‘Know Thy Enemies’: Knowledge of Rivals’ Types and Its Effect on Auctions*

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Abstract

We study auctions in which bidders may know the types of some rival bidders but not others. This asymmetry in bidders’ knowledge about rivals’ types has different effects on the two standard auction formats. In a second-price auction, it is weakly dominant to bid one’s valuation, so the knowledge of rivals’ types has no effect, and the good is allocated efficiently. In a first-price auction, bidders refine their bidding strategies based on their knowledge of rivals’ types, which yields an inefficient allocation. We show that the inefficient allocation in the first-price auction translates into a poor revenue performance. Given a standard regularity condition, the seller earns higher expected revenue from the second-price auction than from the first-price auction, whereas the bidders are better off from the latter.

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1 Introduction

Much of the existing auction literature assumes that bidders possess symmetric knowledge regarding rival bidders' types. For instance, the standard independent private values (IPV) models assume that bidders are ignorant of the rival bidders' types, knowing only their distributions (e.g., Riley and Samuelson (1981) and Myerson (1981)). In a Bertrand model, bidders are assumed to know the realized types of all opponents. The models with affiliation allow bidders to refine their assessment of the rival bidders' types based on their signals, but treatment of a given bidder's knowledge about other bidders is often symmetric (see Milgrom and Weber (1982)).

In practice, bidders' knowledge about their rivals is unlikely to be symmetric. Bidders tend to know more about some rival bidders than others. For instance, in a procurement auction competed by domestic firms and foreign firms, domestic firms are likely to know more about the technical capabilities of their domestic rivals than foreign rivals. In auctions for government assets, such as mineral, timber harvesting, and the spectrum rights, bidders often consist of incumbent firms with long history of operation in the industry and relative newcomers in the area. It is then presumably easier for a bidder to estimate the preferences and technological abilities of the old firms than those of the new ones. A similar distinction may exist with respect to the institutional buyers and noninstitutional buyers in art auctions as well as treasury auctions.

We develop a model that accommodates such an asymmetry in the bidders' knowledge of their rivals' types. Specifically, our model considers, in a symmetric IPV setting, an arbitrary partition of bidders into "knowledge groups" such that bidders know the valuations of their rivals within the same group, while they know only the distribution of types for bidders outside their group. This model offers a simple way of describing the aforementioned asymmetry in a bidder's knowledge of his rivals. Aside from the dichotomous nature of a bidder's information about his rivals, our model is general. In particular, our model allows for an arbitrary partition structure, which includes the two standard assumptions — full information and no information about other bidders — as special cases.

We study how the asymmetric knowledge of a bidder's rivals affects the performance of standard auctions. Toward this goal, we first study existence of an equilibrium in the standard auctions and characterize their properties. While the standard weak dominance argument continues to work for the second-price auction (which is equivalent to an ascending-bid auction in the IPV setting), the standard equilibrium analysis for a first-price auction does not follow, due to two features of our model: (1) an equilibrium bidding strategy depends not just on one's own valuation but also on those of his group members; and (2) given the arbitrary partition structure, the auction becomes generally asymmetric since the group sizes may be different. We adapt Reny (1999) to establish existence of an equilibrium for a first-price auction. Among the properties found is that the highest valuation bidder in a group (henceforth called "a group leader") never loses to lower valuation bidders in the same group, making the competition effectively among the group leaders.

Second, we compare first- and second-price (or equivalently, ascending-bid) auctions in terms of allocative efficiency and expected revenue. We find that the second-price auction dominates the first-price auction on both accounts. The efficiency dominance of the second-price auction can be explained as follows. In a second-price auction (or equivalently ascending bid auctions), it is a weakly dominant strategy to bid one's own valuation, regardless of the knowledge partition. Hence, absent a reserve price, the second-price auction allocates the good efficiently. By contrast, an equilibrium bidding strategy in a first-price auction depends on a bidder's knowledge of his rivals' valuations, since the latter affects the optimal degree to which a given bidder shades his bid. Since the realized types of rival bidders vary across groups, a bidder of the same valuation but facing different valuations of the within-group rivals will bid differently, which results in inefficient allocation of the good, even when the valuations of the goods are symmetrically distributed.

Given the allocational inefficiency arising from first-price auctions, the revenue equivalence theorem would normally imply that that a second-price auction revenue-dominates a first-price auction. Our nonstandard informational assumption, however, makes the revenue equivalence theorem less than immediate. In particular, in a first-price auction, a bidder's winning probability may not depend solely on his bidding strategy, but it may also depend on his valuation. For instance, as a bidder's valuation rises, this may lead his within-group rival bidders to readjust their bids. Hence, even though the former bidder does not change his bid, his winning probability may change. This latter possibility makes the standard envelope theorem — a key step in applying the revenue equivalence theorem — inapplicable in our setting. Our main analysis consists in reestablishing this step to show that the aforementioned revenue ranking holds in our model.

To our knowledge, the current paper is the first to study how bidders' knowledge of their rivals' types — in particular, its asymmetry — matters in auctions. A similar feature is present in an auction with interdependent valuations since a bidder's signal contains information about other bidders' valuations (see Milgrom and Weber (1982), Maskin (1992), Ausubel (1997), Esö and Maskin (2000), Dasgupta and Maskin (2000), Perry and Reny (2001), and Jehiel and Moldovanu (2000)). Despite the similarity, there are a several important distinctions between this literature and the current paper. First, the extent to which a bidder's signal impacts on others' valuations (hence, a bidder's knowledge of his rivals) is symmetric in some of these models (see Milgrom and Weber (1982) and Ausubel (1997), for instance). Second, a bidder's signal is single dimensional in these models, so his signal affects both his valuation and others'.[1] In our model, a bidder observes multidimensional signals, one for his valuation and others for other bidders' valuations. This distinction matters since a single dimensional signal often leads to an efficient allocation in a symmetric first-price auction (see Milgrom and Weber (1982)), whereas an inefficient allocation arises in our model with multidimensional signals.[2] Finally, these papers are purely concerned about allocative efficiency, whereas we study both efficiency and revenue. The current paper's focus on the private-value setting separates the issue of "knowledge of rivals' types" as a distinct problem of its own, which is relevant in many auction environments, and also yields an unambiguous revenue comparison of the standard auctions. In this respect, the current paper complements that literature.

The rest of the paper is organized as follows. In the next section, we set up the model. Section 3 then establishes existence of an equilibrium and characterize its properties. Section 4 then compares the two auction forms. Section 5 concludes.

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[1] In fact, in a typical model of interdependent valuations a signal affect's one own valuation more than others (see the single crossing property assumed in Maskin (1992), Esö and Maskin (2000), Dasgupta and Maskin (2000), Perry and Reny (2001)). In our model, a bidder observes a signal on his own valuation and a signal on others, separately.

[2] See Jehiel and Moldovanu (2000) for the difficulty multidimensional signals pose for general mechanism design.

2 Model

A seller has a single indivisible good to sell to $n \geq 3$ risk neutral bidders. The seller is assumed to put no value on the good. Bidders have independent private values. Specifically, bidder i's valuation v_i is drawn from the interval $[0, 1]$, following a common distribution function $F(\cdot)$ whose density function $f(\cdot)$ is bounded away from zero. F is assumed to be common knowledge. Letting N denote the set of all bidders, the valuation profile $\boldsymbol{v} := (v_i)_{i \in N}$ of the bidders has the joint density $f_N(\boldsymbol{v}) := \prod_{i \in N} f(v_i)$.[3]

We depart from standard IPV models (represented by Myerson (1981) and Riley and Samuelson (1981)) by allowing some bidders to know about each other's valuation. To express this idea formally, we impose a partition structure $\mathcal{G}$ on the set N, where $\mathcal{G}$ is the set of disjoint groups (or disjoint sets) with each group consisting of bidders whose realized valuations are common knowledge among themselves. Also, the partition structure is assumed to be common knowledge. Hereafter, $G \in \mathcal{G}$ denotes a group. Then, letting $\boldsymbol{v}_G := (v_i)_{i \in G}$ denote the vector of the valuations of bidders in the group G, $\boldsymbol{v}_G$ is assumed to be common knowledge among bidders in G. Note that *every realization* of $\boldsymbol{v}_G$ is commonly known to bidders in G while only the *distribution* of $\boldsymbol{v}_G$ is known to bidders in the other groups. A bidder is referred to as a *group leader* if he has the highest valuation in the group. For later use, we let $\boldsymbol{v}_{G/i} := (v_j)_{j \in G/i} \in \boldsymbol{V}_{G/i}$ and $\boldsymbol{v}_{-G} := (v_j)_{j \in N/G} \in \boldsymbol{V}_{-G}$ denote the profile of valuations for group G except for bidder i and the valuation profile for all bidders except for group G, respectively. The joint density for the vector $\boldsymbol{v}_G$ is denoted by

$$f_G(\boldsymbol{v}_G) := \prod_{i \in G} f(v_i),$$

and the joint densities for the vector $\boldsymbol{v}_{G/i}$ and $\boldsymbol{v}_{-G}$ are similarly denoted by $f_{G/i}$ and f_{-G}, respectively.

Throughout, we consider two standard auctions, first-price and second-price auction (which is equivalent to an ascending-bid auction), with no reserve price. The assumption of no reserve price is made purely for simplicity, and all subsequent results will continue to hold even when a binding reserve price is introduced. In our model, bidders of the same group are informed of their types as common knowledge. Clearly,

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[3] Throughout the paper, bold face letters are used to denote vectors.

the seller could elicit such information for free through cross reporting and then use it to extract the entire surplus from the bidders, while implementing any allocation. In this sense, the standard auctions are not optimal at least from the revenue perspective. Nevertheless, we focus on the standard auctions, for several reasons: First, these auctions are most frequently used, so our study will add to the positive analysis on the subject matter — i.e., the knowledge of rivals' types. Second, as will be seen later, focusing on the two formats entails no loss if the goal of mechanism design is allocative efficiency.[4] Third, our model is intended to serve as a metaphor for a general scenario in which bidders have asymmetric information about their rivals' types. Our positive analysis can yield insights of broad applicability beyond the particular model studied (while the cross-reporting mechanism appears to require the particular partition model).

As is well known, in a second-price auction, all bidders bid simultaneously, and the highest bidder wins the good and pays for the highest losing bid. Ties are broken arbitrarily. Given our IPV assumption, the ascending-bid auction (more precisely, the Japanese button auction) is strategically equivalent to the second-price auction (albeit different in the extensive form), and either format is simply referred to as a second-price auction throughout the paper.

In a first-price auction, all bidders bid simultaneously, and the highest bidder wins and pays his bid. Unlike a second-price auction, a tie-breaking rule is important in guaranteeing existence of an equilibrium in this format. For instance, in a standard Bertrand game played by two firms with heterogeneous costs, a Nash equilibrium exists only when the ties are broken in favor of the lower-cost firm. For the same reason, we assume that (1) a tie is broken in favor of a bidder with a higher valuation if there are multiple highest bidders; and that (2) if there are multiple highest bidders with the same valuation, then the object is assigned randomly with equal probability among those bidders. While this tie-breaking rule is endogenous, it can be implemented by performing an auxiliary second-price auction with bidders who submitted the highest bid in first-price auction.[5] In both games, we look for a Nash

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[4] As the literature on interdependent valuations points out, the main objective of many government auctions has been achieving allocative efficient, and not revenue maximization.

[5] See Maskin and Riley (2000) for a similar assumption about tie breaking. Further, our tie-breaking rule can be justified as producing a limiting equilibrium of a game in which bidders must bid in a discrete space and a random tie-breaking rule is used.

equilibrium in weakly undominated strategies.

Our model includes as special cases two extreme partition structures. In the one case, every set in the partition is a singleton, so every bidder knows only his valuation. This is the standard assumption made in the auction literature. The other case has one grand set in the partition, which means that bidders know all other bidders' types. The resulting game is precisely the Bertrand game. As is well known, revenue equivalence holds for these two partition structures.

Proposition 0 *If* $|G| = 1$ *for all* $G \in \mathcal{G}$ *or there is a* $G \in \mathcal{G}$ *such that* $|G| = n$*, then first-price and second-price auctions are revenue equivalent.*[6]

Proof. For the former case, the result follows from Myerson (1981) or Riley and Samuelson (1981). For the latter, it is a unique equilibrium for a group leader to bid the second highest valuation, which results in the same outcome as the weakly dominant equilibrium of the second-price auction does. ▌

To focus on more interesting cases, we impose the following assumption in the sequel.

Assumption 1 *There is a group* G *such that* $1 < |G| < n$*.*

Apart from this assumption, we do not impose any restriction on the partition structure. Naturally, different groups may contain different numbers of bidders, so the partition may be asymmetric. For instance, if there are four bidders with $N = \{1, 2, 3, 4\}$, our model encompasses three different possibilities: two groups of two (e.g., $\{\{1, 2\}, \{3, 4\}\}$), one group of three and one group of one (e.g., $\{\{1, 2, 3\}, \{4\}\}$), two groups of one and one group of two (e.g., $\{\{1\}, \{2\}, \{3, 4\}\}$), while Proposition 0 covers the two special cases: $\{\{1, 2, 3, 4\}\}$ and $\{\{1\}, \{2\}, \{3\}, \{4\}\}$.

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[6] Throughout, $|\cdot|$ denotes the cardinality of a set.

3 Equilibrium Characterizations and Efficiency Comparison

3.1 Second-Price Auction

The well-known argument from Vickrey (1961) works in this game, regardless of the partition structure: Each bidder has a weakly dominant strategy of bidding his valuation. The optimality of this strategy does not depend on rivals' bids, so the knowledge of their types, and hence the partition structure, does not affect the equilibrium behavior. It follows that the good is allocated efficiently in every partition structure. This result will be used for subsequent comparison with the first-price auction.

The equilibrium for first-price auction is not as immediate. Hence, the remainder of this section is devoted to that case.

3.2 First-Price Auction

We first establish several necessary conditions for an equilibrium, given any arbitrary knowledge partition. These conditions will constitute partial characterizations of equilibrium, which will be used for establishing existence of the equilibrium and for comparison with the second-price auction.

To this end, fix an equilibrium (whose existence will be shown later). Fix any bidder $i \in G$ and let $m(i) \in \arg\max_{j \in G/i} v_j$ be the highest valuation bidder, excluding bidder i, in the same group, and let $v_{m(i)}$ be his valuation. Let $b_i(\boldsymbol{v}_G)$ and $\underline{b}_i$ respectively denote an arbitrary selection of bidder i's equilibrium bids and their infimum, given the valuation profile, $\boldsymbol{v}_G$, of group G. Since we restrict the equilibrium strategies to be undominated, we must have $b_i(\boldsymbol{v}_G) \leq v_i$. The following lemma shows that $b_i(\boldsymbol{v}_G) \geq v_{m(i)}$ whenever $v_i > v_{m(i)}$.

Lemma 1 *If $v_i > v_{m(i)}$, then, in any equilibrium of the first-price auction, $\underline{b}_i \geq v_{m(i)}$ and i beats all bidders in G.*

Proof. See the Appendix.

This lemma implies that allocation is efficient within each group in any equilibrium. It also implies that competition is effectively among the group leaders, so the attention can be restricted to group leaders when we search for an equilibrium. The next lemma shows that the equilibrium distributions of their bids have no mass points.[7] It refers to $y_G(b)$, which denotes the probability of outbidding all bidders outside a group G with a bid of b in an arbitrary equilibrium.

Lemma 2 *In any equilibrium of a first-price auction, $y_G(b)$ is continuous in b for every $G \in \mathcal{G}$.*

Proof. See the Appendix.

We next show that any equilibrium in a first-price auction is essentially pure.

Lemma 3 *For each $G \in \mathcal{G}$, the equilibrium bid, $b_i(\boldsymbol{v}_G)$, is unique for almost every $\boldsymbol{v}_G$ such that $v_i > v_{m(i)}$.*

Proof. See the Appendix.

We are now in a position to address the existence issue. Two features of the model make the standard existence result inapplicable in our setting. First, each bidder observes the entire profile of valuations of his group members, so this creates a multi-dimensional signal problem. Second, since we assume an arbitrary knowledge partition, the environment is generally asymmetric. Our proof builds on the existence result of Reny (1999). We will sketch the proof, whose detailed version is contained in the appendix.

Proposition 1 *There exists a pure-strategy equilibrium in which each bidder, say i, employs a bidding strategy, $b_i(\boldsymbol{v}_G)$ that is nondecreasing and takes a value in $[v_{m(i)}, v_i]$ when $v_i \geq v_{m(i)}$, or else $b_i(\boldsymbol{v}_G) = v_i$.*

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[7] While this result would be standard in the auction literature, our informational structure makes it nontrivial and warrants a separate proof. The standard proof is based on the argument along the following line: if a bidder puts mass on a bid b, then there exists an interval $(b-\epsilon, b)$ to which everyone else assigns probability 0, which then proves a profitable deviation from b (see Fudenberg and Tirole (1991) pp.223-225). This argument does not work in our model since, by Lemma 1, a group leader is constrained by the second-highest valuation in his group, so bids will be placed on every interval with some probability. Our proof basically amounts to showing that if a positive mass is put on b by a group leader, the opponent group leaders will submit bids between $(b-\epsilon, b)$ with such a small probability that it pays the former leader to move the mass point.

Proof Sketch: [Formal proof is collected in the Appendix.] We consider a hypothetical first-price auction game in which there is one player representing each group. Specifically, for each group $I \in \mathcal{G}$ in the original game, we assign a single player with a signal, $\boldsymbol{v}_I = (v_i)_{i \in I} \in [0,1]^{|I|}$. That player realizes the highest valuation of $\boldsymbol{v}_I$, upon winning the game (and zero when losing) and chooses a bid $\beta_I(\boldsymbol{v}_I) \in [v_I^2, v_I^1]$, where v_I^r denotes the r^{th} order statistics of vector $\boldsymbol{v}_I$, and $v_I^2 := 0$ if $|I| = 1$.

We then closely follow the arguments of Reny (1999) to prove that the hypothetical game has a pure-strategy equilibrium, $\boldsymbol{\beta}^*$, in which each (hypothetical) player is choosing a nondecreasing bid function $\beta_I^*(\boldsymbol{v}_I) \in [v_I^2, v_I^1]$.

Given the equilibrium, $\boldsymbol{\beta}^*$, of the hypothetical game, we can construct the equilibrium of the original game as follows: Each bidder $i \in I$ bids $\min\{v_i, \beta_I^*(\boldsymbol{v}_I)\}$ for a value realization $\boldsymbol{v}_I$ of the group I. To see that this forms an equilibrium of the original game, suppose first that i is not a group leader. Then, given our endogenous tie-breaking rule and $\beta_I^*(\boldsymbol{v}_I) \geq v_i$, bidder i has no incentive to deviate. If i is a group leader, his equilibrium bid is $\beta_I^*(\boldsymbol{v}_I) \leq v_i$. Given the behavior of other bidders in his group, bidder i cannot benefit from bidding below v_I^2. Given this constraint, bidding $\beta_I^*(\boldsymbol{v}_I)$ is optimal since β_I^* is an equilibrium strategy in the hypothetical game. ▌

Given the private value specification, a group leader's equilibrium bid is likely to depend only on his valuation and the second highest valuation in the group. In fact, it is likely that there exists an unconstrained bidding strategy, $B_I(v)$, for group I such that a leader of group I bids $\max\{B_I(v_I^1), v_I^2\}$, given the first and second-highest valuations, v_I^1 and v_I^2, respectively. The following example illustrates this pattern.

Example 1 *Suppose that there are four bidders in two equal-sized groups:* $\mathcal{G} = \{\{1,2\},\{3,4\}\}$. *Suppose also that each bidder draws his valuation uniformly from* $[0,1]$. *It is then a (symmetric) equilibrium for each bidder to bid* $\min\{v_i, \max\{\frac{2}{3}v_i, v_{m(i)}\}\}$ *when his valuation is* v_i *and that of the other bidder in the group is* $v_{m(i)}$. *In this equilibrium, a group leader with valuation* v_i *adopts an unconstrained bid* $B(v_i) = \frac{2}{3}v_i$, *unless it is less than the valuation of the other bidder in his group.*[8] *That is, a leader behaves much as in a standard first-price auction (i.e., singleton partitions), except that he behaves like a Bertrand player against his within-group rivals. Note also that the unconstrained bid,* $B(v_i) = \frac{2}{3}v_i$*, adopted by the group leader is the same as the equilibrium bid that would be employed in a standard first-price auction game (i.e., singleton knowledge partitions) with* ***only three*** *players. Essentially, the group leader acts as if he faces no competition from the lower valuation bidder in his group as long as* $\frac{2}{3}v_i \geq v_{m(i)}$*. Clearly this will reduce the competition, all else equal. On the other hand, whenever the* $\frac{2}{3}v_i < v_{m(i)}$*, the second highest valuation acts as a constraint, thus raising the intensity of competition. How these two conflicting effects will affect the revenue will be the subject of the next section.*

3.3 Efficiency Comparison of Auctions

We are now in a position to study the allocational features of the first-price auction. Given the properties established above, it is not difficult to see the possibility that the first-price auction may entail an inefficient allocation. As is usually the case with this format, a bidder shades his bid to optimally balance the incremental profit and the incremental reduction in the winning probability, associated with raising his bid. As is clear from Lemma 1, the extent of shading is bounded by the second highest valuation of his group. Hence, a group leader facing within-group rivals with higher valuations will bid more aggressively than a group leader facing ones with lower valuations, even though the former leader has a lower valuation than the latter leader.

This point can be illustrated via Example 1. In that example, bidders' valuations

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[8] That this is an equilibrium can be seen as follows. Since two groups are symmetric,

$$\begin{aligned} y_G(b) &= \text{Prob}\{\boldsymbol{v}_G |\ \max\{B(v_i), v_{m(i)}\} \leq b, \text{ where } i = \arg\max_{j \in G} v_j\} \\ &= \text{Prob}\{\boldsymbol{v}_G |\ v_i \leq B^{-1}(b) \text{ and } v_{m(i)} \leq b, \text{ where } i = \arg\max_{j \in G} v_j\} \\ &= 2F(3b/2)F(b) - F^2(b) = 2b\max\{1, 3b/2\} - b^2. \end{aligned}$$

A group leader with valuation v_i maximizes $y_G(b)(v_i - b)$ subject to the constraint $b \geq v_{m(i)}$. It can be easily verified that with $y_G(b)$ given above, $y_G(b)(v_i - b)$ is increasing with b for $b < \frac{2}{3}v_i$ and decreasing for $b > \frac{2}{3}v_i$, which implies $\max\{\frac{2}{3}v_i, v_{m(i)}\}$ is indeed an equilibrium.

satisfy

$$v_1 > v_3 > v_4 > \frac{2}{3}v_1 > \max\{\frac{2}{3}v_3, v_2\},$$

with positive probability. Given such realizations of valuations, bidders 1 and 3 are group leaders who bid $\frac{2}{3}v_1$ and v_4, respectively, in equilibrium. Since $\frac{2}{3}v_1 < v_4$, bidder 3 then wins the object, which is inefficient since $v_1 > v_3$. As explained in the above paragraph, a low v_2 allows bidder 1 to shade more than bidder 3, who is constrained by a high v_4, so the latter outbids the former. This observation is generalized in the following proposition.

Proposition 2 *Given Assumption 1, a first-price auction allocates the good inefficiently with positive probability in any equilibrium.*

Proof. Recall that $y_G(b)$ denotes the probability of outbidding all bidders outside a group G with a bid of b in a first-price auction. By Lemma 2, $y_G(b)$ is continuous. Hence, there must be an interval $[b_-, b_+]$ where, for some G, $y_G(b)$ is strictly increasing. Otherwise, the continuity of $y_G(b)$ implies every bidder is putting the entire mass on a single bid, which clearly cannot hold in an equilibrium.

Efficiency requires that for almost every $\boldsymbol{v}_G$ with $b_- \leq v_{m(i)} < v_i \leq b_+$,

$$y_G(b_i(\boldsymbol{v}_G)) = F^{n-|G|}(v_i). \tag{1}$$

Since i must receive positive expected payoff, we know from Lemma 1 that $v_{m(i)} \leq b_i(\boldsymbol{v}_G) < v_i$. Fix any $\boldsymbol{v}_G$ satisfying (1) and consider an arbitrary $\boldsymbol{v}'_G = (v'_j)_{j \in G}$ satisfying $b_- \leq b_i(\boldsymbol{v}_G) < v'_{m(i)} < v'_i = v_i \leq b_+$. Clearly, there exists a positive measure of such profiles. For such a $\boldsymbol{v}'_G$,

$$b_i(\boldsymbol{v}_G) < v'_{m(i)} \leq b_i(\boldsymbol{v}'_G),$$

again by Lemma 1. Since $y_G(b)$ strictly increases in b for $b \in [b_-, b_+]$,

$$y_G(b_i(\boldsymbol{v}'_G)) > y_G(b_i(\boldsymbol{v}_G)) = F^{n-|G|}(v_i),$$

which violates (1). ▌

This result immediately implies that the second-price auction dominates the first-price auction in terms of allocative efficiency.

4 Revenue Comparison

In this section, we study the revenue implications of employing alternative auction formats. The revenue equivalence theorem would normally imply that the allocational inefficiency of the first-price auction translates into a poor revenue performance, relative to a second-price auction. As will be seen, our non-standard informational problem renders the standard revenue equivalence argument inapplicable. As mentioned in the introduction, a key step in applying the revenue equivalence theorem — the envelope theorem which enables one to express a bidder's payoff as an integral of his winning probability — does not follow from the standard argument. We proceed in two steps, first establishing the revenue equivalence theorem, assuming the envelope theorem, and then we prove that the envelope theorem works for the first-price auction.

4.1 Revenue Equivalence Theorem

Fix an arbitrary auction form and an associated equilibrium. Suppose that, in that equilibrium, bidder i wins the good with probability $x_i(\boldsymbol{v})$ and makes an expected payment $t_i(\boldsymbol{v})$, when bidders have valuations, $\boldsymbol{v}$. Taking expectation over $\boldsymbol{v}_{-G}$, we obtain their expected values conditional on the group valuation profile $\boldsymbol{v}_G$:

$$\overline{x}_i(\boldsymbol{v}_G) := \int_{\mathbf{V}_{-G}} x_i(\boldsymbol{v}_G, \boldsymbol{v}_{-G}) f_{-G}(\boldsymbol{v}_{-G}) d\boldsymbol{v}_{-G}$$

and

$$\bar{t}_i(\boldsymbol{v}_G) := \int_{\mathbf{V}_{-G}} t_i(\boldsymbol{v}_G, \boldsymbol{v}_{-G}) f_{-G}(\boldsymbol{v}_{-G}) d\boldsymbol{v}_{-G}.$$

Bidder i's expected utility is then expressed as:

$$\overline{\pi}_i(\boldsymbol{v}_G) := v_i \overline{x}_i(\boldsymbol{v}_G) - \bar{t}_i(\boldsymbol{v}_G) \tag{2}$$

The following lemma shows that the expected revenue is completely determined by the allocation of the good, given the envelope theorem result.

Lemma 4 *(Revenue Equivalence Theorem) If*

$$\overline{\pi}_i(\boldsymbol{v}_G) = \int_0^{v_i} \overline{x}_i(s, \boldsymbol{v}_{G/i}) ds \quad \forall \boldsymbol{v}_G, \tag{3}$$

then the seller's revenue can be expressed as

$$\int_{\boldsymbol{V}} \left(\sum_i J(v_i) x_i(\boldsymbol{v}) \right) f_N(\boldsymbol{v}) d\boldsymbol{v} \tag{4}$$

where

$$J(v_i) := v_i - \frac{1 - F(v_i)}{f(v_i)}.$$

Proof. See the Appendix.

Lemma 4 provides an easy method for comparing revenues across different auction formats. As mentioned, condition (3) is not immediate for a first-price auction in our model, and will be the focus of the next subsection. The next lemma explores the revenue implications of the allocative inefficiency. It refers to the following regularity condition, which is standard in the mechanism design literature:

Assumption 2 *$J(v_i)$ is strictly increasing on the interval* $[0, 1]$.

Specifically, the lemma shows that, given this assumption, an allocationally efficient auction dominates an inefficient one in terms of revenue but that, given a slightly stronger assumption, the opposite ranking holds in terms of the expected surplus accruing to the bidders.

Lemma 5 *Suppose that two auction formats, named E and I, satisfy (3). If auction E allocates the good efficiently almost surely and auction I inefficiently with positive probability, then, given Assumption 2, auction E yields strictly higher expected revenue than auction I. Furthermore, if $(1 - F(\cdot))/f(\cdot)$ is strictly decreasing on the interval* $[0, 1]$,[9] *then bidders receive higher expected surplus in auction I than in auction E.*

Proof. See the Appendix.

Given Proposition 2, Lemma 5 will yield an unambiguous revenue comparison between the first- and second-price auctions if condition (3) holds for both auction formats. We turn to this latter issue.

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[9] Note that this assumption is stronger than Assumption 2.

4.2 The Envelope Theorem

To see if condition (3) holds in the two auction formats, fix an auction format (to be one of the two) and an associated equilibrium. Let $\overline{z}_i(b; v_i, \boldsymbol{v}_{G/i})$ and $\overline{\tau}_i(b; v_i, \boldsymbol{v}_{G/i})$ respectively denote bidder i's winning probability and expected payment in that equilibrium, if he observes $\boldsymbol{v}_G$ and bids b (which may not be his equilibrium bid) and all others play their equilibrium strategies. The resulting payoff for bidder i can be expressed as:

$$\pi_i(b; v_i, \boldsymbol{v}_{G/i}) := \overline{z}_i(b; v_i, \boldsymbol{v}_{G/i})v_i - \overline{\tau}_i(b; v_i, \boldsymbol{v}_{G/i}). \tag{5}$$

In equilibrium, bidder i plays his equilibrium strategy $b_i(\boldsymbol{v}_G)$, so he must win with probability, $\overline{x}_i(\boldsymbol{v}_G) = \overline{z}_i(b_i(\boldsymbol{v}_G); \boldsymbol{v}_G)$, and pay $\overline{t}_i(\boldsymbol{v}_G) = \overline{\tau}_i(b_i(\boldsymbol{v}_G); \boldsymbol{v}_G)$, and hence receives the payoff of $\overline{\pi}_i(\boldsymbol{v}_G) = \overline{x}_i(\boldsymbol{v}_G)v_i - \overline{t}_i(\boldsymbol{v}_G)$, given profile $\boldsymbol{v}_G$ of group G valuations.

The standard envelope theorem has two components: (a) $\pi_i(b; v_i, \boldsymbol{v}_{G/i})$ is absolutely continuous in v, which enables one to express it as an integral of its partial derivative with respect to v (see Theorem 1 of Milgrom and Segal (2000), for instance); and (b) that partial derivative, whenever exists, equals bidder i's probability of winning.

These two properties are readily seen to hold in the second-price auction. In this format, the weak dominance property of the equilibrium strategy means that bidder i's winning probability, $\overline{z}_i(b; v_i, \boldsymbol{v}_{G/i})$, is independent of his own valuation, v_i, which implies that both (a) and (b) holds. Hence, condition (3) holds in a second-price auction.

Things are different, however, for the first-price auction. In its equilibrium, bidder i's winning probability, $\overline{z}_i(b; v_i, \boldsymbol{v}_{G/i})$, may depend on his own valuation v_i: Fixing his bid, as v_i rises, for instance, other bidders in the same group may adjust their strategies, which may affect his winning probability. The dependence of $\overline{z}_i$ on v_i makes the standard argument problematic, for neither (a) nor (b) may hold. To see this, reconsider Example 1, in which bidder $j \neq i$ bids $\min\{v_j, \max\{\frac{2}{3}v_j, v_i\}\}$. Figure 1 shows how a bidder j's equilibrium bidding strategy varies with i's valuation v_i, when j's valuation remains fixed at v_j.

[Insert Figure 1 around here.]

Given j's equilibrium bidding strategy,

$$\overline{z}_i(b; v_i, \boldsymbol{v}_{G/i}) = \begin{cases} y_G(b) = 2b\max\{1, 3b/2\} - b^2 & \text{if } v_i < b \\ 0 & \text{if } v_i \geq b \end{cases}. \tag{6}$$

Clearly, i's probability of winning decreases in v_i *discontinuously*, fixing his bid at b. Hence, property (a) does not hold. Property (b) also becomes problematic. To see this, suppose that $\overline{\pi}_i(\boldsymbol{v}_G)$ is differentiable with respect to v_i (which is unclear given the possible discontinuity). Then, an envelope theorem will imply that

$$\frac{\partial \overline{\pi}_i(\boldsymbol{v}_G)}{\partial v_i} = \overline{x}_i(\boldsymbol{v}_G) + \left.\frac{\partial \overline{z}_i(b; v_i, \boldsymbol{v}_{G/i})}{\partial v_i}\right|_{b=b_i(\boldsymbol{v}_G)} [v_i - b_i(\boldsymbol{v}_G)]. \tag{7}$$

If the second term does not vanish, property (b) will not hold.

Despite these possible problems, we show below that condition (3) holds in any equilibrium of the first-price auction. The key observations are that the failures of properties (a) and (b) are out-of-equilibrium phenomena and that condition (3) requires the two properties to hold only on the equilibrium path. For instance, in the above example, it is precisely when $v_i = b_i(\boldsymbol{v}_G)$ that $\overline{z}_i$ falls discontinuously as v_i increases, implying that the second term in (7) indeed vanishes. We present this result below.

Lemma 6 *(Envelope Theorem) In any equilibrium of a first-price auction,*

$$\overline{\pi}_i(\boldsymbol{v}_G) = \int_0^{v_i} \overline{x}_i(s, \boldsymbol{v}_{G/i})ds \quad \forall v_i \in [0, 1].$$

Proof. See the Appendix.

4.3 Revenue Comparison

We are now ready to present our revenue comparison of the two standard auctions. By Lemma 6, condition (3) holds for the two auction formats. We also noted that the first-price auction produces an inefficient allocation in any equilibrium (see Proposition 2), while the allocation is efficient in second price auction. Hence, the revenue dominance of the second-price auction follows from Lemma 5.

Proposition 3 *Given Assumptions 1 and 2, a first-price auction generates strictly lower expected revenue than a second-price auction, absent a reserve price. If we further assume that* $(1 - F(\cdot))/f(\cdot)$ *is strictly decreasing on the interval* $[0, 1]$*, bidders receive greater surplus in a first-price auction than in a second-price auction.*

Several remarks are in order. First, the proposition relies only on necessary conditions of an equilibrium. Hence, the results stated apply to any (undominated) equilibrium in a first-price auction and not just to the one described in Proposition 1.

Second, recall that the two auction formats are revenue equivalent if bidders are completely ignorant of their rivals' types (see Proposition 0). Therefore, Proposition 3 implies that the seller is worse off in a first-price auction from bidders' acquiring asymmetric knowledge of their rivals' types.[10] While the reduced revenue can be attributed to the inefficient allocation caused by their (asymmetric) knowledge, it is important to note that the revenue loss does not coincide with the efficiency loss. Proposition 3 informs us that, given a mild regularity condition, the revenue loss is greater than the efficiency loss, which means that the bidders are collectively better off from acquiring (asymmetric) knowledge of their rivals' types — hence the title of this paper.

Third, our revenue result can be also interpreted in terms of the two effects discussed in Example 1. As was seen there, the knowledge of rivals' types reduces the effective number of competitors a group leader faces. This effect blunts the competition. On the other hand, the knowledge of the rivals' types may raise the leader's bid in a way not possible in the standard case. Our revenue result suggests that the former effect dominates the latter effect, so the net effect is to reduce the competition. The same intuition suggests that if the partition structure is changed to increase the former effect (more members within a group), the overall competition would fall. While this point cannot be generally shown, it is illustrated in the next example.

Example 2 *Consider a first-price auction with 6 bidders, each valuing the good at* v *uniformly drawn from* $[0, 1]$*. Consider two partition structures: two groups of three*

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[10] As Proposition 0 suggests, if all bidders' types become common knowledge to all of them, then revenue equivalence is restored. Hence, the effect of acquiring information about rivals' types is ultimately nonmonotonic.

bidders and three groups of two bidders. In the former partition, a bidder i's equilibrium bidding function is $\min\{v_i, \max\{\frac{3}{4}v_i, v_{m(i)}\}\}$*, and it is* $\min\{v_i, \max\{\frac{4}{5}v_i, v_{m(i)}\}\}$ *in the latter partition. Comparison of the unconstrained bidding functions,* $\frac{3}{4}v_i$ *and* $\frac{4}{5}v_i$*, reveals that the competition-reduction effect is severer when the group size is bigger for a given total number of bidders. This is intuitive since a group leader can afford to bid lower when he knows that there are more lower valuation bidders. Meanwhile, the bigger the group size is, the more likely is* $v_{m(i)}$ *to take a large value, thus more likely to constrain the leader's bid. Consistent with our analytical result, the former effect turns out to dominate the latter effect: The expected revenue generated is 0.6998 in the former partition and 0.7067 in the latter partition.*

5 Concluding Remarks

We have shown that the bidders' asymmetric knowledge about their rivals' types adversely affects the first-price auction, both in terms of allocative efficiency and expected revenue, thus favoring the second-price auction relative to the first-price auction. We conclude by commenting on some extensions.

- *Correlated Private Values:* If bidders' valuations are correlated but private, then the revenue/efficiency ranking found in this paper still holds. In particular, the weak dominance property of the second-price auction continues to hold, and produces an efficient allocation. Likewise, Proposition 2 continues to hold, so inefficient allocation arises in any equilibrium of a first-price auction. As in the current model, this inefficiency implies a poor revenue performance of the first-price auction, relative to the second-price auction. Furthermore, the well-known linkage effect reinforces the revenue dominance of the second price auction (see Milgrom (1989)).

  Our model is not immediately generalizable to the affiliated/interdendent valuations case. The within-group knowledge of valuations will render allocations inefficient even in the second price auction. While the linkage effect will continue to favor the second-price auction, the precise comparison remains unclear.

- *Costs of Learning Rivals' Types:* In practice, the knowledge of rivals' types may not be freely available but may rather require a costly learning process. Costly learning of rivals' types can reinforce the results of the current paper. To fix the idea, suppose that bidders can learn the types of his within-group rivals at some small cost $c > 0$, while it is prohibitively costly to learn the types of rivals outside his group. The presence of such learning cost will not change the revenue comparison. In a second-price auction, the weak dominance property implies that the bidders have no incentive to learn their rivals' types. By contrast, bidders will have incentives to learn their (within-group) rivals' types in the first-price auction. Learning rivals' types enables a bidder to refine his assessments of his rivals' strategies and tailor his own strategy based on the refined assessments. While this feature presents some modelling challenges since different types of bidders will have differing incentives to learn their rivals' types,[11] some types of bidders will likely learn in equilibrium if c is sufficiently small. Since the learning cost adds to the social costs, the inferiority of a first-price auction will then persist and may even be severer.

- *Within-Group Collusion:* It is natural to suspect that the mutual knowledge of valuations within a group may facilitate collusion among group members. This will simply mean in our framework that all bidders except for group leaders drop out of competition in each auction format. Then, competition will be simply among group leaders without any within-group challenge. If there are K groups, then it becomes a K-bidder auction in which group $G \in \mathcal{G}$ leader draws his valuation from cdf $F^{|G|}$. Clearly, if the group size is the same across the groups, then efficient allocation will be attained under both formats, and their revenue equivalence will be restored.[12] If the group sizes are heterogeneous, then the ensuing auctions become standard asymmetric auctions. Incidentally, Marshall,

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[11] For instance, a bidder with zero valuation will never learn. Our conjecture is that, for a sufficiently small c there exists a threshold level of valuation for each bidder such that in equilibrium that bidder learns if and only if his valuation exceeds that threshold value. Our inefficiency result will continue to hold, given this outcome. Further, some learning costs will be expended. This conceptual problem can be avoided, at some loss of realism, if one models the learning of rivals' types as taking place before or at the same time of learning one's own valuation.

[12] It is easy to see that the revenues will be lower with this type of collusion than without the collusion.

Meurer, Richard, and Stromquist (1994) considered asymmetric auctions in which bidders' valuations are drawn from a cdf of the form, F^{m_i}, for some integer m_i for each $i \in N$. Interestingly, their numerical analysis reveals that a first-price auction revenue dominates the second-price auction, when F takes a uniform distribution. Whether this reversal of ranking holds more generally (in the case of collusion) remains an issue.

- *Imperfect Knowledge of Rivals' Types:* An avenue of extension is to relax the dichotomous nature of a bidder's knowledge. In the current model, each bidder either knows his rival's type completely (if his rival is inside his group) or not at all (if his rival is outside his group). If a bidder were to obtain imperfect signals about his rivals' types, this would not affect the comparison in terms of allocative efficiency. These signals will affect the bidders' strategies in a way that will disrupt efficient allocation in the first-price auction, while the weak dominance argument will continue to imply that these signals will have no impact on the bidding behavior in the second-price auction. The revenue comparison becomes nontrivial, however, since the envelope argument does not hold.[13] Generalization along this line thus awaits further research.

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[13] The first-price auction creates a linkage effect somewhat reminiscent of the one arising in the affiliated value setting. The presence of this effect makes the overall ranking ambiguous.

Appendix

Proof of Lemma 1. Bidder i must receive strictly positive (expected) payoff in equilibrium (since he can bid slightly higher than $v_{m(i)}$, which will win with positive probability). If $\underline{b}_i < v_{m(i)}$, then bidder $m(i)$ must also earn strictly positive payoff in equilibrium. For both bidders to earn positive payoffs, their infimum must coincide and each must put mass point there. But then it pays either one of them to raise the mass point slightly above, which will increase the probability of winning discontinuously while lowering his payoff conditional on winning only slightly. Hence, we have a contradiction, so we must have $\underline{b}_i \geq v_{m(i)}$. The last statement follows directly from the first statement and our tie-breaking rule. ▌

Proof of Lemma 2. Suppose to the contrary that $y_G(b)$ jumps up at b for some group G. This can only occur if the leader of another group, say $\tilde{G} \neq G$, puts mass on b. We must have only one such group since, otherwise, one of leaders of those groups would want to bid slightly above b to increase the winning probability discontinuously. Therefore, $y_{\tilde{G}}$ is continuous at b. Since the leader of $\tilde{G}$ bids b for a positive measure of type profiles of $\tilde{G}$ bidders, there must exist such a profile, $\boldsymbol{v}_{\tilde{G}}$, with $v^2_{\tilde{G}} < b \leq v^1_{\tilde{G}}$, where $v^1_{\tilde{G}}$ and $v^2_{\tilde{G}}$ are the first and second order statistics of $\boldsymbol{v}_{\tilde{G}}$. The leader of $\tilde{G}$ can ensure himself a positive surplus given $\boldsymbol{v}_{\tilde{G}}$, so we must also have $y_{\tilde{G}}(b) > 0$.

In equilibrium, the group G leader should have no incentive to deviate by bidding below b given $\boldsymbol{v}_{\tilde{G}}$, which requires that, for $\epsilon > 0$,

$$[v^1_{\tilde{G}} - (b-\epsilon)]y_{\tilde{G}}(b-\epsilon) \leq [v^1_{\tilde{G}} - b]y_{\tilde{G}}(b),$$

or

$$[v^1_{\tilde{G}} - b]\frac{y_{\tilde{G}}(b) - y_{\tilde{G}}(b-\epsilon)}{\epsilon} \geq y_{\tilde{G}}(b-\epsilon). \tag{A.1}$$

For sufficiently $\epsilon > 0$, $y_{\tilde{G}}(b-\epsilon)$ must be strictly positive since $y_{\tilde{G}}(\cdot)$ is continuous at b and $y_{\tilde{G}}(b) > 0$. Hence, to prove that such deviation is profitable, it suffices to show

$$\limsup_{\epsilon \downarrow 0} \frac{y_{\tilde{G}}(b) - y_{\tilde{G}}(b-\epsilon)}{\epsilon} \leq 0. \tag{A.2}$$

We prove this in the remainder.

Consider again any group $G \neq \tilde{G}$. If G consists of a single bidder, then for small enough ϵ, he would assign probability 0 to the interval $[b-\epsilon, b)$. Hence, no single-bidder group can contribute to $y_{\tilde{G}}(b) - y_{\tilde{G}}(b-\epsilon)$, for a sufficiently small ϵ, and we are done if all groups other than $\tilde{G}$ have single bidders. Assume therefore that there exists a group $G \neq \tilde{G}$, which contains more than one bidder. We show below that even such a group chooses almost zero probability in the interval $[b-\epsilon, b)$ for a small $\epsilon > 0$.

To prove this, we find an upper bound for $y_{\tilde{G}}(b) - y_{\tilde{G}}(b-\epsilon)$ for a small ϵ, which is accomplished by identifying a set of $\boldsymbol{v}_G$ for which a leader of G should not make a bid between b and $b-\epsilon$. To begin, note that since y_G jumps up at b as mentioned above, we have $\overline{p} := \lim_{b' \uparrow b} y_G(b') < y_G(b)$. Let $r := \frac{y_G(b)}{\overline{p}} > 1$, and take any $K_1 > \frac{1}{r-1}$. Then, for any $\epsilon > 0$, a leader of G with $v_G^1 > b + K_1\epsilon$ strictly prefers b to any $\tilde{b} \in [b-\epsilon, b)$ since

$$\frac{[v_G^1 - b]y_G(b)}{[v_G^1 - \tilde{b}]y_G(\tilde{b})} \geq \left(\frac{v_G^1 - b}{v_G^1 - \tilde{b}}\right) r \geq \left(\frac{v_G^1 - b}{v_G^1 - b + \epsilon}\right) r > 1,$$

where the numerator and denominator are the payoffs from the bidding b and $\tilde{b}$, respectively, the first inequality follows from $y_G(\tilde{b}) \leq \overline{p}$, the second inequality follows from $\tilde{b} \geq b - \epsilon$, and the last inequality follows from $v_G^1 > b + K_1\epsilon$ and from $K_1 > \frac{1}{r-1}$. It follows that a bid $\tilde{b} \in [b-\epsilon, b)$ can only be made by the group G leader if $v_G^1 \in [b-\epsilon, b+K_1\epsilon]$.

Next, set $K_2 := K_1 + 3$ for K_1 chosen above and assume that the group G leader has $v_G^1 \in [b-\epsilon, b+K_1\epsilon]$ — the only possibility that causes the leader to bid in $[b-\epsilon, b)$. Suppose that the second-highest rival in the group has $v_G^2 \leq b - K_2\epsilon$. Then, the group G leader will face no within-group challenge by bidding $b - K_2 < b - \epsilon$. In fact, for a sufficiently small $\epsilon > 0$, a group G leader with $v_G^1 \in [b-\epsilon, b+K_1\epsilon]$ strictly prefers $b - K_2$ to any bid $\tilde{b} \in [b-\epsilon, b)$, since

$$\frac{[v_G^1 - \tilde{b}]y_G(\tilde{b})}{[v_G^1 - (b - K_2\epsilon)]y_G(b - K_2\epsilon)} \leq \frac{(K_1+1)y_G(\tilde{b})}{(K_1+2)y_G(b-K_2\epsilon)} < 1,$$

where the numerator and the denominator represent the payoffs from bidding $\tilde{b} \in [b-\epsilon, b)$ and $b - K_2\epsilon$, respectively, and the first inequality holds (for $\epsilon < 1$ say) since $v_G^1 \in [b-\epsilon, b+K_1\epsilon]$ and $\tilde{b} \geq b - \epsilon$, and the second inequality holds since $y_G(\cdot)$ is continuous at b. It follows that, for a sufficiently small $\epsilon > 0$, the group G leader will never bid in $[b-\epsilon, b)$ if $v_G^2 \leq b - K_2\epsilon$.

Combining the two arguments, we conclude that a group G bidder will bid in $[b-\epsilon, b)$ only if $v_G^1 \in [b-\epsilon, b+K_1\epsilon]$ and $v_G^2 \in (b-K_2\epsilon, b)$. The probability of this latter event is no greater than $\binom{|G|}{2}\left(F(b+K_1\epsilon) - F(b-K_2\epsilon)\right)^2$. Therefore,

$$y_{\tilde{G}}(b) - y_{\tilde{G}}(b-\epsilon) \leq \sum_{\substack{G \neq \tilde{G} \\ |G| \geq 2}} \binom{|G|}{2} \left(F(b+K_1\epsilon) - F(b-K_2\epsilon)\right)^2. \tag{A.3}$$

Hence, we obtain

$$\begin{aligned}
& \limsup_{\epsilon \downarrow 0} \frac{y_{\tilde{G}}(b) - y_{\tilde{G}}(b-\epsilon)}{\epsilon} \\
\leq \; & \lim_{\epsilon \downarrow 0} \sum_{\substack{G \neq \tilde{G} \\ |G| \geq 2}} \binom{|G|}{2} \frac{\left(F(b+K_1\epsilon) - F(b-K_2\epsilon)\right)^2}{\epsilon} \\
= \; & \lim_{\epsilon \downarrow 0} \sum_{\substack{G \neq \tilde{G} \\ |G| \geq 2}} 2\binom{|G|}{2} \left(F(b+K_1\epsilon) - F(b-K_2\epsilon)\right)\left(K_1 f(b+K_1\epsilon) + K_2 f(b-K_2\epsilon)\right) \\
= \; & 0,
\end{aligned}$$

where the inequality follows from (A.3), and the first equality follows from the L'hopital's rule.

The last string of inequalities implies that it pays the group $\tilde{G}$ leader to move down the mass point, which yields a contradiction to the conjectured equilibrium. Hence, we conclude that $y_G(\cdot)$ is continuous for all $G \in \mathcal{G}$. ▌

Proof of Lemma 3.

Let $b_G(\boldsymbol{v}_G)$ be an arbitrary selection from the support of a group G leader's (possibly mixed) equilibrium bids when the valuation profile of group G members is $\boldsymbol{v}_G$. Consider two valuation profiles $\boldsymbol{v}_G$ and $\overline{\boldsymbol{v}}_G$ with $\boldsymbol{v}_G \leq \overline{\boldsymbol{v}}_G$ and $v_G^1 < \overline{v}_G^1$, where v_G^1 (resp. $\overline{v}_G^1$) refers to the group G leader's valuation, given profile $\boldsymbol{v}_G$ (resp. $\overline{\boldsymbol{v}}_G$). Let $b = b_G(\boldsymbol{v}_G)$ and $\overline{b} = b_G(\overline{\boldsymbol{v}}_G)$, and then we show that $b \leq \overline{b}$; i.e., an arbitrary equilibrium bidding strategy is nondecreasing. Since $\overline{b} \geq \overline{v}_{m(i)} \geq v_{m(i)}$, we are done if $b \leq \overline{v}_{m(i)}$. Hence, assume that $b > \overline{v}_{m(i)}$. This means that, given the profile of $\overline{\boldsymbol{v}}_G$, the group G leader could beat all of his within group rivals by bidding b, so his winning probability would be simply that of outbidding other group leaders, $y_G(b)$. Likewise, given the profile of $\boldsymbol{v}_G$, the group G leader would face the winning probability of $y_G(\bar{b})$ when bidding $\bar{b}$. Then, incentive compatibility requires $y_G(b)[v_G^1 - b] \leq y_G(\bar{b})[v_G^1 - \bar{b}]$ and we have $y_G(\bar{b})[\bar{v}_G^1 - \bar{b}] \leq y_G(b)[\bar{v}_G^1 - b]$. Combining last two inequalities gives

$$(\bar{v}_G^1 - v_G^1)(y_G(\bar{b}) - y_G(b)) \geq 0. \tag{A.4}$$

Suppose, to the contrary, that $b > \bar{b}$. Then, since $y_G(\cdot)$ is nondecreasing, we must have $y_G(\bar{b}) = y_G(b) > 0$. But this cannot hold since the group G leader would strictly prefer to bid $\bar{b}$ when $\boldsymbol{v}_G$ is realized. We therefore conclude that an arbitrary selection from the equilibrium strategies must be non-decreasing. Because there can be only countably many jumps in a non-decreasing and bounded correspondence, the equilibrium bidding strategy of i is almost pure when he is a leader. ▌

Proof of Proposition 1.

Fix an arbitrary partition structure and suppose that there are K groups in that partition structure. As outlined in the text, we first consider a *hypothetical game* in which there are only K players, one for each group. In this hypothetical game, player $I \in \mathcal{G}$ observes as private information $\boldsymbol{v}_I = (v_i)_{i \in I}$, the valuation profile of bidders in group I in the original game, and bids

$$\beta_I(\boldsymbol{v}_I) \in [v_I^2, v_I^1], \tag{A.5}$$

where v_I^r denotes the r^{th} order statistics of the vector $\boldsymbol{v}_I$, and $v_I^2 = 0$ if $|I| = 1$. All bidders bid simultaneously, and the good is allocated according to the first-price auction rule. Ties are broken according to our endogenous sharing rule. Formally, given the profile of bids submitted, $\boldsymbol{b} = (b_1, \cdots, b_K)$, let $W(\boldsymbol{b}, \boldsymbol{v}) = \arg\max_J \{v_J^1 | J \in \arg\max_{H \in \mathcal{G}} b_H\}$ denote the set of highest-valuation bidders (in the hypothetical game) who submitted the highest bid. Then, the payoff of player I is described as:

$$U_I(\boldsymbol{b}; \boldsymbol{v}_I) := \begin{cases} \frac{1}{|W(\boldsymbol{b},\boldsymbol{v})|}(v_I^1 - b_I) & \text{if } I \in W(\boldsymbol{b}, \boldsymbol{v}) \\ 0 & \text{otherwise,} \end{cases} \tag{A.6}$$

when the players observed $\boldsymbol{v}$ and bid $\boldsymbol{b}$. Notice that each player only realizes the highest valuation of his group. If the bidders play a strategy profile $\boldsymbol{\beta} = (\beta_I)_{I \in \mathcal{G}}$, then bidder I receives payoff: $u_I(\boldsymbol{\beta}) := \int U_I(\boldsymbol{\beta}(\boldsymbol{v}); \boldsymbol{v}_I) f_N(\boldsymbol{v}) d\boldsymbol{v}$.

Given this description of hypothetical game, we turn to existence of the Nash equilibrium in this game. Reny (1999) provides us with conditions for the existence of a mixed strategy equilibrium (see Corollary 5.2 of Reny (1999)). First of all, as Reny (1999) did in the case of multi-unit pay-your-bid auction, we study a *restricted* version of this hypothetical game where bid functions are restricted to be nondecreasing. This latter restriction ensures that the strategy space is compact if endowed with the pointwise convergence topology, thereby making the set of mixed strategies compact with the weak* topology. An equilibrium of the *restricted hypothetical game* will be shown to be an equilibrium of the hypothetical game later when we show that there exists a best response satisfying the monotonicity constraint when all other bidders play the restricted equilibrium strategies.

Given the compactness, *better-reply security* (as defined in Reny (1999)) is sufficient to establish the existence of a mixed strategy equilibrium of the restricted hypothetical game. We prove that its sufficient conditions, reciprocal upper semicontinuity and payoff security, hold for the restricted hypothetical game.[14]

Step 1: *The payoffs of the players the restricted hypothetical game satisfy reciprocal upper semicontinuity in mixed strategies.*

Proof. We prove reciprocal upper semicontinuity in the players' pure strategies, which is sufficient for reciprocal uppersemicontinuity in the mixed strategies. The former is in turn proven by showing that $u(\boldsymbol{\beta}) = \sum_I u_I(\boldsymbol{\beta})$ is upper semicontinuous in $\boldsymbol{\beta}$. To this end, we first show that $U(\boldsymbol{b}; \boldsymbol{v}) := \sum_I U_I(\boldsymbol{b}; \boldsymbol{v}_I)$ is upper semicontinuous in $\boldsymbol{b}$ for every $\boldsymbol{v}$. For a $\boldsymbol{v}$, pick an arbitrary $\boldsymbol{b}$ and a sequence $\boldsymbol{b}^t = (b_I^t)_{I \in \mathcal{G}}$ converging to $\boldsymbol{b}$. It suffices to show that for any given $\epsilon > 0$, there exists T such that $U(\boldsymbol{b}; \boldsymbol{v}) + \epsilon \geq U(\boldsymbol{b}^t; \boldsymbol{v})$ for all $t \geq T$. Let $b = b_I$ and $v = v_I^1$ for $I \in W(\boldsymbol{b}, \boldsymbol{v})$. Then, $U(\boldsymbol{b}; \boldsymbol{v}) = v - b$. For a sufficiently large t, we must have (i) $W(\boldsymbol{b}^t, \boldsymbol{v}) \subset \arg\max_{J \in \mathcal{G}} b_J$, and that (ii)

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[14] A standard first-price auction does not satisfy reciprocal upper semicontinuity, given a random tie-breaking rule (see Reny (1999), p.1040). Reciprocal upper semicontinuity holds here because of our endogenous sharing rule, which allocates the good efficiently within tying bidders.

$b_I^t \geq b - \epsilon$ for every I. It then follows that

$$\begin{aligned} U(\boldsymbol{b}^t;\boldsymbol{v}) &= \sum_{I\in W(\boldsymbol{b}^t,\boldsymbol{v})} \frac{1}{|W(\boldsymbol{b}^t,\boldsymbol{v})|}(v_I^1 - b_I^t) \\ &\leq \sum_{I\in W(\boldsymbol{b}^t,\boldsymbol{v})} \frac{1}{|W(\boldsymbol{b}^t,\boldsymbol{v})|}(v - b_I^t) \\ &\leq v - b + \epsilon \\ &= U(\boldsymbol{b};\boldsymbol{v}) + \epsilon, \end{aligned}$$

where the first inequality follows from (i) since $v = \max_{J'}\{v_{J'}^1 \mid J' \in \arg\max_{J\in\mathcal{G}} b_J\} \geq v_I^1$ for any $I \in W(\boldsymbol{b}^t,\boldsymbol{v}) \subset \arg\max_{J\in\mathcal{G}} b_J$, and the second inequality follows from (ii).

To show the upper semicontinuity of u, consider a sequence $\boldsymbol{\beta}^t$ converging to $\boldsymbol{\beta}$ pointwise. Then,

$$\begin{aligned} \limsup_{t\to\infty} u(\boldsymbol{\beta}^t) &= \limsup_{t\to\infty} \int U(\boldsymbol{\beta}^t(\boldsymbol{v}),\boldsymbol{v}) f_N(\boldsymbol{v}) d\boldsymbol{v} \\ &\leq \int \limsup_{t\to\infty} U(\boldsymbol{\beta}^t(\boldsymbol{v}),\boldsymbol{v}) f_N(\boldsymbol{v}) d\boldsymbol{v} \leq \int U(\boldsymbol{\beta}(\boldsymbol{v}),\boldsymbol{v}) f_N(\boldsymbol{v}) d\boldsymbol{v} = u(\boldsymbol{\beta}), \end{aligned}$$

where the first inequality follows from the Fatou's Lemma (see Ash (1972), p.295, for instance) and the second inequality from the upper semicontinuity of U. ▌

Step 2: *The restricted hypothetical game is payoff secure in mixed strategies.*

Proof. Let m_I denote the bidder I's mixed strategy and $\boldsymbol{m} = (m_I)_{I\in\mathcal{G}}$ its profile for all players. Note that m_I is a mixing over non-decreasing pure strategy bid functions satisfying (A.5). Then, our game is payoff secure if for every $\boldsymbol{m}$ and every $\epsilon > 0$, each player i has a strategy $\overline{m}_I$ such that $u_I(\overline{m}_I, \boldsymbol{m}'_{-I}) \geq u_I(\boldsymbol{m}) - \epsilon$ for all $\boldsymbol{m}'_{-I}$ in some open neighborhood of $\boldsymbol{m}_{-I}$. This part of the proof follows precisely the same argument as in Reny (1999). The key step is to observe that, given $\boldsymbol{m}_{-I}$ and ϵ, a player I can achieve a payoff within $\epsilon/2$ of his supremum payoff by adopting a bidding strategy that is strictly increasing in v_I^1.[15] Since the latter strategy does not

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[15] To see this point, suppose hypothetically that player I wins the auction whenever he makes the highest bid *even if a tie occurs* at that bid. Given this presumption, I's payoff is upper semicontinuous in his bid, so the maximum is well defined and is attained by a bidding function which is nondecreasing in v_I^i. The resulting maximum must constitute an upper bound for I's payoff (since he will not always win at a tie in the true game). This payoff can be arbitrarily closely approximated by a modifying the bid function slightly to raise the bid at a tie and to avoid constant bids.

put any mass on a single bid, $u_I(\overline{m}_I, \cdot)$ is continuous in $\boldsymbol{m}_{-I}$. Thus, we can take a neighborhood of $\boldsymbol{m}_{-I}$ where $u_I(\overline{m}_I, \cdot)$ is at least $u_I(\boldsymbol{m}) - \epsilon$. ∎

Given that the two conditions are met, Corollary 5.2 of Reny (1999) implies that there exists a mixed strategy equilibrium, denoted $\boldsymbol{m}^*$, whose support consists of non-decreasing bid functions. Now, we complete the proof by showing that I has a best response which is non-decreasing, which implies that m_I^* must be a best response overall.

Step 3: *When all other players play their equilibrium strategies of the restricted hypothetical game, player I has a best response strategy which is non-decreasing.*

Proof. As before, let $y_I(b)$ denote player I's probability of winning when bidding b. By Lemma 2,[16] the best response set $M_I(\boldsymbol{v}_I) := \arg\max_{v_I^2 \le b \le v_I^1} y_I(b)[v_I^1 - b]$ is nonempty. Further, since $y_I(\cdot)$ nondecreasing, the objective function satisfies the single crossing property in $(b, \boldsymbol{v}_I)$. By Theorem 4 of Milgrom and Shannon (1994), then one can select a best response function, $\beta_I(\boldsymbol{v}_I)$ that is nondecreasing in $\boldsymbol{v}_I$. ∎

This last step implies that $\boldsymbol{m}^*$ is an equilibrium of the (unrestricted) hypothetical game. Furthermore, Lemma 3 guarantees that $\boldsymbol{m}^*$ is almost pure. Hence, there exists a pure strategy equilibrium $\boldsymbol{\beta}^*$. Given the equilibrium, $\boldsymbol{\beta}^*$, of the hypothetical game, one can construct the equilibrium strategies for the original game, as described in the main text. ∎

Proof of Lemma 4. Combining (2) and (3) yields

$$\bar{t}_i(\boldsymbol{v}_G) = v_i \overline{x}_i(\boldsymbol{v}_G) - \int_0^{v_i} \overline{x}_i(s, \boldsymbol{v}_G) ds \tag{A.7}$$

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[16] While Lemma 2 establishes the continuity for the original game, the same proof applies to the restricted hypothetical game.

Next, integrating (A.7) over $\boldsymbol{v}_{G/i}$ gives

$$\begin{aligned}
T(v_i) &:= \int_{\mathbf{V}_{G/i}} \bar{t}_i(v_i, \boldsymbol{v}_{G/i}) f_{G/i}(\boldsymbol{v}_{G/i}) d\boldsymbol{v}_{G/i} \\
&= v_i \int_{\boldsymbol{V}_{G/i}} \overline{x}_i(v_i, \boldsymbol{v}_{G/i}) f_{G/i}(\boldsymbol{v}_{G/i}) d\boldsymbol{v}_{G/i} - \int_{\boldsymbol{V}_{G/i}} \int_0^{v_i} \overline{x}_i(s, \boldsymbol{v}_{G/i}) ds f_{G/i}(\boldsymbol{v}_{G/i}) d\boldsymbol{v}_{G/i} \\
&= v_i X(v_i) - \int_0^{v_i} \int_{\boldsymbol{V}_{G/i}} \overline{x}_i(s, \boldsymbol{v}_{G/i}) f_{G/i}(\boldsymbol{v}_{G/i}) d\boldsymbol{v}_{G/i} ds \\
&= v_i X(v_i) - \int_0^{v_i} X(s) ds, \qquad \text{(A.8)}
\end{aligned}$$

where $X(v_i) := \int_{\boldsymbol{V}_{G/i}} \overline{x}_i(v_i, \boldsymbol{v}_{G/i}) f_{G/i}(\boldsymbol{v}_{G/i}) d\boldsymbol{v}_{G/i}$. The remaining steps are the same as Myerson (1981) and is omitted. ▮

Proof of Lemma 5. Let $(x_i, t_i)_{i=1}^n$ and $(\tilde{x}_i, \tilde{t}_i)_{i=1}^n$ respectively denote the allocation-payment pairs arising in the equilibria of auctions E and I. Similarly, and let R and $\tilde{R}$ denote the expected revenues, $y(\boldsymbol{v})$ and $\tilde{y}(\boldsymbol{v})$ denote the probabilities of a bidder with the highest valuation winning the object given $\boldsymbol{v}$, from auctions E and I, respectively.

By assumption, a set

$$\boldsymbol{S} := \{\boldsymbol{v} \in \boldsymbol{V} : \tilde{y}(\boldsymbol{v}) = y(\boldsymbol{v}) = 1\}$$

has a measure less than one. For almost every $\boldsymbol{v} \in S$,

$$\sum_i J(v_i)(x_i(\boldsymbol{v}) - \tilde{x}_i(\boldsymbol{v})) = 0. \qquad \text{(A.9)}$$

Let v^k denotes kth order statistic of $\boldsymbol{v}$. Then, for almost every $\boldsymbol{v} \in \boldsymbol{V}/\boldsymbol{S}$,

$$\sum_i J(v_i)(x_i(\boldsymbol{v}) - \tilde{x}_i(\boldsymbol{v})) \geq (J(v^1) - J(v^2))(1 - \tilde{y}(\boldsymbol{v})) > 0, \qquad \text{(A.10)}$$

where the first inequality in the equation (A.10) holds since

$$\begin{aligned}
&\sum_i J(v_i) x_i(\boldsymbol{v}) = J(v^1) \quad \text{and} \\
&\sum_i J(v_i) \tilde{x}_i(\boldsymbol{v}) \leq J(v^1)\tilde{y}(\boldsymbol{v}) + J(v^2)(1 - \tilde{y}(\boldsymbol{v})),
\end{aligned}$$

and the second inequality holds since $J(\cdot)$ is strictly increasing and $\tilde{y}(\cdot) < 1$ in $\boldsymbol{V}/\boldsymbol{S}$.

Combining (A.9) and (A.10), we have

$$\begin{aligned} R - \tilde{R} &= \int_{\boldsymbol{V}} \sum_i J(v_i)(x_i(\boldsymbol{v}) - \tilde{x}_i(\boldsymbol{v}))f(\boldsymbol{v})d\boldsymbol{v} \\ &= \int_{\boldsymbol{V}/\boldsymbol{S}} \sum_i J(v_i)(x_i(\boldsymbol{v}) - \tilde{x}_i(\boldsymbol{v}))f(\boldsymbol{v}) > 0, \end{aligned}$$

since $\boldsymbol{V}/\boldsymbol{S}$ has a strictly positive measure.

We now prove the last statement. Since the net social surplus is seller's revenue plus bidders' surplus, decomposing the equation (4) gives bidders' expected surplus:

$$\int_{\boldsymbol{V}} \left(\sum_i \frac{1 - F(v_i)}{f(v_i)} x_i(\boldsymbol{v}) \right) f_N(\boldsymbol{v}) d\boldsymbol{v}.$$

If the inverse hazard rate $\frac{1-F(\cdot)}{f(\cdot)}$ is strictly increasing, then the same proof as the one in revenue comparison yields the stated ranking in terms of the bidders' expected surplus. ▌

Proof of Lemma 6. Recall that in equilibrium bidder i receives

$$\overline{\pi}_i(\boldsymbol{v}_G) = \overline{x}_i(\boldsymbol{v}_G)[v_i - b_i(\boldsymbol{v}_G)]. \tag{A.11}$$

There are two cases.

Case 1: $v_i \leq v_{m(i)}$. Note first that $\overline{\pi}_i(\boldsymbol{v}_G) = 0$ if $v_i < v_{m(i)}$, by Lemma 1. We now prove that $\overline{\pi}_i(\boldsymbol{v}_G) = 0$ if $v_i = v_{m(i)}$. Suppose to the contrary that bidder i earns strictly positive payoff. Then, his infimum bid must be less than v_i but weakly greater than the infimum bid of bidder $m(i)$. If it is strictly greater, then bidder $m(i)$ can profitably deviate by outbidding i's infimum bid. If the infimums coincide, then both bidders must put mass points there, which yields a contradiction. Hence, bidder i cannot make strictly positive payoff, which implies that bidder i makes zero payoff in that case. Since Lemma 1 also implies that $\overline{x}_i(v_i, \boldsymbol{v}_{G/i}) = 0$ for any $v_i < v_{m(i)}$, for any $v_i \leq v_{m(i)}$,

$$\overline{\pi}_i(\boldsymbol{v}_G) = 0 = \int_0^{v_i} \overline{x}_i(s, \boldsymbol{v}_{G/i})ds.$$

Case 2: $v_i > v_{m(i)}$. Since no bidder in G, other than i, bids strictly greater than $v_{m(i)}$, bidder i beats all other bidders in his group by bidding $v_{m(i)}$ or more, given our tie-breaking rule. Hence, bidder i will win with probability $y_G(b)$ by bidding $b \geq v_{m(i)}$;

i.e., $\overline{z}_i(b;\boldsymbol{v}_G) = y_G(b)$ for any $b \geq v_{m(i)}$. Furthermore, by Lemma 1, bidder i never bids below $v_{m(i)}$. Hence, $\overline{x}_i(\boldsymbol{v}_G) = y_G(b_i(\boldsymbol{v}_G))$ since, by Lemma 3, $b_i(\boldsymbol{v}_G)$ is unique for almost every $\boldsymbol{v}_G$. Further, we can write:

$$\overline{\pi}_i(\boldsymbol{v}_G) = \max_{b \geq v_{m(i)}} \overline{z}_i(b;\boldsymbol{v}_G)[v_i - b] = \max_{b \geq v_{m(i)}} y_G(b)[v_i - b],$$

and $b_i(\boldsymbol{v}_G)$ must be a solution to this constrained maximization problem. Observe that a function, $\phi(b, v_i) := y_G(b)[v_i - b]$, has a derivative, $\phi_{v_i}(b, v_i) = y_G(b)$, for all $v_i \geq v_{m(i)}$, and that the derivative is uniformly bounded (by 1). Hence, $\overline{\pi}_i(\boldsymbol{v}_G)$ is absolutely continuous in v_i and can be expressed as an integral of $y_G(b_i(\boldsymbol{v}_G)) = \overline{x}_i(\boldsymbol{v}_G)$. The result then follows since

$$\begin{aligned}
\overline{\pi}_i(\boldsymbol{v}_G) &= \int_{v_{m(i)}}^{v_i} \phi_{v_i}(b_i(\boldsymbol{v}_G), v_i)ds + \overline{\pi}_i(v_{m(i)}) \\
&= \int_{v_{m(i)}}^{v_i} \overline{x}_i(s, \boldsymbol{v}_{G/i})ds + \overline{\pi}_i(v_{m(i)}) \\
&= \int_{v_{m(i)}}^{v_i} \overline{x}_i(s, \boldsymbol{v}_{G/i})ds + \int_0^{v_{m(i)}} \overline{x}_i(s, \boldsymbol{v}_{G/i})ds \\
&= \int_0^{v_i} \overline{x}_i(s, \boldsymbol{v}_{G/i})ds,
\end{aligned}$$

where the first equation follows from Theorem 2 of Milgrom and Segal (2000), the second from $\phi_{v_i}(b_i(\boldsymbol{v}_G), v_i) = y_G(b_i(\boldsymbol{v}_G)) = \overline{x}_i(\boldsymbol{v}_G)$, and the third from the result in Case 1. ▌

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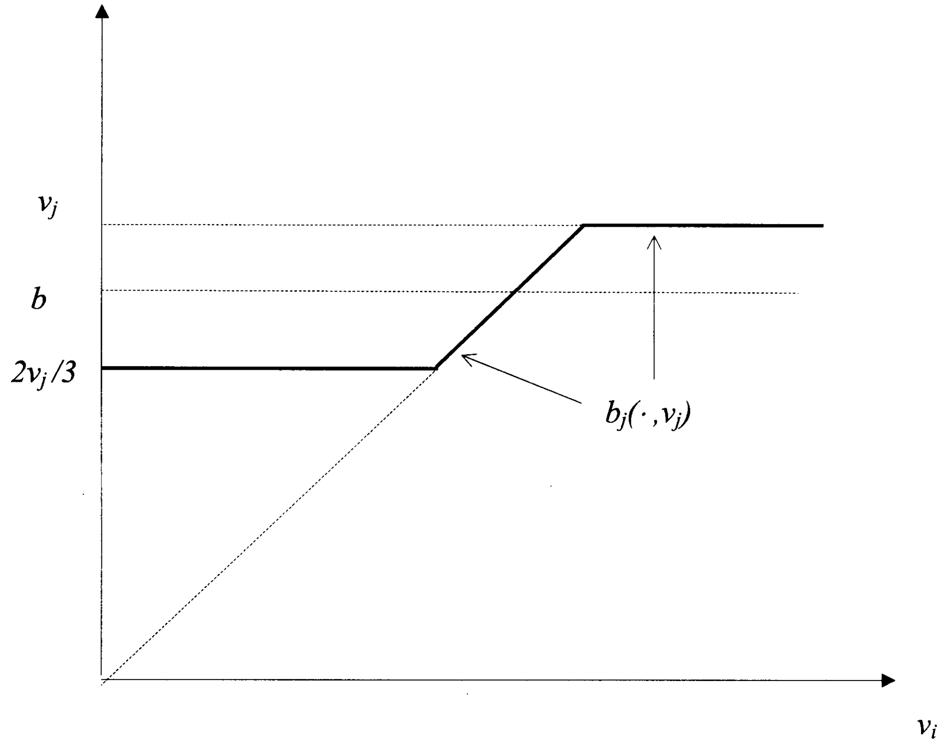

Figure 1