Notes on Large Deviations in Economics and Finance

Noah Williams
Princeton University
and NBER

http://www.princeton.edu/∼noahw

Introduction

- **What is large deviation theory?**
- Loosely: a theory of rare events. Typically provide exponential bound on probability of such events and characterize them.
- Large growth area in math/applied probability in last ≈ 20 years. Seeing some applications in economics.
- **Why is it useful?**
- Often interested in characterizing extreme events in themselves (crashes, failures, busts).
- Bounding probability of extreme events can characterize likelihood of "typical" events.

Formal Definition of LDP

Consider $\{Z^\epsilon\}$ on $(\Omega, \mathcal{F}, P)$, taking values in $\mathcal{X}$.

A *rate function* $S : \mathcal{X} \to [0, \infty]$ has the property that for any $M < \infty$ the level set $\{x \in \mathcal{X} : S(x) \leq M\}$ is compact.

Definition: A sequence $\{Z^\epsilon\}$ satisfies a *large deviation principle* (LDP) on $\mathcal{X}$ with *rate function* S and *speed* ϵ if:

1. For each closed subset F of $\mathcal{X}$,:

$$\limsup_{\epsilon \to 0} \epsilon \log P\{Z^\epsilon \in F\} \leq - \inf_{x \in F} S(x).$$

2. For each open subset G of $\mathcal{X}$,:

$$\liminf_{\epsilon \to 0} \epsilon \log P\{Z^\epsilon \in G\} \geq - \inf_{x \in G} S(x).$$

Interpretation

- Under regularity, upper & lower hold with equality:

$$\lim_{\epsilon \to 0} \epsilon \log P\{Z^\epsilon \in F\} = -\inf_{x \in F} S(x) = -\bar{S}.$$

i.e. ~~log~~ $P\{Z^\epsilon \in F\} \approx C \exp(-\bar{S}/\epsilon)$.

- Note F is independent of ϵ. If $Z^\epsilon \to \bar{Z} \notin F$ then $P \to 0$.
- Compare w/CLT. Let $\epsilon = 1/n$, $Z_n = \sum_i X_i/n$, X_i, i.i.d., $a > 0$.

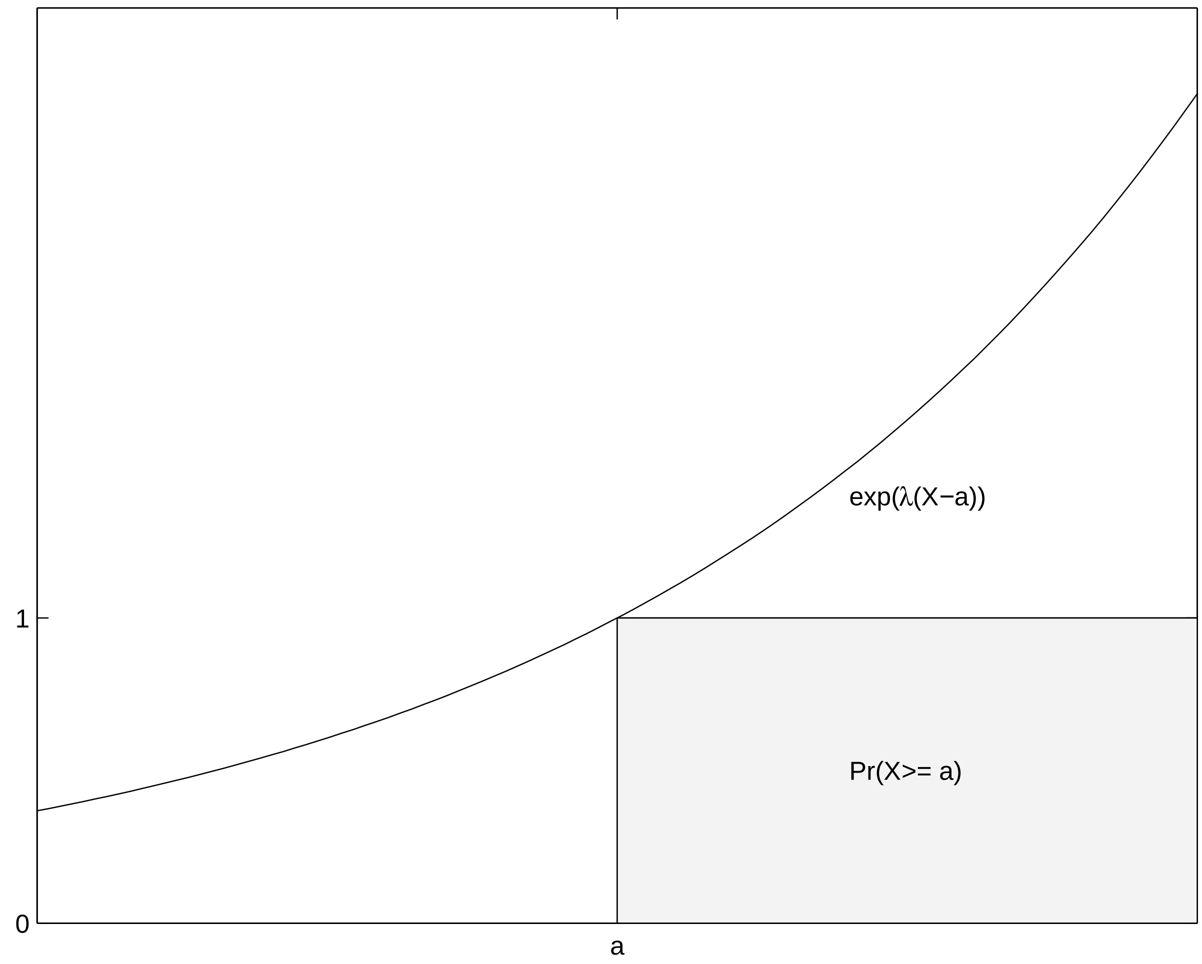
$$\begin{aligned} \text{CLT} &: \quad P\left(|Z_n - \bar{Z}| \geq \frac{a}{\sqrt{n}}\right) \to 2\Phi(-a) \\ \text{LDP} &: \quad P\left(|Z_n - \bar{Z}| \geq a\right) \to C \exp(-\bar{S}_a n) \end{aligned}$$

Contexts & Applications

- Performance of decision rules (and estimators, robustness)
- Performance of portfolios (outperforming benchmarks, large losses)
- Proofs of convergence (implies weak LLN)
- Rare events: extreme or non-equilibrium outcomes (large business cycles, escape dynamics)
- Transitions between equilibria and equilibrium selection (evolutionary games)

Chebyshev

$$P(X_i \in [a, \infty)) \leq E\left[\exp\left(\lambda\left(X_i - a\right)\right)\right]$$



Basic idea: Cramer's theorem

Take $Z_n = \sum X_i/n$, X_i i.i.d. Fix $a > E(X)$. Define:

$$\begin{aligned} H(\lambda) &= \log E \exp(\langle \lambda, X \rangle) \\ S(x) &= \sup_{\lambda} \{ \langle \lambda, x \rangle - H(\lambda) \} \end{aligned}$$

Then Z_n satisfies an LDP with rate function I and speed $1/n$.

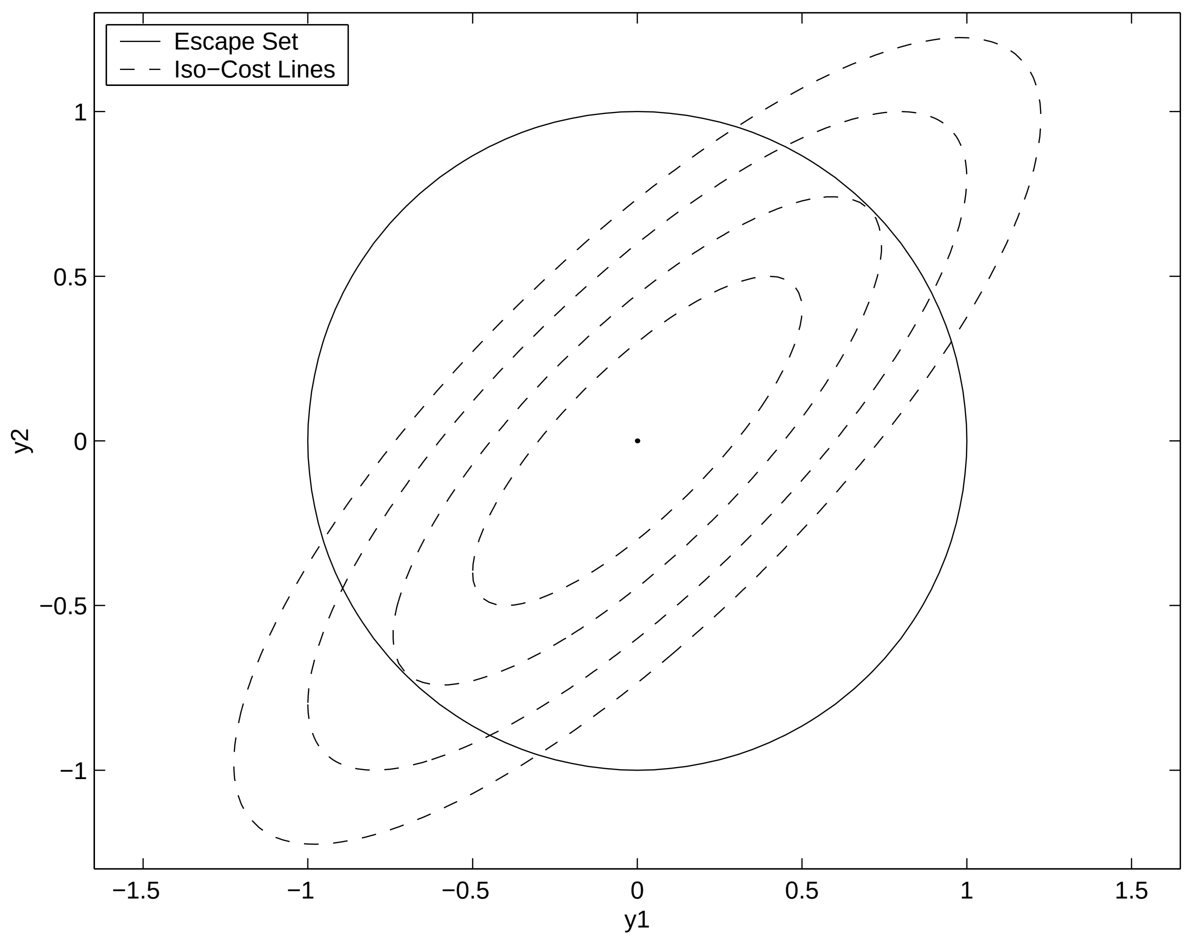
Sketch of proof for upper bound:

Consider scalar case, and fix $a > EX$.

$$\begin{aligned} P(Z_n \in [a, \infty)) &= E\left[1\{Z_n - a \geq 0\}\right] \\ (\mathit{Chebyshev}) &\leq E\left[\exp\left(n\lambda\left(X_i - a\right)\right)\right] \text{ for all } \lambda \geq 0 \\ &= \exp(-n\lambda a) \prod_{i=1}^{n} E \exp(\lambda X_i) \\ \log P(Z_n \in [a, \infty)) &\leq -n \sup_{\lambda > 0} \left[\lambda a - \log E \exp(\lambda X_i)\right] = -nS(a) \end{aligned}$$

Illustration: Bivariate $N(0, \Sigma)$

$$\overline{S} = \min_x \frac{1}{2} x' \Sigma^{-1} x \quad \text{s.t. } ||x|| \geq a$$



Some notes

- Lower bound is also key: uses (exponential) change of measure.
- Low probability event $\rightarrow$ deterministic optimization problem.
- Let $x^* = \arg\min_{x\in F} S(x)$. Then for all $\delta > 0$:

$$\lim_{n\to\infty} P\left(|Z_n - x^*| < \delta \;||Z_n \in F\right) = 1.$$

- Tom's favorite quotes:
  1. Rare events are exponentially rare.
  2. "If an unlikely event occurs, it is very likely to occur in the most likely way."

Some Extensions

- Gärtner-Ellis: varying statistics. Let $Z^\epsilon \sim \mu^\epsilon$. Define:

$$H(\lambda) = \lim_{\epsilon \to 0} \epsilon \log E \exp\left(\langle \lambda/\epsilon, Z^\epsilon \rangle\right)$$

  Provided limit exists, rate function S is Legendgre of H.

- Sanov's Theorem: Gives LDP for empirical measures of i.i.d. sample. Rate function is relative entropy w.r.t. original meas.

- Donsker-Varadhan: extend to empirical measures of Markov

Some Relationships

- Entropy: information theory, statistical mechanics
- Perturbation methods and sharp LD:

$$\log P \approx C + \epsilon \bar{S}_1 + \epsilon^2 \bar{S}_2 + \dots$$

- Risk sensitivity and (asymptotic) robustness via Varadhan.
  Let L be a (continuous) loss function $L : \mathcal{X} \to \mathbb{R}$.

$$\lim_{\epsilon \to 0} \epsilon \log E[\exp(L(Z^\epsilon)/\epsilon)] = \sup_{x \in \mathcal{X}} \{L(x) - S(x)\}$$

- Idempotent probability/Possibility theory/fuzzy measures.
  Puhalskii: "Deviabilities" $\Pi(F) = \sup_{x \in F} \exp(-S(x))$.
  A nonadditive measure: $\Pi(A \cup B) = \Pi(A) \vee \Pi(B)$.

$$\text{LDP}: \quad \lim_{\epsilon \to 0} \left(\int_{\mathcal{X}} h(x)^{\frac{1}{\epsilon}} d\mu^\epsilon(x) \right)^\epsilon = \sup_{x \in \mathcal{X}} h(x) \Pi(x)$$

Applications: i.i.d. or cross-section

- Hypothesis testing: Say accept hypothesis when (sample) log likelihood ratio above a cutoff. Asymptotics of type I and II errors for fixed cutoffs determined by LDP (Chernoff).
- Krasa-Villamil (1992) - Suppose a bank pools (i.i.d.) risks. Probability of default decreases exponentially with size. If cost increases slower w/size, interior optimal size.
- Dembo-Deuschel-Duffie (2003) - Portfolio consists of positions subject to losses Z_i at exposures U_i. Evaluate portfolio losses $L_n = \sum_i Z_i U_i$, characterize $P(L_n \geq nx)$.
- Stutzer (2003): Minimize probability that portfolio will grow at rate lower than some benchmark. Use LDP to derive long run decay rate. Show that yields result similar to max utility of wealth with endogenous preference power.

Convergence of Exchange Economies

- Random excess demands $z_i(\omega, p) : \Omega \times \mathbb{R}^d \mapsto \mathbb{R}^d$. Aggregate: $Z_n(\omega, p) = \sum_{i=1}^n z_i(\omega, p)$. Prices: $\pi_n(\omega) = \{p : Z_n(\omega, p) = 0\}$.
- "Expectation Economy": $z(p) = \lim_{n\to\infty} Z_n(\omega, p)/n$. Prices π.
- Define $H(\lambda, p)$, $S(x, p)$ as in Gärtner-Ellis. Then under some continuity conditions, Nummelin (2000) shows that $\{\pi_n\}$ satisfies an LDP with speed $1/n$ and rate function $S(0, p)$.
- Some simple conditions insure π is nonempty, gives asymptotic existence (a.s. equilibrium exists eventually) and convergence to expectation economy.

Sample Path Large Deviations

- So far: LDP for single realization or sample average.
  Now: Realization of event in a (dynamic) sample path.
- Most complete theory: Freidlin-Wentzell for diffusions
- Simplest to present: discrete time analogues
  $x^{\sigma}_{t+1} = g^{\sigma}(x_t) + \sigma W_{t+1}$. $W_t \sim N(0,1)$, $g^{\sigma} \to g^0$, $x^* = g^0(x^*)$.
- Gärtner-Ellis: given $x_t = x$, 1-Step LDP at rate σ^2 for x_{t+1} with rate function: $S_1(x,y) = \frac{1}{2}(y - g^0(x))^2$
- For multi-step rate function, sum up the one-step transitions:

$$S(x,y,T) = \inf_{\{x_t\}_{t=0}^{T}} \frac{1}{2} \sum_{t=0}^{T-1} (x_{t+1} - g^0(x_t))^2 \quad \text{s.t. } x_0 = x,\ x_T = y$$

Multi-Step Sample Path LDP

- LDP conditional on $x_0^\sigma = x$ is then:

$$\lim_{\sigma\to 0} \sigma^2 \log P\left(\sup_{1\le t\le T} \left|x_t^\sigma - x_t^0\right| > a\right) = -\inf_{\{y:|y-x_t^0|>a\}} S(x,y,T)$$

- Define escape time from stable point x^*:

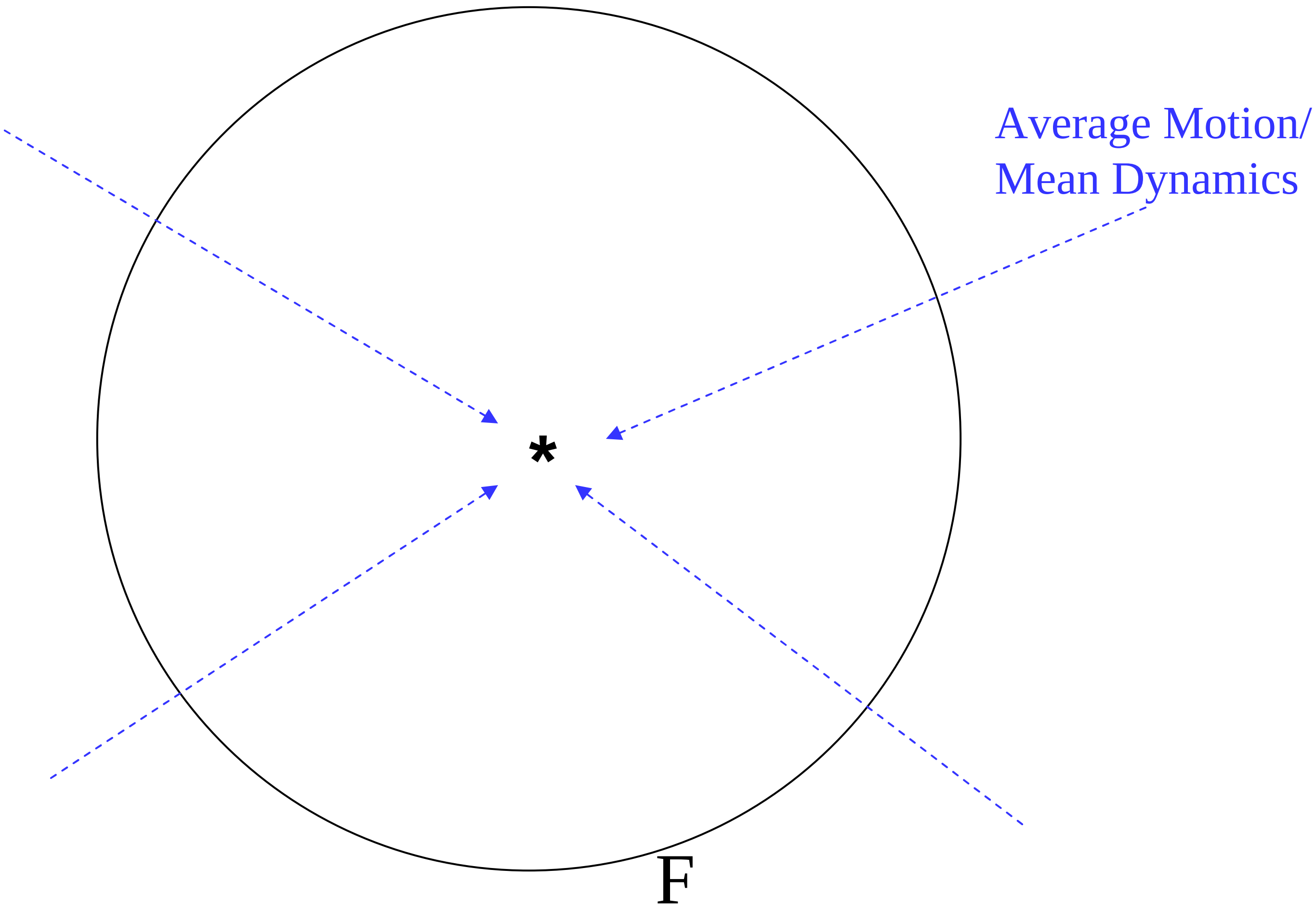
$$\tau^\sigma = \min\{t > 0 : \ |x_t^\sigma - x^*| \ge a, \ |x_0^\sigma - x^*| < a\}$$

- Let $\bar{S} = \inf_{\{y\in x^*\pm a, T<\infty\}} S(x,y,T)$, y^* the minimizer.
- Then for $\eta > 0$ we have:

$$\lim_{\sigma\to 0} P\left(\exp\left(\frac{\overline{S}-\eta}{\sigma^2}\right) \le \tau^\sigma \le \exp\left(\frac{\overline{S}+\eta}{\sigma^2}\right)\right) = 1.$$

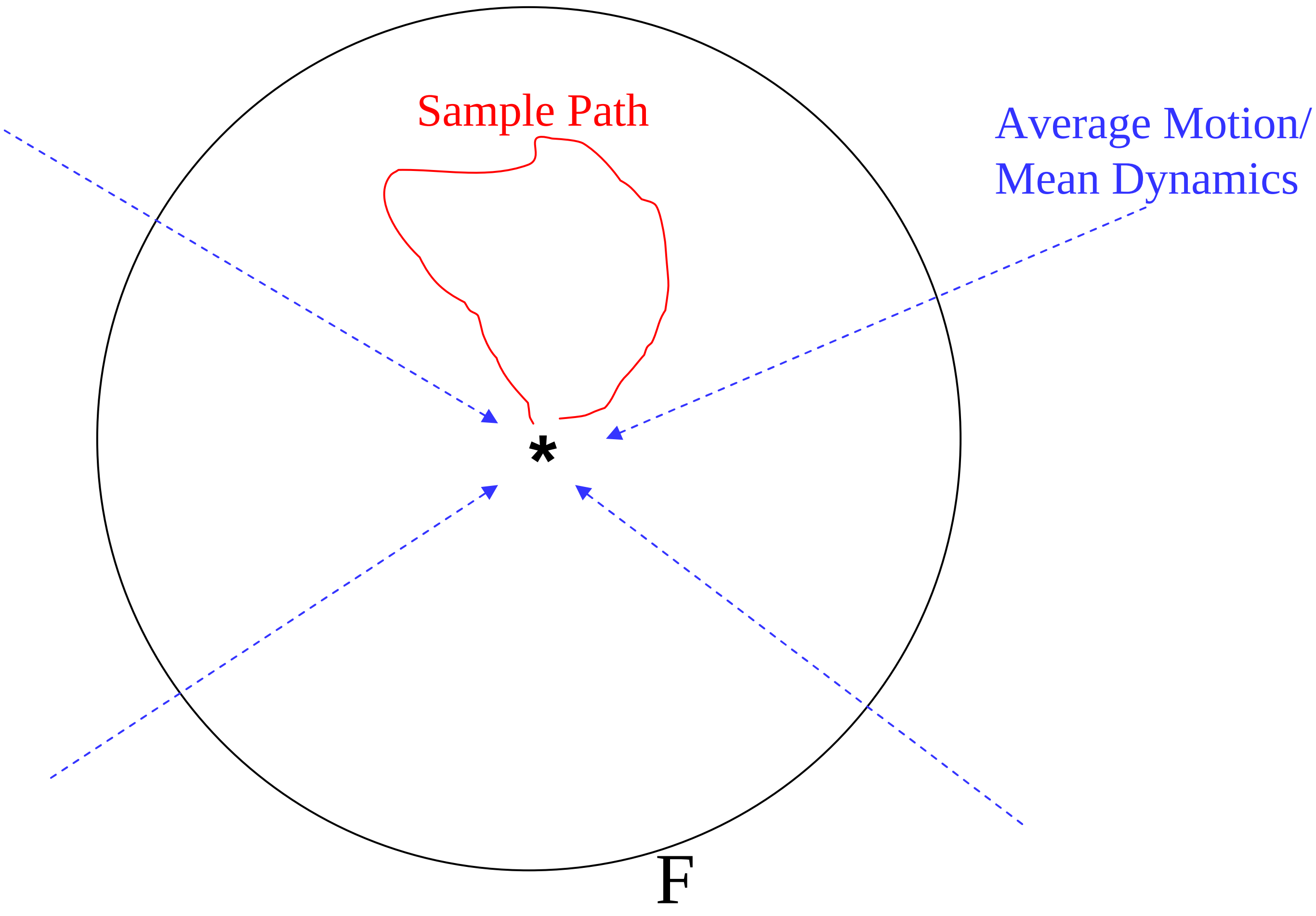
$$\lim_{\sigma\to 0} P(|x_{\tau^\sigma}^\sigma - y^*| < \eta) = 1.$$

The Exit/Escape Problem I

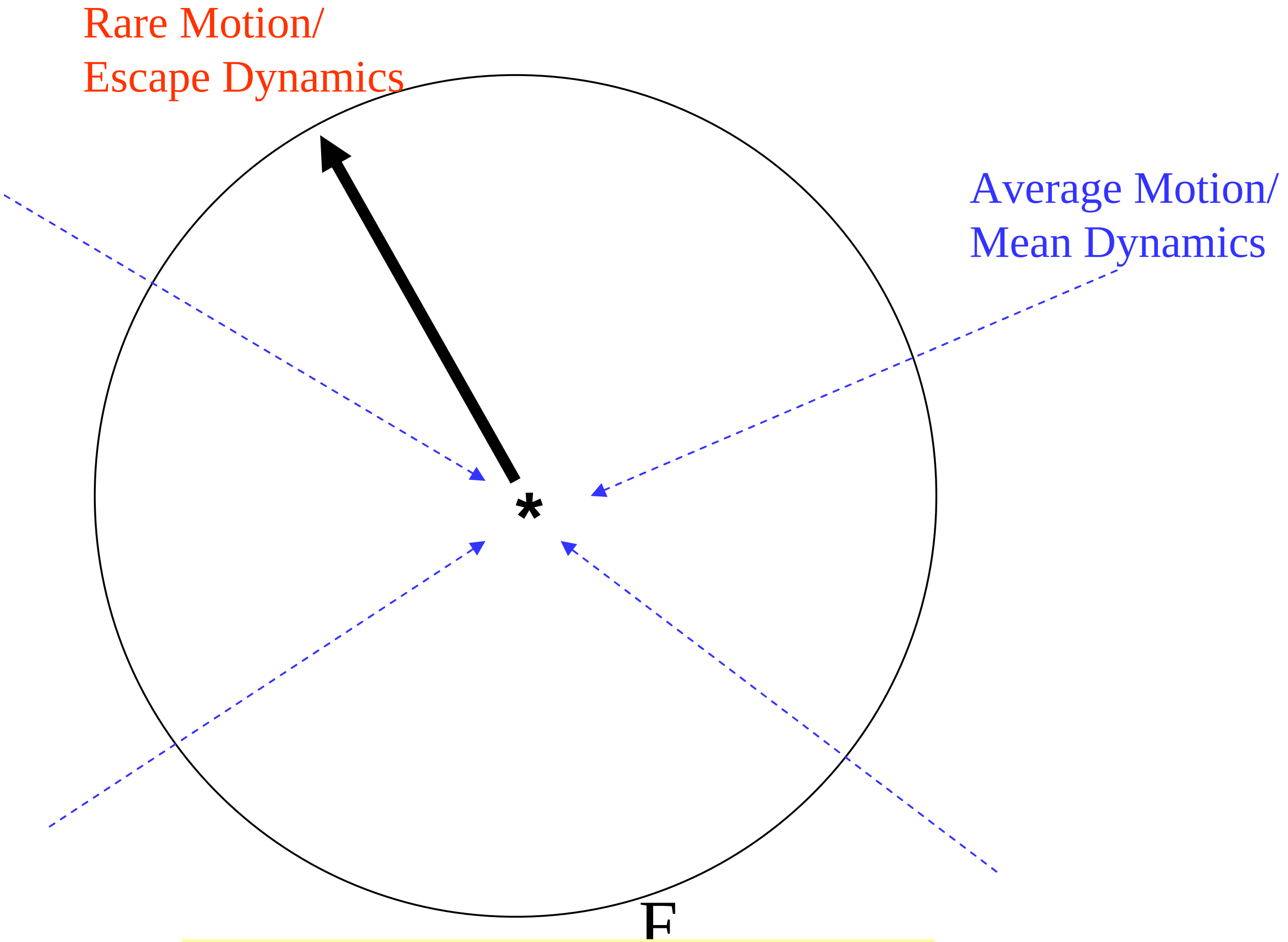


The Exit/Escape Problem I

When pushed away, tend to return to equilibrium



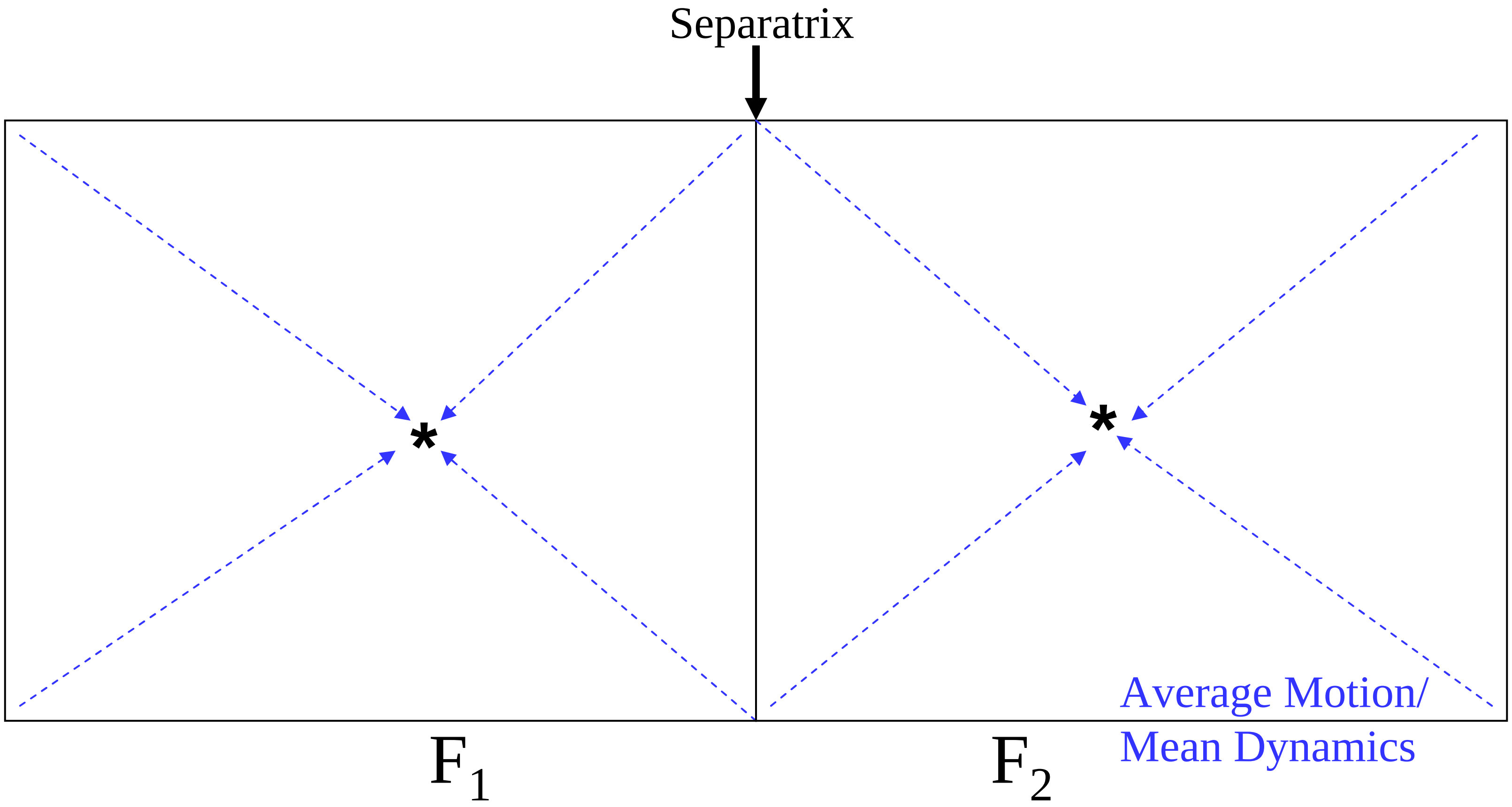
The Exit/Escape Problem I



Occasionally, escape to boundary of set

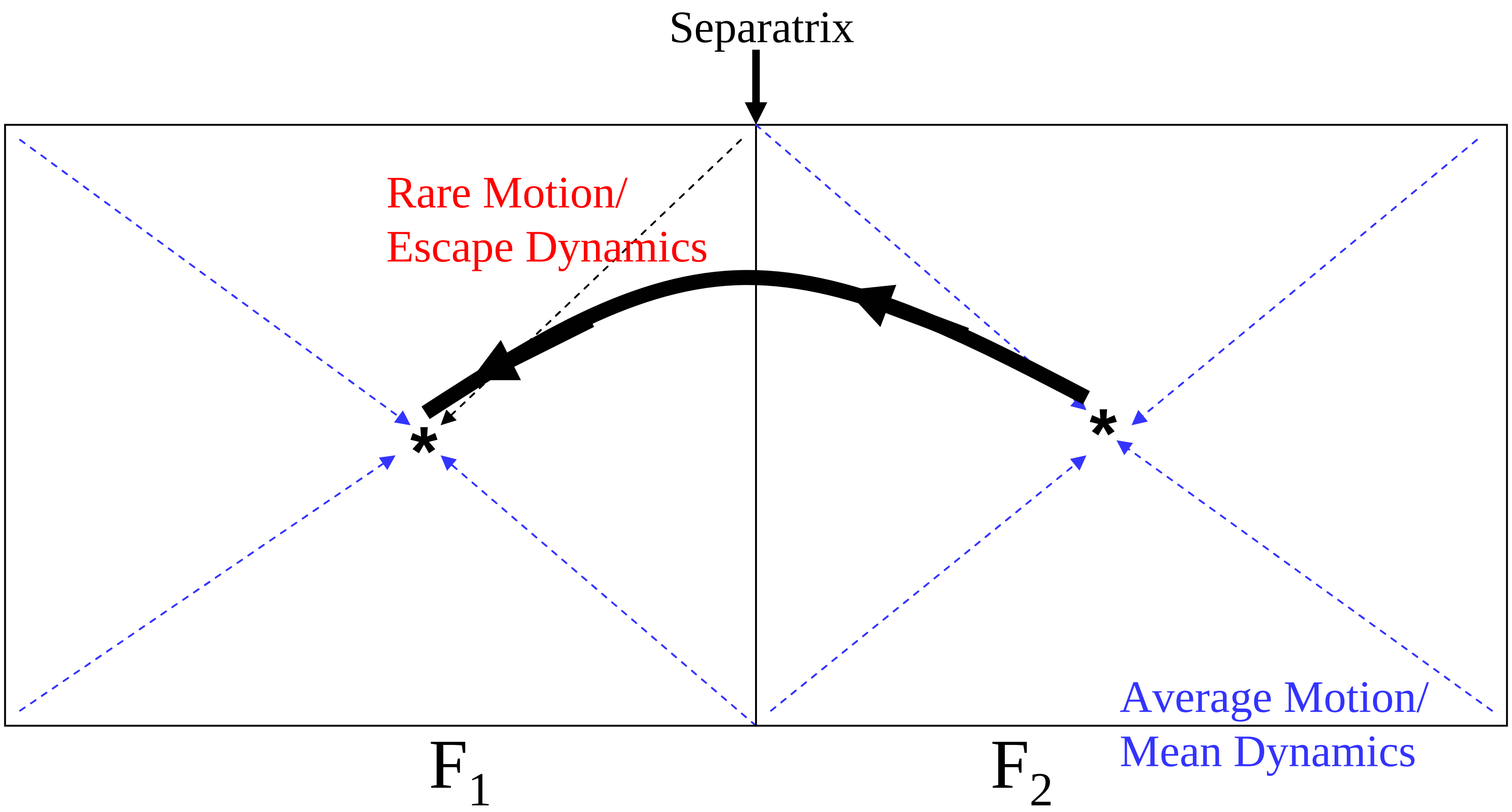
The Exit/Escape Problem II: Multiple Equilibria

2 stable equilibria with basins of attraction

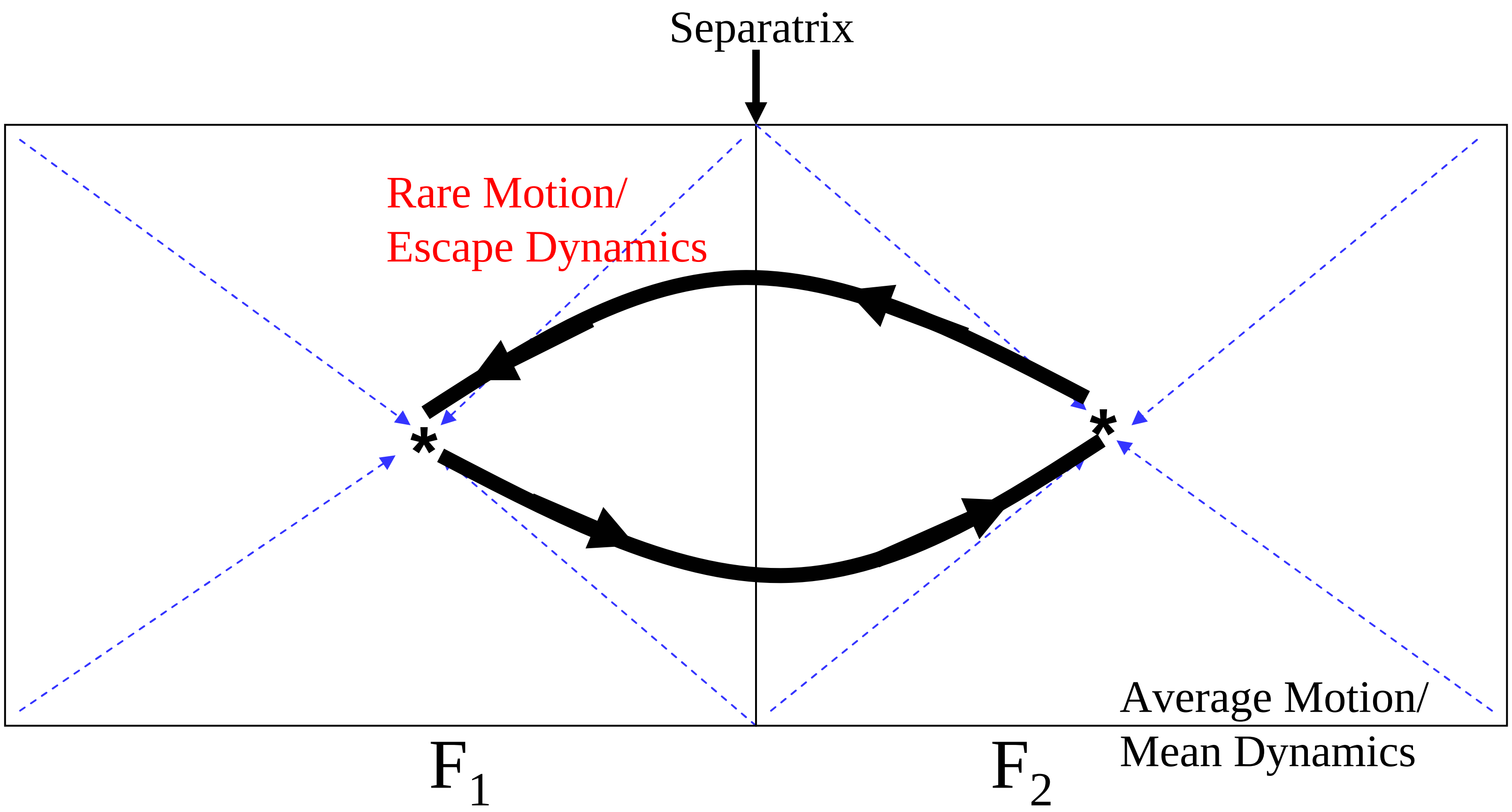


The Exit/Escape Problem II: Multiple Equilibria

Occasionally escape from one eq to the other



The Exit/Escape Problem II: Multiple Equilibria



Generates a Markov chain over equilibria

Application: Large Business Cycles

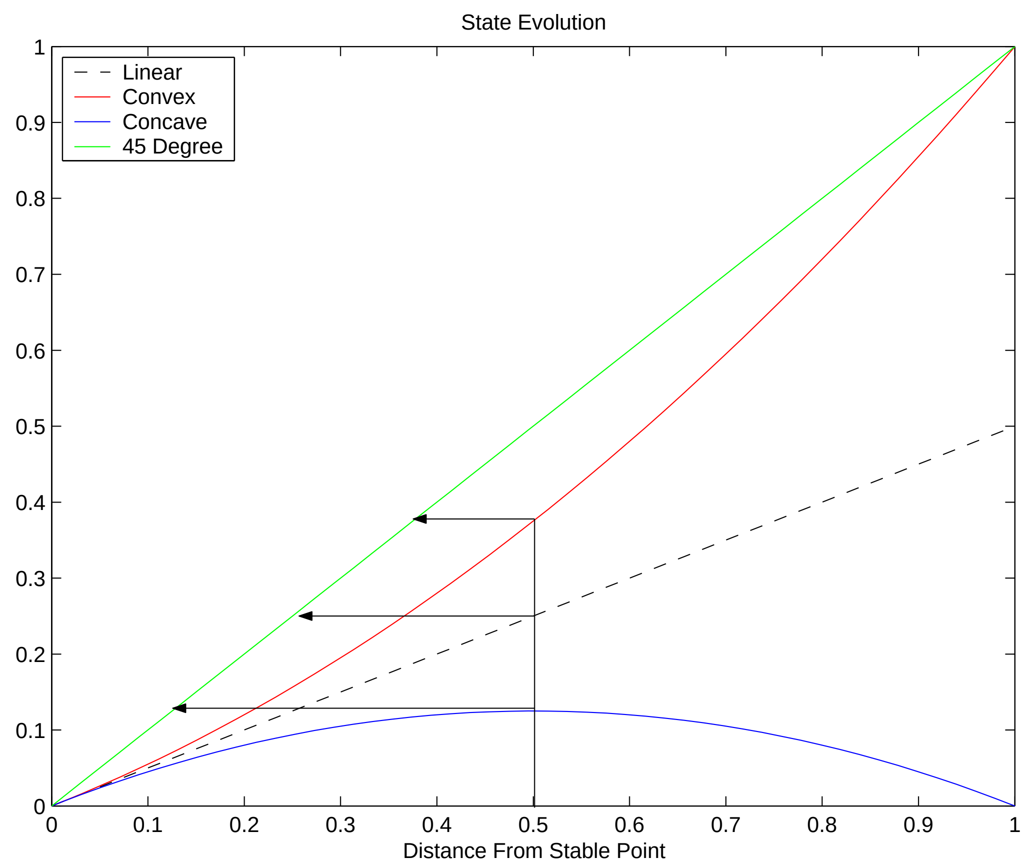
- "Small Noise Asymptotics ..." studies stochastic growth model. Characterize prob. and freq. of large movements away from deterministic steady state.
- Nonlinearity in policy function determines expected direction of large movements. Convex: positive movements more likely. Concave: negative more likely.
- The way state, policy defined in model implies sharp recessions (slightly) more likely than large booms.

"Intuition" for Asymmetry

Consider 2 step for simplicity. Say $x^* = 0, x_1 = 0.5$, $a = x_2 = 1$.

Start of date 1: at $x^* = 0$. End of date 1: at 0.5.

Start of date 2: at $g^0(0.5)$. End of date 2: at 1.



Application: Escape Dynamics

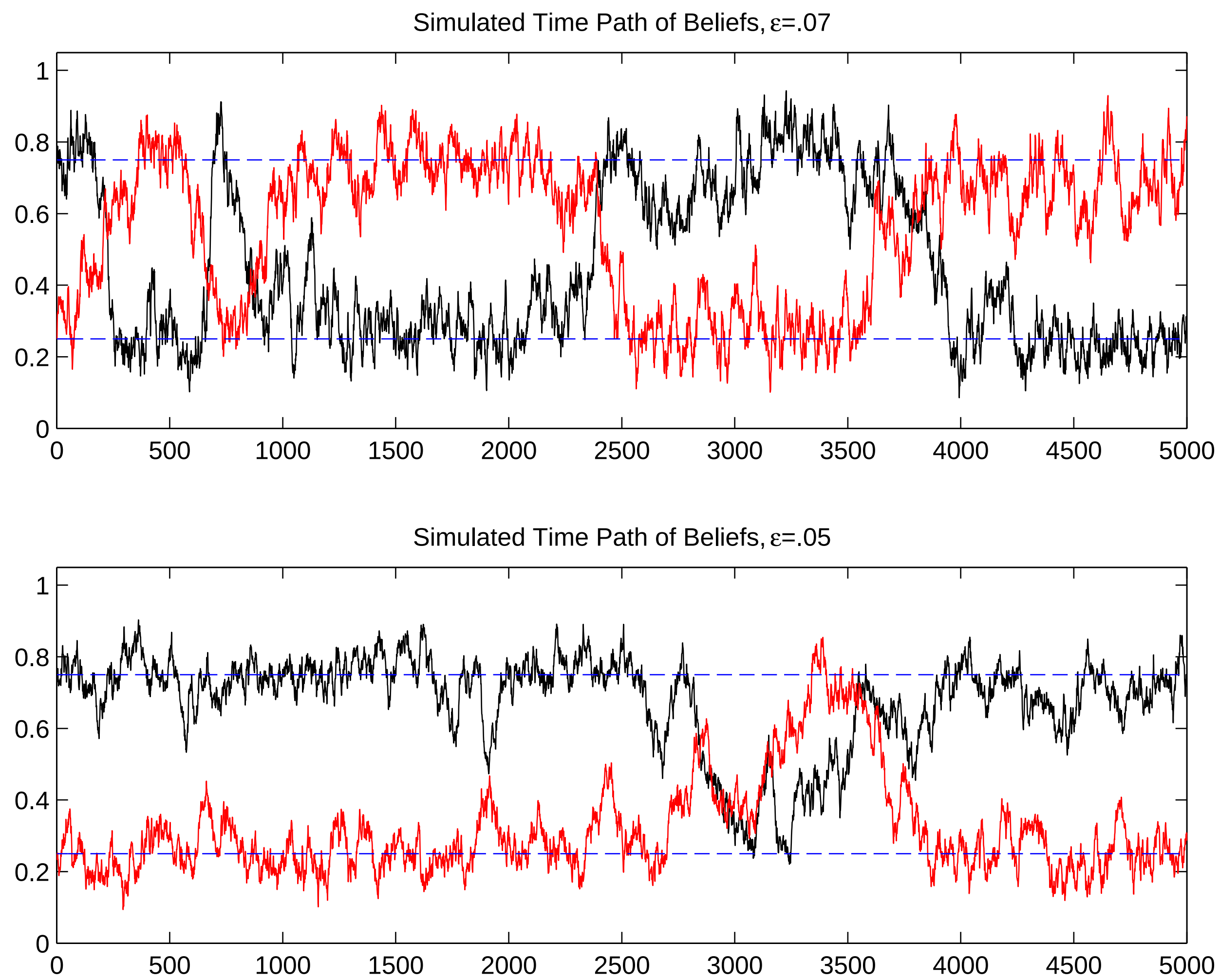
- In some learning models, recurrent escapes from a stable limit point are a key feature of time series.
- Example: Single agent learning a vector γ. Truth depends on agent's action, and beliefs.

$$
\begin{aligned}
\text{Belief} : y_{in} &= \gamma_i x_{in} + \eta_{in}, \; i = 1, 2, \\
\text{Action} : x_n &= b_n + W_n, \; W_n \sim N(0, \sigma^2 I) \\
\text{Decision} : b_n &= b(\gamma) \\
\text{Truth} : y_{in} &= f_i(b_n) + \bar{\gamma}_i x_{in} \\
\text{Updating} : \gamma_{n+1} &= \gamma_n + \varepsilon(y_n - \gamma x_n) x_n
\end{aligned}
$$

- Specify so for $\sigma > 0$ unique stable equilibrium $\bar{\gamma}$.

An Example

Convergence: as $\varepsilon \to 0$, $\gamma_n \Rightarrow \bar{\gamma} = [.75, .25]$. Recurrent escapes.

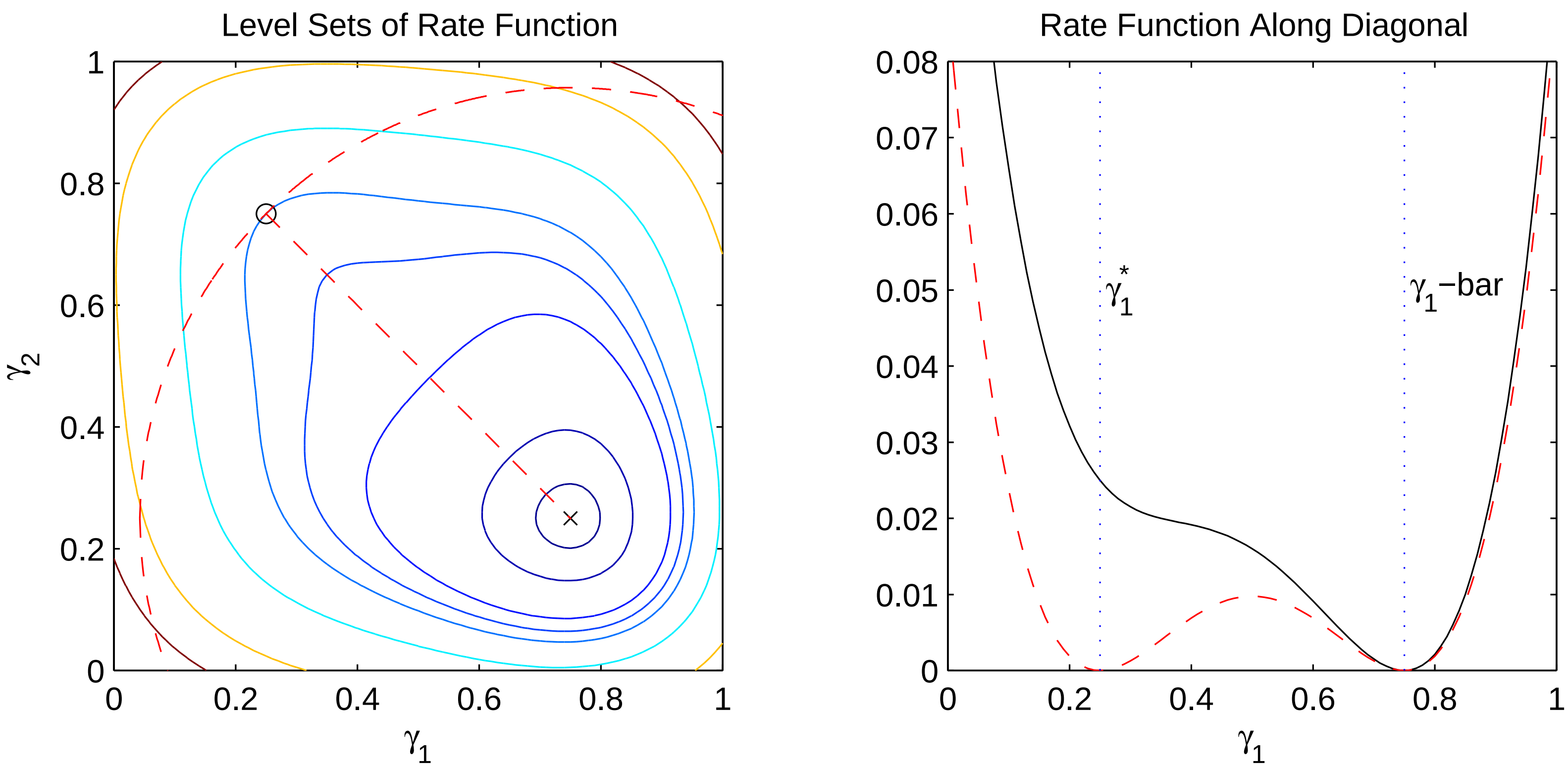


Overview: Escape Dynamics

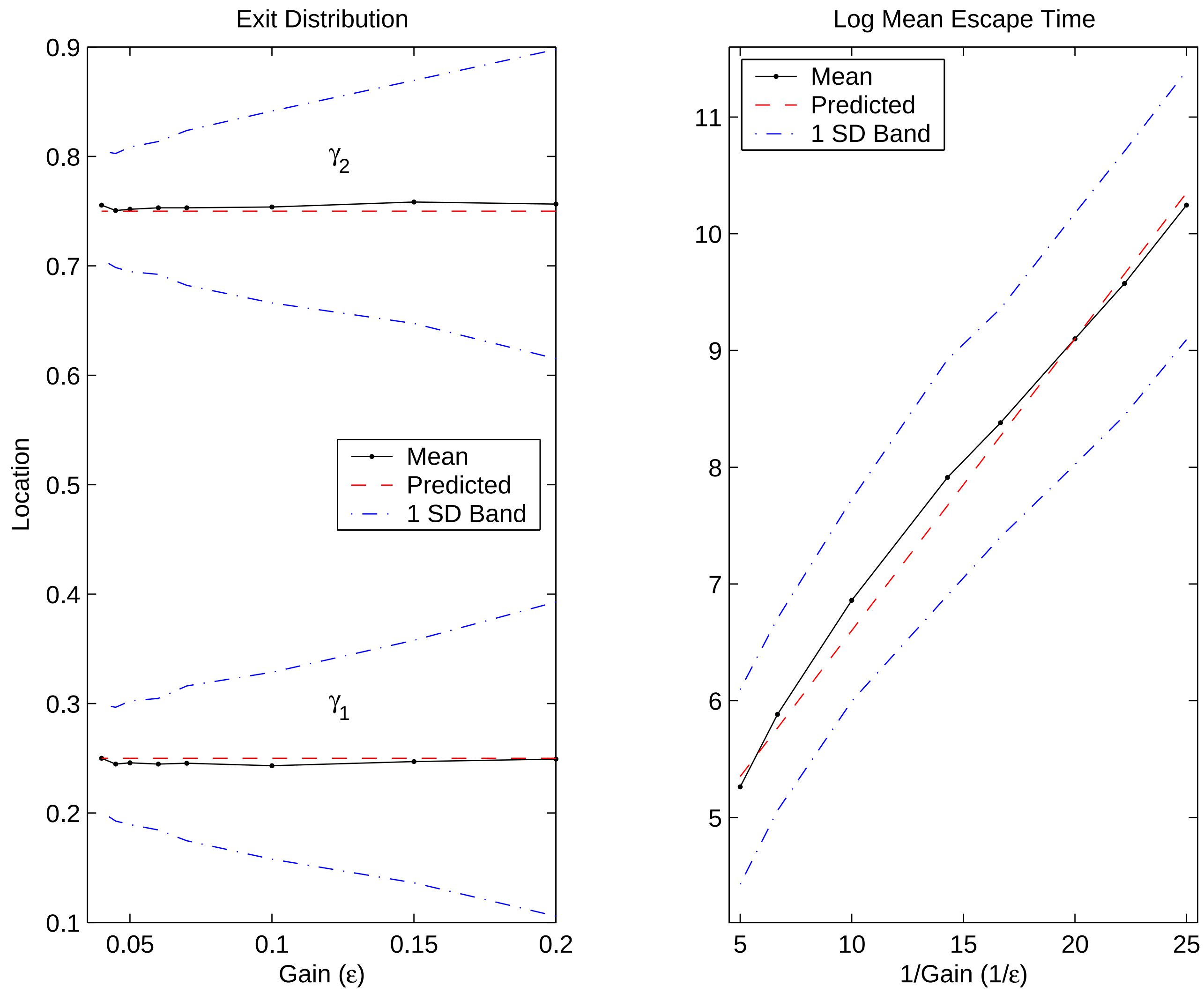
- Similar type of LDP applies, gives predictions of probability and frequency of escape, location of dominant escape path.
- With multiple equilibria, can calculate limit transition rates between equilibria. Limit distribution over equilibria collapses on "stochastically stable" or long-run equilibrium.
- With unique equilibrium, can still have interesting escape dynamics (as in the example). Often have escapes toward a "near equilibrium".

Analysis: "Near Equilibrium"

As noise → 0, there are 2 (stable) equilibria, 1 of which gets "discovered" by escape dynamics.



Comparing the Predictions to Simulations



Some References

- Books: Dembo-Zeitouni (1998, Springer, 2nd ed.), Freidlin-Wentzell (1998, Springer, 2nd ed.), Puhalskii (2001, Chapman), Dupuis-Ellis (1997, Wiley)
- Econ - decisions etc.: Krasa-Villamil (1992 JET, 1992 Oxford Ec. Papers), Nummelin (2000, Ann. App. Prob.), Williams (2003, "Small Noise")
- Econ - multiple eq.: Foster-Young (1990, J. Pop Bio), Kandori-Mailath-Rob (1993, Ec.), Young (1993, Ec.), Ellison (1993, Ec., 2000 ReStud), Kasa (2004, IER), Williams (2002, unpub "Stochastic Ficititious Play"), Benaïm-Weibull (2003, Ec.)
- Econ - escape dynamics: Sargent (1999, Conquest book), Cho-Williams-Sargent (2002, ReStud), Williams (2004, unpub, "Escape Dynamics"), Bullard-Cho (2002, unpub), Cho-Kasa (2003, unpub)
- Finance: Stutzer (2003, J. Econometrics), Dembo-Deuschel-Duffie (2004, Fin. & Stoch.), Pham (2003, Fin. & Stoch.), Callen-Govindaraj-Xu (2000, Ec. Theory)