

Final Exam Solutions

1. Since we are only assuming common knowledge of rationality, we need to look for the rationalizable strategies. Since this is a two player game, it is equivalent to look for strategies which survive iterated removal of strictly dominated strategies. It turns out to be easiest to first use iterated strict dominance to get rid of pure strategies, and then to get rid of mixed strategies which are never a best response.

We can first remove c , which is strictly dominated by d . After this, we can remove A , which is then strictly dominated by B . Then, we can remove a , which is strictly dominated by $\frac{2}{5}b + \frac{3}{5}d$. No other pure strategies can be removed.

Since pure strategies B and C are both best responses to b , these strategies and all mixtures between them are rationalizable. If player 1 plays B , then d and e are both best responses for player 2; if $\sigma_1(B) \in (\frac{3}{5}, 1)$, then d is the unique best response; if $\sigma_1(B) = \frac{3}{5}$, then b and d are both best responses; if $\sigma_1(B) \in [0, \frac{3}{5})$, b is the unique best response. Thus, it is never a best response for player 2 to put positive weight on both b and e ; but all other mixtures between b , d , and e are best responses. Thus:

Rationalizable strategies for 1: All mixtures of B and C .

Rationalizable strategies for 2: All mixtures of b , d , and e which do not put positive weight on both b and e .

2. For this to be a subgame perfect equilibrium, we need to check that no one wants to deviate from any of the three possible future play paths. One of these paths is the punishment path $\{(C, c), (C, c), \dots\}$. By deviating today only, a player can only lower his payoff today, and cannot affect his payoff in the future. Hence, staying on the path is optimal.

The initial play path is $\{(A, a), (B, b), (A, a), (B, b), \dots\}$, which yields the payoff stream $\{3, 2, 3, 2, \dots\}$ for both players. The easiest way to compute the value of this payoff stream is to find the value c such that $\{3, 2, 3, 2, \dots\}$ is worth the same as $\{c, c, c, c, \dots\}$, which we know is worth c . It's not hard to see that the c that yields indifference between the two payoff streams is the one which solves $3 + \delta(2) = c + \delta(c)$, which holds when $c = (3 + 2\delta)/(1 + \delta)$. (One can also compute this value by evaluating $\{3, 2, 3, 2, \dots\}$ as the sum of $\{3, 0, 3, 0, \dots\}$ and $\{0, 2, 0, 2, \dots\}$.)

If either player deviates in the initial period, he can obtain 5 in that period and obtains 1 thereafter. Hence, following the equilibrium is optimal when

$$\begin{aligned} (3 + 2\delta)/(1 + \delta) &\geq (1 - \delta)5 + \delta(1) \\ \Leftrightarrow 4\delta^2 + \delta - 2 &\geq 0 \end{aligned}$$

$$\Leftrightarrow \delta \geq (\sqrt{33} - 1)/8.$$

The other play path to consider is $\{(B, b), (A, a), (B, b), (A, a), \dots\}$, which yields the payoff stream $\{2, 3, 2, 3, \dots\}$ for both players. This payoff stream is worth $(2 + 3\delta)/(1 + \delta)$. Deviating yields a payoff of 3 today and 1 thereafter. Hence, following the equilibrium is optimal when

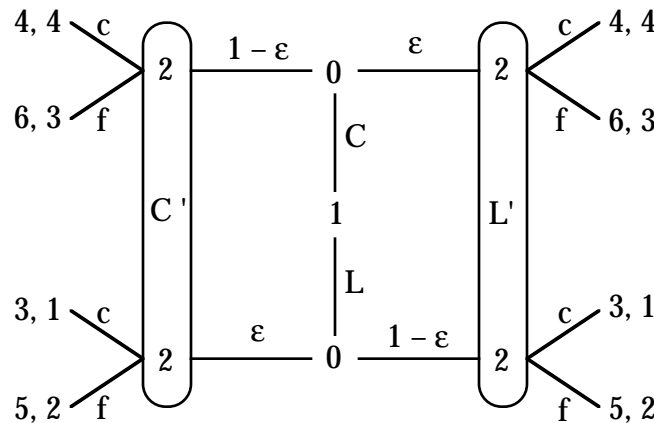
$$\begin{aligned} (2 + 3\delta)/(1 + \delta) &\geq (1 - \delta)3 + \delta(1) \\ \Leftrightarrow 2\delta^2 + 2\delta - 1 &\geq 0 \\ \Leftrightarrow \delta &\geq (\sqrt{3} - 1)/2. \end{aligned}$$

Observe that $(\sqrt{33} - 1)/8 > (\sqrt{3} - 1)/2$ (to see this, notice that $(\sqrt{33} - 1)/8 > \frac{1}{2} > (\sqrt{3} - 1)/2$). Hence, the strategy profile is sustainable whenever $\delta \geq (\sqrt{33} - 1)/8$.

3. The statement is true. Every normal form game has a perfect equilibrium, and all perfect equilibria are Nash equilibria which do not use weakly dominated strategies.

4. This example is due to Bagwell (1995).

- (i) The unique Nash equilibrium is (C, c) .
- (ii) The unique SPE is $(L, (c, f))$, yielding payoffs of $(5, 2)$.
- (iii) The extensive form is as follows:



To find the pure sequential equilibria of this game, consider the two possible pure strategies of player 1. If player 1 plays L , then player 2 always believes that L has been played, and so always plays f ; this makes player 1 prefer to deviate to C . On the other hand, if player 1 chooses C , then player 2 always believes that C has been played, and so always plays c ; if she does this, then C is optimal for player 1. Hence $(C, (c, c))$ is the unique pure sequential equilibrium.

(iv) All information sets are reached with positive probability however the players behave. Therefore, consistency reduces to the requirement that beliefs are Bayesian, and so all perfect Bayesian equilibria are sequential equilibria.

(v) Suppose that player 2 plays f after seeing the signal L' . We know from the argument in part (iii) that there are no equilibria of this kind in which player 1 chooses a pure strategy, so player 1 must be mixing. For him to be willing to do this, he must receive the same payoff from playing C or L . If we let p be the probability that 2 plays c if she sees the signal C' , then 1 is indifferent when

$$\begin{aligned} u_1(C) &= u_1(L) \\ \Leftrightarrow 4((1 - \varepsilon)p) + 6(1 - (1 - \varepsilon)p) &= 3(\varepsilon p) + 5(1 - \varepsilon p) \\ \Leftrightarrow p &= 1/(2 - 4\varepsilon). \end{aligned}$$

Since $\varepsilon \in (0, \frac{1}{4})$, $p \in (\frac{1}{2}, 1)$, so player 2 is randomizing after seeing C' . For 2 to be willing to do this, she must be indifferent between c and f in this event. This is true when

$$\begin{aligned} u_2(c \mid C') &= u_2(f \mid C') \\ \Leftrightarrow 4\sigma_1(c)(1 - \varepsilon) + 1(1 - \sigma_1(c))\varepsilon &= 3\sigma_1(c)(1 - \varepsilon) + 2(1 - \sigma_1(c))\varepsilon \\ \Leftrightarrow \sigma_1(c) &= \varepsilon. \end{aligned}$$

We have now specified behavior at all information sets. To ensure that the specified behavior is an equilibrium, we need to check that f is optimal for player 2 if she observes L' . This will be true if

$$\begin{aligned} u_2(c \mid L') &\leq u_2(f \mid L') \\ \Leftrightarrow 4\sigma_1(c)\varepsilon + 1(1 - \sigma_1(c))(1 - \varepsilon) &\leq 3\sigma_1(c)\varepsilon + 2(1 - \sigma_1(c))(1 - \varepsilon) \\ \Leftrightarrow 4\varepsilon^2 + (1 - \varepsilon)^2 &\leq 3\varepsilon^2 + 2(1 - \varepsilon)^2 \\ \Leftrightarrow \varepsilon &\leq \frac{1}{2}. \end{aligned}$$

Since $\varepsilon < \frac{1}{4}$, player 2 (strictly) prefers to play f after seeing L' . Thus, the strategy profile we have constructed is a sequential equilibrium.

Sequential eq'm: $(\varepsilon C + (1 - \varepsilon)L, (1/(2 - 4\varepsilon)c + ((1 - 4\varepsilon)/(2 - 4\varepsilon))f$ (vs. C'), f (vs. L')).