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Midterm Solutions – Economics 713

1. Dave's preferences fail to satisfy the independence axiom. Let $q = (0, 1, 0)$. Then $p^3 = \frac{1}{2}p^1 + \frac{1}{2}q$, and $p^4 = \frac{1}{2}p^2 + \frac{1}{2}q$. Hence, the independence axiom implies that Dave's preference between p^3 and p^4 must be the same as his preference between p^1 and p^2, but the question states that this is not the case. (Alternatively, one could show that if the utilities u_a, u_b, and u_c satisfy $u_a = \frac{1}{2}u_b + \frac{1}{2}u_c$, they must also satisfy $\frac{1}{2}u_a + \frac{1}{2}u_b = \frac{3}{4}u_b + \frac{1}{4}u_c$. This implies that Dave's preferences do not admit an expected utility representation, and so cannot satisfy all three axioms.)

2. The extensive form perfect equilibria of Γ are those strategy profiles which correspond to normal form perfect equilibria of the agent normal form game $A(\Gamma)$. Moreover, it is known that all extensive form perfect equilibria of Γ correspond to sequential equilibria of Γ. But if no player controls more than one information set in Γ, then the agent normal form $A(\Gamma)$ is identical to the reduced normal form $G(\Gamma)$. Therefore, perfect equilibria of $G(\Gamma)$ correspond to perfect equilibria of Γ, and hence to sequential equilibria.

3. All beliefs for player 3 are consistent. Let ε_B, ε_C, and ε_D be the weights placed on B, C, and D in the perturbed strategy profiles. Then $\mu^\varepsilon(x) = \varepsilon_B/(\varepsilon_B + \varepsilon_C\varepsilon_D)$. Assume that $\varepsilon_C = \varepsilon_D = \varepsilon$. If $\varepsilon_B = \varepsilon$, then $\mu^\varepsilon(x) = 1/(1 + \varepsilon) \to 1$, so $\mu(x) = 1$ is consistent. If $\alpha \in (0, 1)$ and $\varepsilon_B = \varepsilon^2\alpha/(1 - \alpha)$, then $\mu^\varepsilon(x) = \alpha$, so $\mu(x) = \alpha$ is consistent. Finally, if $\varepsilon_B = \varepsilon^3$, then $\mu^\varepsilon(x) = \varepsilon/(\varepsilon + 1) \to 0$, so $\mu(x) = 0$ is consistent.

4. (i) To begin, observe that B is strictly dominated by $\frac{7}{10}A + \frac{3}{10}C$, and so is not played in any Nash equilibrium. If 1 plays A, then 2 plays b, which makes 1 prefer C, which is a contradiction. If 1 plays C, then 2 plays a, which makes 1 prefer A , which is again a contradiction. Thus, in any Nash equilibrium, 1 mixes between A and C. He is willing to do this if

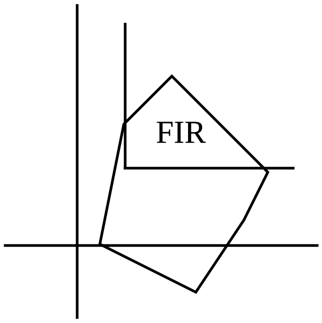
$$\begin{aligned} & 7\sigma_2(a) + 4\sigma_2(b) + 2\,\sigma_2(c) = 2\sigma_2(a) + 8\sigma_2(b) + 2\,\sigma_2(c) \\ \Leftrightarrow\quad & 5\sigma_2(a) = 4\sigma_2(b). \end{aligned}$$

Hence, 2 must either play c with probability 1 or must put positive probability on both a and b. Given that 1 does not play B, such behaviors by 2 are optimal precisely when 1 plays $\frac{1}{4}A + \frac{3}{4}C$. Therefore, the Nash equilibria of G are the strategy profiles of the form $(\frac{1}{4}A + \frac{3}{4}C, \sigma_2)$, where σ_2 satisfies $5\sigma_2(a) = 4\sigma_2(b)$.

(ii) Every feasible, individually rational payoff vector is attainable in a Nash equilibrium of $G^\infty(\delta)$ for a high enough value of δ. A payoff vector is individually rational if each component of the vector strictly exceeds the corresponding player's minmax value. It is easy to see that $\underline{v}_1 = 2$, which is ensured if player 2 chooses c. On the other hand, $\underline{v}_2 = \frac{13}{4}$, which is ensured if player 1 chooses $\frac{1}{4}B + \frac{3}{4}C$.

(To determine the strategy which 1 uses to minmax 2, we must check the payoff that 2 obtains by playing a best response at each vertex of his best response correspondence. Player 1's three pure strategies are at vertices, and player 2's best responses to these strategies yield payoffs of 7, 4, and 5. Strategy a is a best response on one side of the segment between $\frac{1}{4}A + \frac{3}{4}C$ and $\frac{1}{4}B + \frac{3}{4}C$, and strategy b is a best response on the other side of the segment; c is a best response only against $\frac{1}{4}A + \frac{3}{4}C$. Thus, $\frac{1}{4}A + \frac{3}{4}C$ and $\frac{1}{4}B + \frac{3}{4}C$ are the only two mixed strategies we need consider, and 2's optimal payoffs at these are 4 and $\frac{13}{4}$.)

The feasible equilibrium payoffs are sketched below.



(iii) If σ is a strategy profile for $G^\infty(\delta)$ which yields payoffs outside of the closure of the set FIR, then at least one player gets strictly less than his minmax value. Suppose without loss of generality that player 1 is such a player, obtaining a payoff of less than $\underline{v}_1$ under σ. If instead of playing σ_1 in response to σ_2 player 1 played his myopic best response after every history, he would obtain no less than his minmax value $\underline{v}_1$ in the repeated game. Thus, this strategy is a profitable deviation from σ_1, and so σ is not a Nash equilibrium.

5. (i) There are two sets of pooling equilibria.

(I) $\sigma_1(G|\theta_s) = \sigma_1(G|\theta_t) = \sigma_1(G|\theta_f) = 1$ and $\sigma_2(B|G) = 1$ along with

either [$\sigma_2(A|S) = 1$ with $\mu(\theta_s|S) \geq \mu(\theta_t|S) + 2\mu(\theta_f|S)$]

or [$\sigma_2(A|S) \geq \frac{2}{3}$ with $\mu(\theta_s|S) = \mu(\theta_t|S) + 2\mu(\theta_f|S)$].

(II) $\sigma_1(S|\theta_s) = \sigma_1(S|\theta_t) = \sigma_1(S|\theta_f) = 1$ and $\sigma_2(L|S) = 1$ along with

either [$\sigma_2(A|G) = 1$ with $\mu(\theta_s|G) \geq \mu(\theta_t|G)$ and $\mu(\theta_s|G) \geq 2\mu(\theta_f|G)$]

or [$\sigma_2(A|G) \geq \frac{2}{3}$ and $\sigma_2(C|G) = 0$ with $\mu(\theta_s|G) = \mu(\theta_t|G)$ and $\mu(\theta_f|G) \leq \frac{1}{5}$]
or [$\sigma_2(A|G) \geq \frac{2}{3}$ and $\sigma_2(B|G) = 0$ with $\mu(\theta_s|G) = 2\mu(\theta_f|G)$ and $\mu(\theta_t|G) \leq \frac{2}{5}$]
or [$\sigma_2(B|G) \leq \frac{1}{3}$ and $\sigma_2(C|G) \leq \frac{1}{3}$ with $(\mu(\theta_s|G), \mu(\theta_t|G), \mu(\theta_f|G)) = (\frac{2}{5}, \frac{2}{5}, \frac{1}{5})$].

(iii) The equilibria in set (I) satisfy all signaling game refinements, while the equilibria in set (II) fail equilibrium dominance. To see why the first claim is true, note that the spineless type and the ferocious type could benefit from switching from G to S ($D(S) = \{\theta_t\}$, so $T(S) - D(S) = \{\theta_s, \theta_f\}$). Player 2's best response to the spineless type is to attack, and if she does this neither θ_s nor θ_f benefits from deviating ($u_1^*(\theta_s) = 0 > -1 = u_1(\theta_s, S, A)$ and $u_1^*(\theta_f) = -1 > -3 = u_1(\theta_f, S, A)$).

To establish the second claim, observe that the only types who want to deviate from the second set of equilibria are the tough and ferocious types. Player 2's best responses to beliefs putting all weight on these types are bowing, crying, and randomizing between these options. Neither the tough nor the ferocious type is better off deviating against all of player 2's best responses (for example, the tough type is better off deviating if this will cause player 2 to bow but not if it will cause her to cry), and for this reason the equilibria satisfy the intuitive criterion. However, for each best response of player 2, at least one of the types strictly prefers to deviate, and so the equilibria fail equilibrium dominance. More formally,

$D(G) = \{\theta_s\}$, so $T(G) - D(G) = \{\theta_t, \theta_f\}$.
$MBR(\{\theta_t, \theta_f\}, G) = \{\sigma_2(\cdot|G)\colon\ \sigma_2(A|G) = 0\}$.
If $\sigma_2(A|G) = 0$ and $\sigma_2(B|G) > \frac{1}{3}$, then $u_1^*(\theta_t) = 0 < u_1(\theta_t, G, \sigma_2)$,
while if $\sigma_2(A|G) = 0$ and $\sigma_2(C|G) > \frac{1}{3}$, then $u_1^*(\theta_f) = 0 < u_1(\theta_f, G, \sigma_2)$.

Therefore, the equilibria in set (II) fail equilibrium dominance.