Economics 712
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Lecture Notes 5. Frequency Domain Approach to Time Series

i. spectral densities

The spectral density of a time series is defined as

$$f_x(\omega) = \frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\sigma_x(j)e^{-ij\omega} \quad \omega\in[-\pi,\pi] \tag{5.1}$$

Note that since $\sin(\omega) = -\sin(-\omega)$ and $\sigma_x(j) = \sigma_x(-j)$, then

$$\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\sigma_x(j)e^{-ij\omega} = \frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\sigma_x(j)\cos j\omega = \frac{\sigma_x(0)}{2\pi}+\frac{1}{\pi}\sum_{j=1}^{\infty}\sigma_x(j)\cos j\omega \tag{5.2}$$

The spectral density is therefore the Fourier transform of the autocovariance function. By the Riesz-Fischer theorem, this means that the spectral density function and the autocovariance function contain the same information about x_t. One can recover the autocovariances via the formula

$$\int_{-\pi}^{\pi} f_x(\omega)e^{-ij\omega}d\omega = \sigma_x(j) \tag{5.3}$$

Notice a special case of the recovery formula is

$$\int_{-\pi}^{\pi} f_x(\omega)d\omega = \sigma_x(0) \tag{5.4}$$

which will prove to have a deep interpretation.

We now turn to some examples of spectral density functions.

Example 5.1. white noise process

If $x_t = \varepsilon_t$, then the spectral density equals

$$\frac{\sigma_\varepsilon^2}{2\pi} \tag{5.5}$$

The function is shaped as a rectangle for the interval $[-\pi, \pi]$. Each frequency produces the same value, which is the origin of the term white noise.

Example 5.2. AR(1) Process

If $x_t = \rho x_{t-1} + \varepsilon_t$, then the spectral density equals

$$\frac{\sigma_\varepsilon^2}{2\pi\left(1-\rho e^{-i\omega}\right)\left(1-\rho e^{i\omega}\right)} = \frac{\sigma_\varepsilon^2}{2\pi\left(1-2\rho\cos\omega+\rho^2\right)} \tag{5.6}$$

When $\rho > 0$, then the maximum of the function is $\omega = 0$. If $\rho < 0$, then $\omega = 0$ minimum. Notice that as $\rho \Rightarrow 1$, the spectral density function becomes arbitrarily large at $\omega = 0$.

Example 5.3. MA(1) Process

If $x_t = \varepsilon_t + \rho\varepsilon_{t-1}$, then the spectral density equals

$$\frac{\sigma_\varepsilon^2\left(1+\rho e^{-i\omega}\right)\left(1+\rho e^{i\omega}\right)}{2\pi} = \frac{\sigma_\varepsilon^2\left(1+2\rho\cos\omega+\rho^2\right)}{2\pi} \tag{5.7}$$

If $\rho > 0$, then the function is maximal at $\omega = 0$ wheras if $\rho < 0$ the function is minimal. Notice that the qualitative shape of the spectral density of an MA(1) is the same as for the AR(1).

Example 5.4. AR(2) Process

If $x_t = \rho_1 x_{t-1} + \rho_2 x_{t-2} + \varepsilon_t$, then the spectral density equals

$$\begin{aligned} &\frac{\sigma_\varepsilon^2}{2\pi\left(1-\rho_1 e^{-i\omega}-\rho_2 e^{-i2\omega}\right)\left(1-\rho e^{i\omega}-\rho_2 e^{i2\omega}\right)} = \\ &\frac{\sigma_\varepsilon^2}{2\pi\left(1+\rho_1^2+\rho_2^2+2\left(\rho_1\rho_2-\rho_1\right)\cos\omega-2\rho_2\cos 2\omega\right)} \end{aligned} \tag{5.8}$$

the maximum value of this function is difficult to compute, except when ρ_1 and ρ_2 are both positive, in which case the spectral density must have a maximum at $\omega = 0$.

ii. spectral representation of a time series

This section describes a decomposition of a time series into frequency-specific components. The decomposition will illustrate a deep relationship between a structure of a time series and its spectral density.

Theorem 5.1 Spectral representation theorem (Cramér's theorem)

Let x_t be a zero mean L^2 process. The x_t may be expressed as

$$x_t = \int_{-\pi}^{\pi} e^{it\omega} dz_x(\omega) \tag{5.9}$$

where $dz_x(\omega)$ is a complex valued random process such that

$$
\begin{aligned}
&i.\ \mathrm{E}\left(dz_x(\omega)\right) = 0 \\
&ii.\ \mathrm{E}\left(dz_x(\omega)\overline{dz_x(\omega)}\right) = dF_x(\omega) \\
&iii.\ \mathrm{E}\left(dz_x(\omega_i)\overline{dz_x(\omega_j)}\right) = 0,\ \ \omega_i \neq \omega_j \\
&iv.\ z_x(\omega)\ \text{ is unique, outside a set of measure zero.}
\end{aligned}
\tag{5.10}
$$

The domain of integration is $(-\pi, \pi]$. (The use of the half-open interval is a minor technicality which I will ignore.)

What is $dz_x(\omega)$? This is an example of a random function. This means that for each fixed frequency ϖ $z_x(\varpi)$ is a random variable. Suppose that for *any* fixed frequencies $\omega_1 < \omega_2 \leq \omega_3 < \omega_4$

$$
\mathrm{E}\left(\left(z_x(\omega_4) - z_x(\omega_3)\right)\left(\overline{z_x(\omega_2) - z_x(\omega_1)}\right)\right) = 0 \tag{5.11}
$$

then the process is said to possess orthogonal increments. A standard example of such a process is Brownian motion. A Brownian motion $B(t)$ $t \geq 0$ is random function such that $B(\bar{t}) \sim N(0, \bar{t})$.

What does this theorem tell us about the underlying structure of a time series? Recall (5.4), which stated that the variance of a process is the integral of its spectral density. Using (5.9), the variance can also be written as

$$
\sigma_x(0) = Ex_t^2 = E\left(\int_{-\pi}^{\pi} e^{it\omega} dz_x(\omega) \overline{\int_{-\pi}^{\pi} e^{it\omega} dz_x(\omega)}\right). \tag{5.12}
$$

The use of the complex conjugate is allowed since x_t is real. Since $dz_x(\omega)$ is composed of orthogonal increments, it must be the case that

$$E\left(\int_{-\pi}^{\pi} e^{it\omega} dz_x(\omega) \overline{\int_{-\pi}^{\pi} e^{it\omega} dz_x(\omega)}\right) =$$
$$E\int_{-\pi}^{\pi} e^{it\omega} dz_x(\omega) \overline{e^{it\omega} dz_x(\omega)} = \tag{5.13}$$
$$E\int_{-\pi}^{\pi} dz_x(\omega) \overline{dz_x(\omega)}$$

The expected value operator can be moved inside the integral (since the integral is a linear operator), which means

$$E\int_{-\pi}^{\pi} dz_x(\omega) \overline{dz_x(\omega)} = E\int_{-\pi}^{\pi} E\left(dz_x(\omega) \overline{dz_x(\omega)}\right) = \int_{-\pi}^{\pi} dF_x(\omega) \tag{5.14}$$

This indicates that the spectral density reveals how each stochastic bit of x_t constributes to its overall variance.

Finally, note that if x_t is real, one can rewrite the spectral representation as follows. First, observe that in order for integral in (5.9) to be real with probability 1, it is necessary for $dz_x(\omega) = \overline{dz_x(-\omega)}$. Why? Take a fixed ϖ and $-\varpi$. In order for the sum

$$e^{it\varpi} dz_x(\varpi) + e^{-it\varpi} dz_x(-\varpi) \tag{5.15}$$

to be real, it is necessary that $e^{it\varpi} dz_x(\varpi) = \overline{e^{-it\varpi} dz_x(-\varpi)}$. Of course, $e^{it\varpi} = \overline{e^{-it\varpi}}$. Since $z(\omega)$ is random, $dz_x(\omega) = \overline{dz_x(-\omega)}$ is needed to hold with probability 1 in order to ensure the integral is real.

Rewriting $dz_x(\omega) = du_x(\omega) - i dv_x(\omega)$ the requirement that $dz_x(\omega) = \overline{dz_x(-\omega)}$ is real means that

$$du_x(\omega) = du_x(\omega) \text{ and } dv_x(\omega) = -du_x(-\omega) \tag{5.16}$$

Recall that by the spectral representation theorem $\mathrm{E}\left(dz_x(\omega)dz_x(-\omega)\right)=0$. This can thus be rewritten

$$\begin{aligned}&\mathrm{E}\left(du_x(\omega)du_x(-\omega)\right)-i\mathrm{E}\left(du_x(\omega)dv_x(-\omega)\right)-\\&i\mathrm{E}\left(du_x(-\omega)v_x(\omega)\right)+\mathrm{E}\left(dv_x(\omega)dv_x(-\omega)\right)=0\end{aligned}\tag{5.17}$$

Using (5.16), for (5.17) to hold at all frequencies, it must be the case that $\mathrm{E}\left(u_x(\omega)u_x(-\omega)\right)-\mathrm{E}\left(v_x(\omega)v_x(\omega)\right)=0$, implirs $\mathrm{E}\left(du_x(\omega)^2\right)=\mathrm{E}\left(dv_x(\omega)^2\right)$.

We now rewrite the spectral representation of x_t as

$$\begin{aligned}x_t&=\int_{-\pi}^{\pi}e^{it\omega}dz_x(\omega)=\int_{-\pi}^{\pi}\left(\cos\omega t+i\sin\omega t\right)\left(du_x(\omega)-dv_x(\omega)\right)\\&\int_{-\pi}^{\pi}\cos\omega t du_x(\omega)-\int_{-\pi}^{\pi}\cos\omega t dv_x(\omega)+i\int_{-\pi}^{\pi}\sin\omega t du_x(\omega)+i\int_{-\pi}^{\pi}\sin\omega t dv_x(\omega)\end{aligned}\tag{5.18}$$

The first and fourth integrals in the third line of (5.18) are even functions, meaning $g(-y)=g(y)$, whereas the second and third are odd functions, meaning that $g(-y)=-g(y)$. Therefore

$$\begin{aligned}x_t&=\int_{-\pi}^{\pi}\left(\cos\omega t du_x(\omega)+\sin\omega t dv_x(\omega)\right)=\\&\int_{0}^{\pi}\left(\cos\omega t dU_x(\omega)+\sin\omega t dV_x(\omega)\right)\end{aligned}\tag{5.19}$$

where

$$dU_x(\omega)=du_x(\omega)+du_x(-\omega)=2du_x(\omega)\tag{5.20}$$

and

$$dV_x(\omega)=dv_x(\omega)-dv_x(-\omega)=2dv_x(\omega).\tag{5.21}$$

Example 5.5. deterministic seasonal

If $x_t = \varepsilon_t + \cos\bar{\omega}t$, then the spectral density will not exist at $\bar{\omega}$. The reason for this is that the one frequency will contribute a non eligible amount of variance at $\bar{\omega}$. In this case, dF_x will still exist; in this case $\bar{\omega}$ represents a jump point.

One way to think about the spectral density function is to define it as

$$f_x(\omega) = \alpha \frac{\sigma_\varepsilon^2}{2\pi} + \beta\delta(\omega - \bar{\omega}) \tag{5.22}$$

where $\delta(\omega - \bar{\omega})$ is the so-called Dirac delta function. This is a function with the properties i. $\delta(0) = \infty$; 0 otherwise. ii. $\int_{-\pi}^{\pi} \delta(\omega - \bar{\omega}) g(\omega)\, d\omega = g(0)$. This is a so-called generalized function. The Dirac function allows one to work with jumps in the spectral distribution function.

In general, we can write the spectral density function as

$$f_x(\omega) = \alpha f(\omega) + \sum_k \beta_k \delta\left(\omega - \bar{\omega}_k\right) \tag{5.23}$$

where $f(\omega)$ is continuous.

iii. filters

Often, one works with a time series that is a transformation of another, i.e.

$$y_t = \beta(L)x_t \tag{5.24}$$

In this case, $\beta(L)$ is known as a filter. The frequency domain allows for a number of insights into the effects of filters.

The MA representation of a process may be thought of as describing a given process created by applying a filter to white noise. The following theorem describes how this filtering applies to the spectral representation of the process.

Theorem 5.2. Construction of spectral representation from white noise spectral representation.

Suppose that ε_t is a white noise process such that

$$\varepsilon_t = \int_{-\pi}^{\pi} e^{it\omega} dz_{\varepsilon}(\omega) \tag{5.25}$$

Let $x_t = \beta(\mathrm{L})\varepsilon_t$. Then

$$x_t = \int_{-\pi}^{\pi} e^{it\omega} \beta\left(e^{-i\omega}\right) dz_{\varepsilon}(\omega) \tag{5.26}$$

which means that

$$dz_x(\omega) = \beta\left(e^{-i\omega}\right) dz_{\varepsilon}(\omega) \tag{5.27}$$

in the spectral representation of x_t.

Pf.

$$\begin{aligned}
\sum_{j=-\infty}^{\infty} \beta_j \varepsilon_{t-j} &= \sum_{j=-\infty}^{\infty} \beta_j \int_{-\pi}^{\pi} e^{i(t-j)\omega} dz_{\varepsilon}(\omega) \\
&= \int_{-\pi}^{\pi} e^{it\omega} \sum_{j=-\infty}^{\infty} \beta_j e^{-ij\omega} dz_{\varepsilon}(\omega) \\
&= \int_{-\pi}^{\pi} e^{it\omega} \beta\left(e^{-i\omega}\right) dz_{\varepsilon}(\omega) \\
&= \int_{-\pi}^{\pi} e^{it\omega} dz_x(\omega)
\end{aligned} \tag{5.28}$$

where

$$z_x(\lambda) = \int_{-\pi}^{\lambda} \beta\left(e^{-i\omega}\right) dz_\varepsilon(\omega) \tag{5.29}$$

It is straightforward to verify that $dz_x(\lambda)$ possesses all the necessary requirements for the spectral representation. In particular,

$$\begin{aligned}
&i.\ \mathrm{E}\left(dz_x(\omega)\right) = \beta\left(e^{-i\omega}\right)\mathrm{E}\left(dz_\varepsilon(\omega)\right) = 0.\\
&ii.\ \mathrm{E}\left(dz_x(\omega)\overline{dz_x(\omega)}\right) = \beta\left(e^{-i\omega}\right)\beta\left(e^{i\omega}\right)\mathrm{E}\left(dz_\varepsilon(\omega)\overline{dz_\varepsilon(\omega)}\right)\\
&\quad =\beta\left(e^{-i\omega}\right)\beta\left(e^{i\omega}\right)\frac{\sigma_\varepsilon(0)}{2\pi}\\
&iii.\ \mathrm{E}\left(dz_x(\omega_i)\overline{dz_x(\omega_j)}\right) = \ \beta\left(e^{-i\omega}\right)\beta\left(e^{-i\omega}\right)\mathrm{E}\left(dz_\varepsilon(\omega_i)\overline{dz_\varepsilon(\omega_j)}\right)\text{=0 if } \omega_i \neq \omega_j.
\end{aligned} \tag{5.30}$$

The relationship between the properties of $dz_x(\omega)$ and $dz_\varepsilon(\omega)$ can be as follows. Let $\beta\left(e^{-i\omega}\right) \triangleq \gamma(\omega)e^{i\phi(\omega)}$. Observe that one can always do this by the standard properties of complex numbers.

We can first identify an effect of $\gamma(\omega)$ on the $dz_\varepsilon(\omega)$ term in the spectral representation. In particular, $\mathrm{E}\left(dz_x(\omega)\overline{dz_x(\omega)}\right)^{1/2} = \gamma(\omega)\mathrm{E}\left(dz_\varepsilon(\omega)\overline{dz_\varepsilon(\omega)}\right)$. The first feature of any filter is that changes the length of $dz_\varepsilon(\omega)$ by $\gamma(\omega)$. This is referred to as the gain of the filter at ω and illustrates how the filter alters the variance contributions of different frequencies.

Second, consider the effect of $e^{i\phi(\omega)}$ on $e^{it\omega}$. From the form of the spectral representation, the effect is to alter the complex exponential in the sense that $e^{it\omega}e^{i\phi(\omega)} = e^{i(t\omega+\phi(\omega))}$. This shifts the sine and cosine functions by $\phi(\omega)$. This is called the phase shift.

There is nothing in this argument that does not immediately generalize to any filter. Hence for $y_t = \beta(L)x_t$

$$y_t = \int_{-\pi}^{\pi} e^{it\omega}\beta(e^{-i\omega})dz_x(\omega) \tag{5.31}$$

and one can make the same sorts of arguments about the effects of the filter.

Example 5.5. differencing

If $y_t = \left(1-L\right)x_t$, then $f_y\left(\omega\right) = \left(2-2\cos\omega\right) f_x\left(\omega\right)$

Notice that for a differenced series, $f_y\left(0\right) = 0$. This makes intuitive sense. Differencing eliminates the part of the process common to all observations.

Example 5.6. averaging

Suppose that we define $\beta\left(L\right)$ such that

$$\begin{aligned} &\beta_j = 1/\mathrm{T},\ j = 0 \ldots \mathrm{T}\text{-}1, \\ &0 \text{ otherwise} \end{aligned} \tag{5.32}$$

This means that the filter averages the x_t process. In this case, $\beta\left(e^{-i\omega}\right) = \mathrm{T}^{-1}\sum_{j=0}^{\mathrm{T}-1} e^{-ij\omega}$.

Note that $\beta\left(e^{-i\omega}\right) = 1$ if $\omega = 0$.

$$\begin{aligned} &\beta\left(e^{-i\omega}\right) = \mathrm{T}^{-1}\frac{1-e^{-i\mathrm{T}\omega}}{1\text{-}e^{-i\omega}} \quad \omega \neq 0 \\ &\beta\left(e^{-i\omega}\right) = 1 \quad \omega = 0. \end{aligned} \tag{5.33}$$

Further,

$$\beta\left(e^{-i\omega}\right)\beta\left(e^{i\omega}\right) = \mathrm{T}^{-2}\cdot\left(\frac{2-2\cos\mathrm{T}\omega}{2-2\cos\omega}\right) = \mathrm{T}^{-2}\cdot\frac{\sin^2\left(\mathrm{T}\omega/2\right)}{\sin^2\left(\omega/2\right)} \tag{5.34}$$

This is a function which converges to $I_{\omega=0}$. The spectral representation of an averaged series will therefore converge to $d_x(0)$. This makes sense; when we average all the elements of x_t, the part common to all elements is what remains; this is what the zero frequency element $d_x(0)$ captures.

The spectral density function of the averaged series is

$$f_x(\omega)\cdot\mathrm{T}^{-2}\cdot\frac{\sin^2\left(\mathrm{T}\omega/2\right)}{\sin^2\left(\omega/2\right)} \tag{5.35}$$

a function whose integral over $[-\pi,\pi]$ will converge to zero. This implies that a form of the law of large numbers holds for weakly stationary time series, so long as the spectral density is bounded at 0.

Example 5.7. band pass filter.

Take a time series x_t and suppose we wish to create a series y_t that removes the part of the Cramér representation that corresponds to those frequencies above some specified value $\bar{\omega}$, so that

$$\begin{aligned} f_y(\omega) &= f_x(\omega) \text{ if } |\omega| \le \bar{\omega} \\ f_y(\omega) &= 0 \text{ if } |\omega| > \bar{\omega}. \end{aligned} \tag{5.36}$$

This would imply that there exists a filter $\beta(L)$ such that

$$\beta\left(e^{-i\omega}\right)\beta\left(e^{i\omega}\right)=1 \text{ if } |\omega|\le\bar{\omega},\ 0 \text{ otherwise.} \tag{5.37}$$

What polynomial $\beta(\cdot)$ has this property? We construct it as follows. Let $\beta\left(e^{-i\omega}\right)=\gamma\left(\omega\right)$ which requires that $\beta\left(e^{-i\omega}\right)$ is symmetric and 2-sided. Hence, in order to recover β_j, we can use the Fourier inversion formula,

$$\beta_j=\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{I}_{|\omega|\le\bar{\omega}}e^{i\omega j}d\omega=\frac{1}{\pi}\int_{0}^{\pi}\mathrm{I}_{|\omega|\le\bar{\omega}}\cos\left(\omega j\right)d\omega=\frac{\sin\left(\bar{\omega}j\right)}{\pi j} \tag{5.38}$$

This filter was proposed by Robert Engle to allow for band spectrum regression. The idea was to allow one to analyze regressions based on the “long run” parts of various time series Notice that the filter does not preserve various temporal relationships in the data, so that filtered data will not obey restrictions that economic theory (for example) places on the original series.