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Midterm Exam Solutions

1. Behavior strategies are more appropriate here. In the extensive form drawn, player 1 is supposed to forget his initial behavior before choosing an action at his second information set. However, by playing a mixed strategy (e.g., $\frac{1}{2}Ll + \frac{1}{2}Rr$), player 1 can correlate his behavior at the two information sets. Doing so enables him to remember his choice between L and R when deciding between l and r. Since he is not supposed to be able to remember this choice, behavior strategies, which utilize independent randomizations at each information set, are preferable.

2. Each player's unique rationalizable strategy is zero. Thus, the prediction that all players choose zero can be justified by common knowledge of rationality. (In fact, if all long enough (length ≤ 1000 is far more than enough) statements of the form "i knows that j knows that …. that k is rational" are true, that is actually sufficient.)

The proof is as follows.. Since the highest possible entry is 1000, the highest possible target integer is 900. Therefore, because choosing the target integer is the only way to obtain a positive payoff, it is never a best response to choose an integer higher than 900. So no one does this.

Hence, under CKR, everyone realizes that no one will choose an integer higher than 900. Once these strategies are removed, the target integer can not exceed 810, and so no one chooses an integer higher than this.

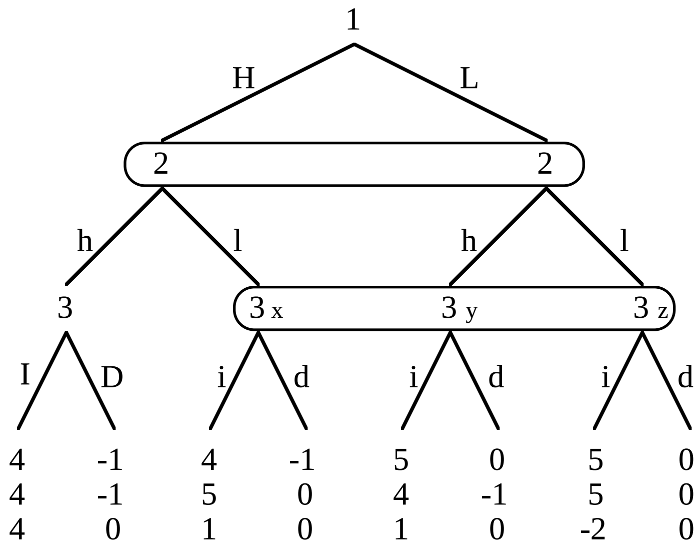
Iterating in this way, we see that once players realize that no one will choose an integer higher than $z > 0$, no will choose an integer higher than $\text{int}(\frac{9}{10}z)$, which by the rounding assumption is strictly less than z. Thus, as long as $z > 0$, the next iteration of removing of strategies which are never a best response will lower the upper bound by at least 1. We therefore conclude that each player's unique rationalizable strategy is to choose zero. (It follows that everyone choosing zero is the game's unique Nash equilibrium.)

3. A strategy which is not pure cannot be strictly dominant. To see this, suppose that σ_i is a strictly dominant strategy whose support $C(\sigma_i) \subset S_i$ is not a singleton. Then σ_i is not equivalent to any $s_i \in C(\sigma_i)$. Therefore, by assumption, σ_i dominates each $s_i \in C(\sigma_i)$: $u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i})$ for all s_{-i}. But then it follows that for each s_{-i},

$$u_i(\sigma_i, s_{-i}) = \sum_{s_i \in C(\sigma_i)} u_i(s_i, s_{-i})\sigma_i(s_i) < \sum_{s_i \in C(\sigma_i)} u_i(\sigma_i, s_{-i})\sigma_i(s_i) = u_i(\sigma_i, s_{-i}),$$

which is a contradiction.

4. (i)



(ii) The only pure sequential equilibrium is (*L*, *l*, (*I*, *d*)). There are additional pure perfect Bayesian equilibria: ((*H*, *h*, (*I*, *d*)), with μ satisfying $\mu(z) \geq \frac{1}{3}$).

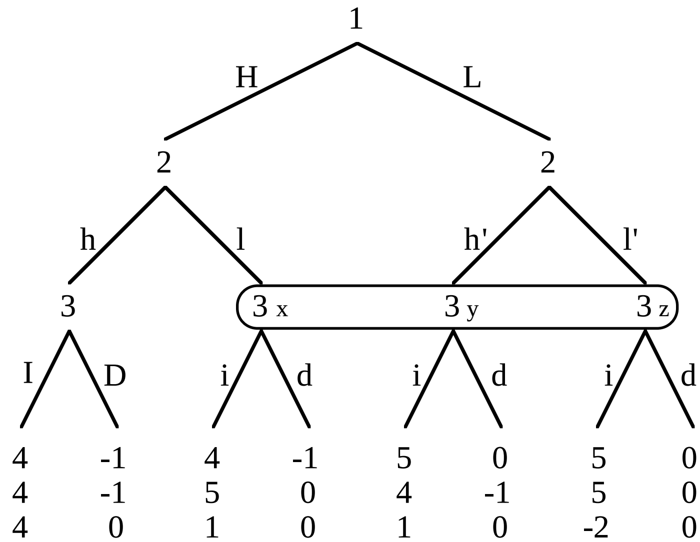
Clearly, 3 must play *I* in any PBE. We divide the remaining analysis into cases according to the behavior of players 2 and 3.

(·, *i*) Then 2 plays *l*, which implies that 1 plays *L*, which implies that 3 plays *d* – a contradiction

(*h*, *d*) Since 2 plays *h*, 1 plays *H*, making *h* a best response. Since 3's right information set is unreached, this is a PBE as long as $\mu(z) \geq \frac{1}{3}$. However, parsimony implies that $\mu(z) = 0$, so this is not sequential.

(*l*, *d*) Then 1 plays *L*. This makes *l* and *d* best responses, so this is a sequential equilibrium.

(iii)



(iv) The sequential equilibria of this game are:

$((H, (h, l'), (I, d))$, with μ satisfying $\mu(y) = 0$ and $\mu(z) \geq \frac{1}{3}$);

$((H, (h, l'), (I, \sigma_3(d) \geq \frac{1}{5}))$, with μ satisfying $\mu(y) = 0$ and $\mu(z) = \frac{1}{3}$).

The perfect Bayesian equilibria are the same, minus the restrictions that $\mu(y) = 0$.

Observe that in any PBE, 3 plays I and 2 plays l'. We divide the remaining analysis into cases according to 3's behavior at her right information set.

3 plays i: Implies that 2 plays l, which implies that 1 plays L, which implies that 3 plays d – a contradiction.

3 plays d: Implies that 2 plays h, which implies that 1 plays H. 3 is willing to play d if $\mu(z) \geq \frac{1}{3}$. This defines a set of PBE. Moreover, the consistent beliefs are those with $\mu(y) = 0$, so this additional restriction yields a set of sequential equilibria.

(Proof of consistency: $\mu(y) = 0$ by parsimony. To get $\mu(z) = 1$, set $\varepsilon_L^k = \frac{1}{k}$ and $\varepsilon_l^k = \varepsilon_{h'}^k = \frac{1}{k^2}$. To get $\mu(z) = c \in (0, 1)$, set $\varepsilon_L^k = \varepsilon_{h'}^k = \frac{1}{k}$ and $\varepsilon_l^k = \frac{1}{k}(1 - c - \frac{1}{k})/c(1 - \frac{1}{k})$.)

3 mixes ($\Rightarrow \mu(z) = \frac{1}{3}$):

Suppose 3 is unreached. Then 1 plays H and 2 plays h. These both require that $\sigma_3(d) \geq \frac{1}{5}$. These are PBE as long as $\mu(z) = \frac{1}{3}$; for sequential equilibrium, we have the additional consistency requirement that $\mu(y) = 0$.

Suppose 3 is reached and 2 plays h. Then 1 plays H, so 3 is not reached – a contradiction.

Suppose 3 is reached and 2 plays l. Then 1 plays L, so $\mu(z) = 1$, which is a contradiction.

Suppose 3 is reached and 2 mixes at her left information set.

Then $\sigma_3(d) = \frac{1}{5}$. Thus, 1 prefers H to L if $\sigma_2(h) \geq 1$. Since $\sigma_2(h) < 1$ by assumption, 1 plays L, and so $\mu(z) = 1$, which is a contradiction.

(v) In the simultaneous move game, the unique sequential equilibrium outcome is the low effort/no investment after low effort outcome (0, 0, 0), while in the sequential move game, the unique sequential equilibrium outcome is high effort outcome (4, 4, 4).

In the simultaneous move game, the good outcome cannot be obtained because the venture capitalist cannot commit to punishing low effort if he expects high effort: if a deviation occurs, consistency requires him to believe that one entrepreneur is still choosing high effort, which means that investing is still a best response. On the other hand, when moves are sequential, then the second entrepreneur will choose low effort whenever the first one does. Thus, if both entrepreneurs are supposed to choose high effort in equilibrium, the venture capitalist can believe that a deviation means that both chose low effort, which enables him to commit to punishing deviations.

In contrast, the bad outcome cannot be a sequential equilibrium in the sequential move game, as punishment of low effort leads player 2 to choose high effort at her left information set, which leads player 1 to choose high effort as well. In the simultaneous move game, if L, l and d are supposed to be played, player 2 does not benefit from deviating to h: since she does not observe player 1's choice, she must play her best response l to his expected play of L. Thus, sequence of deviations which breaks the bad equilibrium in the sequential move case is impossible here.