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Midterm Exam – Economics 713

1. (20 points)

(*i*) State the Classic Folk Theorem. Be sure to present explicit definitions of any terminology related to payoffs that you use.

(*ii*) Prove that the conclusion of the theorem holds for any payoff vector v that corresponds to a pure action profile of the stage game (and that satisfies the requirements of the theorem).

2. (35 points)

Consider the two player normal form game G below, in which player 1 is the row player and player 2 is the column player. (Only player 1's payoffs are shown.)

	a	b	c	d
T	5, ·	3, ·	1, ·	0, ·
B	–1, ·	0, ·	4, ·	7, ·

Define the following sets:

$$J = \{v \in \mathbf{R}^2 : \ v = (u_1(T, s_2), u_1(B, s_2)) \text{ for some } s_2 \in S_2\};$$
$$K = \text{the convex hull of } J;$$
$$L(c) = \{w \in \mathbf{R}^2 : \ w_1 \leq c \text{ and } w_2 \leq c\}.$$

(*i*) Explain in game theoretic terms what it means for a vector to be an element of the set K.

(*ii*) Let c^* be the smallest number such that the point (c^*, c^*) is contained in K. What is the value of c^*? Relate this number to player 1's minmax payoff in G, explaining the reason for the relationship you describe.

(*iii*) Specify the normal vector $p^* \in \mathbf{R}^2$ and the intercept d^* of the hyperplane $H = \{v \in \mathbf{R}^2 : \ p^* \cdot v = d^*\}$ that separates the set $L(c^*)$ from the set K, choosing the vector p^* to have components that are nonnegative and sum to one.

(*iv*) Interpret the fact that $p^* \cdot v \geq d^*$ for all $v \in K$ in game theoretic terms. What conclusions can we draw about player 1's maxmin payoff in G?

(*v*) Let G_n be a two player normal form game in which player 1 has $n \geq 2$ strategies. Sketch a proof of the fact that player 1's minmax payoff in G_n and his maxmin payoff in G_n are equal. (When G_n is zero-sum, this result is an alternate form of the Minmax Theorem.)

3. (15 points)

Let G be a two player normal form game with action sets A_1 and A_2 and payoff functions u_1 and u_2. Assume that in this game, neither player is indifferent between any pair of action profiles. Let Γ be an extensive form game in which (a) player 1 chooses an action from A_1, and then (b) player 2, having observed player 1's choice, chooses an action from A_2; payoffs in Γ are also described by u_1 and u_2.

(*i*) Suppose that a^* is a pure Nash equilibrium of G. Prove that player 1's payoff must be at least $u_1(a^*)$ in any subgame perfect equilibrium of Γ. This property is sometimes called "first mover advantage".

(*ii*) Even in the normal form game G, we know that *in equilibrium*, player 2 *correctly anticipates* player 1's move. Thus, in both games, player 2 knows what player 1 will do in equilibrium, and she responds optimally. Given this, why is it still possible for the equilibrium outcomes of G and Γ to differ? Answer this question by contrasting player 1's equilibrium incentives in G with his equilibrium incentives in Γ. (Hint: A little notation will go a long way.)

4. (30 points)

Arthur and Beatrix compete in a race. At the start of the race, both players are 6 steps away from the finish line. Who gets the first turn is determined by a toss of a fair coin; the players then alternate turns, with the results of all previous turns being observed before the current turn occurs.

During a turn, a player chooses from these four options:

Do nothing at cost 0;
Advance 1 step at cost 2;
Advance 2 steps at cost 7;
Advance 3 steps of at cost 15.

The race ends when the first player crosses the finish line. The winner of the race receives a payoff of 20, while the loser gets nothing. Finally, there is discounting: after each turn, payoffs are discounted by a factor of δ, where δ is less than but very close to 1.

(*i*) Find all subgame perfect equilibria of this game. (Hint: In all subgame perfect equilibria, a player's choice at a decision node only depends on the number of steps he has left and on the number of steps his opponent has left. To help take advantage of this you might want to write down a table.)

(*ii*) Suppose that Arthur wins the coin toss. Compare his equilibrium behavior with his optimal behavior in the absence of competition. Provide intuition for any similarities or differences you find.