady state, and call the value function \( w_0 \). Similarly solve for the value (also as above) as though \( i_1 \) is an optimal steady state, and call the value function \( w_1 \). Then the solution is

\(^{11}\) See Judd (1998) for a discussion of these techniques.
\[ w(i) = \max\{w_0(i), w_1(i)\}. \]

For more of a discussion of this method and other examples see Judd (1998, Chapter 12).

As a note on computational methods, solving (4.1) is an \(O(m^2)\) algorithm. How about the technique here? Note that it depends on \(h(i)\), and we can compute the number of trial evaluations that are made by

\[
\phi(m) = \sum_{i=0}^{i=i_0-1} (h(i + 1) - i + 1) + 1 + \sum_{i=i_0+1}^{i=m} (i - h(i - 1) + 1).
\]

For the parameter values in Section 3.1, we ran the program for a range of values of \(m\) from 1024 to 131072. For this problem (and within the range of \(m\) for which \(\phi(m)\) was computed) the algorithm is approximately \(O(m^{1.6})\). For large \(m\), say \(m = 50,000\),

\[
\frac{m^{1.6}}{m^2} = m^{-0.4} = 0.013
\]

which represents a large gain in computational speed.

### 4.1 Numerical Solutions

For the parameter values above,

\[
\begin{align*}
    b &= 0.52 \\
    c &= 1.0 \\
    h &= 0.05 \\
    \rho &= 0.003
\end{align*}
\]

We have plotted the optimal loading, \(N_{\tau_i}\), for \(N = 2, 3\) and \(4\) communities as well as the socially optimal loading. The results are in Figures 4.1 – 3, where the social optimum loading is always less than the total loading in the dynamic game. As a reference, the curve where \(x_{t+1} = x_t\) has also been plotted. When the total loading is below this curve the level of phosphorus is falling \((x_{t+1} < x_t)\), while above the curve it is rising.

Note that the optimal loading when there are two communities is very close to the social optimum level, especially when \(x \leq 1\). With three communities we see the presence of the Skiba point. For levels of phosphorus below the Skiba point, the dynamic game and the social optimum are close, while there is a large disparity for values above the Skiba point. Finally, when there are four communities, the divergence between the dynamic game solution and the social optimum becomes extreme.
In general, there are four different “phases” that the system can be in with respect to the parameters $\beta, b, k$. They are

(i) LSS: only the lower of the two steady states is an optimal steady state;

(ii) USS: only the upper of the two steady states is an optimal steady state;

(iii) SSS: there is only one steady state, which is therefore the optimal steady state; and

(iv) Skiba: both steady states are optimal steady states, and there is a Skiba point in between them.

In Figure 4.1 we have plotted the phase diagram in $b - k$ space, with $\beta$ fixed at $1/(1.003)$. From the figure, some general principles are evident. For a given value of $b > 0.5$, as $k$ increases, the optimal solution moves towards the lower (oligotrophic) steady state. There is no similar phenomenon for fixed $k$ as $b$ increases. Not that the boundaries between phases SSS-USS and USS-Skiba are at positive slope throughout. However, the boundaries between SSS-LSS and LSS-Skiba are “U” shaped. So, for example at a value of $k = 0.4$, as $b$ increases from 0.5 to 0.8 the system passes through the phases Skiba-LSS-Skiba-USS-SSS, passing through the Skiba phase at two different ranges of values of $b$. At a value of $k = 0.52$ the system passes through the phases Skiba-LSS-SSS-Skiba-USS-SSS, passing through both Skiba and SSS for different ranges of values of $b$. This phenomenon is one of the properties of this system that are brought out by the numerical solutions; they cannot be obtained from theoretical considerations.\footnote{At least we know of no way in general to analytically derive the boundaries of the phase portrait.}
References


Figure 1: Social Optimum Loading

Figure 2: 2 Community Loading
Figure 3: 3 Community Loading

Figure 4: 4 Community Loading
Figure 5: Phase Plot for $\bar{\rho} = 0.003$