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Problem Set 5 due 5/2/08

Question 1

A monopolist produces quantities $q \in \mathbb{R}_0^+$ of a good with costs $c(q) = q^2$. Consumers' valuation of this good depends on their type $\theta \in \{\theta_L, \theta_H\}$, where $0 < \theta_L < \theta_H < \infty$ and $2\theta_L > \theta_H$. Both types are equally likely ex ante. Given a monetary transfer $t \in \mathbb{R}_0^+$ and quantity q, consumers' utility from a purchase is $\theta q - t$, and the monopolist's profit is $t - c(q)$. The monopolist uses the usual direct revelation mechanism.

a) Set up the monopolist's optimization problem if he wants both types of customers to buy. Which constraints will bind at the optimum? Why?

b) Derive the optimal menu of contracts $(t_H^*, q_H^*), (t_L^*, q_L^*)$.

Straightforward.

c) Suppose the monopolist had tried using a two-part tariff instead of a direct revelation mechanism; that is, he sets prices equal to $t = a + bq$ for some a, b, and leaves the choice of q to the consumers. Could he implement $(t_H^*, q_H^*), (t_L^*, q_L^*)$ with such a tariff? If not, are there other nonlinear pricing schemes $t = T(q)$ that might work?

A linear pricing scheme can't work as consumer's indifference curves are also linear (they will choose corner solutions when maximizing their utility). It is easy to see that a strictly convex function with $T'(q_H^*) = \theta_H$, $T'(q_L^*) = \theta_L$, $T(q_H^*) = t_H^*$, $T(q_L^*) = t_L^*$ will work.

Question 2

A social planner is regulating a monopolist whose type $\theta \in \Theta = [0, 1]$ he cannot observe. The monopolist produces two different goods, 1 and 2. A monopolist of type θ has marginal costs θ of producing good 1; his marginal cost of producing good 2 is $(1-\theta)$. A monopolist who produces quantity $q = (q_1, q_2)$ and receives transfer t from the planner has profits $\pi(t, q; \theta) = t - \theta q_1 - (1-\theta)q_2$. The monopolist's reservation utility is zero. The social planner maximizes consumer surplus $S(q_i)$ in both markets, net of the transfer; his objective function is $V(t, q) = S(q_1) + S(q_2) - t$, where $S(q_i) = 1 - (1 - q_i)^2$. θ is distributed uniformly on Θ.

A *direct revelation contract* is a tuple of functions (t, q_1, q_2), $t : \Theta \to \mathbb{R}$, $q_i : \Theta \to \mathbb{R}$, $i = 1, 2$. Rewrite the monopolist's profit from such a contract as $\pi(\hat{\theta}; \theta) = \pi(t(\hat{\theta}), q(\hat{\theta}); \theta)$, where $\hat{\theta}$ is the monopolist's announcement of his type, θ is his true type. The optimal direct revelation contract maximizes the social planner's expected utility, subject to the monopolist's participation and revelation constraints:

$$\max_{q,t} \int_{\Theta} V(t(\theta), q(\theta)) f(\theta) d\theta \text{ s.t. } \pi(\theta;\theta) \geq 0, \forall\theta, \tag{P_θ}$$

$$\theta \in \arg\max_{\hat{\theta}} \pi(\hat{\theta};\theta), \forall\theta, \tag{R_θ}$$

You may use the facts that for any $\bar{\theta} \in \Theta$, $g : \Theta \to \mathbb{R}$, $\int_0^{\bar{\theta}} [\int_\theta^{\bar{\theta}} g(y)dy] f(\theta)d\theta = \int_0^{\bar{\theta}} g(\theta)F(\theta)d\theta$ and $\int_{\bar{\theta}}^1 [\int_{\bar{\theta}}^{\theta} g(y)dy] f(\theta)d\theta = \int_{\bar{\theta}}^1 g(\theta)(1-F(\theta))d\theta$.

a) First suppose that production of good 2 is prohibited for some reason. Derive the function q_1^* of quantities set by the optimal direct revelation contract (t^*, q_1^*). Point out the monopolist's information rent along the way. Is the contract for the best type distorted away from the first-best? What about all other types?

The first-order condition $t'(\theta) = \theta q'(\theta)$ is necessary and sufficient for the revelation constraint (R) if also $q'(\theta) \leq 0$. In that case, (R) implies (P) for all $\theta < 1$; (R) and the quasilinearity of the monopolist's profits imply that (P) holds with equality for $\theta = 1$. Using (P) at $\theta = 1$, we can then rewrite (R) for any θ as

$$\begin{aligned} t'(\theta) &= \theta q'(\theta) = \frac{\partial[\theta q(\theta)]}{\theta} - q(\theta) \\ t(1) - t(\theta) &= q(1) - \theta q(\theta) - \int_\theta^1 q(y)dy, \text{ and hence} \\ t(\theta) &= \theta q(\theta) + \int_\theta^1 q(y)dy, \end{aligned} \tag{T}$$

where $\int_\theta^1 q(y)dy$ is the information rent. Ignoring the monotonicity constraint (M) $q'(\theta) \leq 0$ and using the partial integration fact provided in the question, we write the maximization problem as

$$\begin{aligned} &\max_q \int_\Theta [S(q(\theta)) - \theta q(\theta) - \int_\theta^1 q(y)dy] f(\theta)d\theta = \\ &\max_q \int_\Theta [S(q(\theta)) - q(\theta)[\theta + \frac{F(\theta)}{f(\theta)}]] f(\theta)d\theta, \end{aligned}$$

so that using functional forms, we have first-order conditions

$$\begin{aligned} 2(1 - q(\theta)) &= 2\theta, \text{ or} \\ q^*(\theta) &= 1 - \theta. \end{aligned}$$

Indeed we have $q' < 0$, so that ignoring this constraint was legitimate. The best type is $\theta = 0$, and we have $S'(q(\theta)) = \theta$ at $\theta = 0$ (no distortion at the top). All other types' *quantities are distorted downwards as* $q^{CI}(\theta) = 1 - \frac{1}{2}\theta$.

b) Now suppose the monopolist is producing both goods, 1 and 2. The optimal direct revelation contract is a tuple of functions $(t^* q_1^*, q_2^*)$.

i. In the optimal contract, there exists (at least one) $\tilde{\theta} \in (0,1)$ such that $q_1^*(\tilde{\theta}) = q_2^*(\tilde{\theta})$. Show that the revelation constraint of any θ, together with the participation constraint of type $\tilde{\theta}$, implies the participation constraint for type θ. *(Hint: A change of variables to make this problem look like the problem in a) may be useful).*

ii. Rewrite the revelation constraint in the usual manner, taking care to note whether you have to impose additional conditions on the optimal contract. Then use your result from (i) to derive expressions for $t(\theta)$ for all θ in an incentive compatible contract.

iii. (optional - do this part only if you want practice) Derive the optimal quantities q_1^*, q_2^*. Which types receive informational rents in the optimal contract? Why?

(i) Rewrite π *by defining* $b(\hat{\theta}) = t(\hat{\theta}) - q_2(\hat{\theta})$, $\Delta(\hat{\theta}) = q_1(\hat{\theta}) - q_2(\hat{\theta})$. *Then* $\pi(\hat{\theta};\theta) = b(\hat{\theta}) - \theta\Delta(\hat{\theta})$, *which is the standard form of profits; for* $\theta = \tilde{\theta}$, $\Delta(\theta) = 0$. *For all* θ, *(R) implies* $\pi(\theta;\theta) = b(\theta) - \theta\Delta(\theta) \geq \pi(\tilde{\theta};\theta) = b(\tilde{\theta}) = \pi(\tilde{\theta};\tilde{\theta})$, *so that (R) and (P) at* $\tilde{\theta}$ *indeed imply (P) everywhere else.*

(ii) As above, the first-order condition to (R) is $b'(\theta) = \Delta'(\theta)$, *which is necessary and sufficient for R if also (M)* $\Delta'(\theta) \leq 0$. *It is useful from now on to distinguish between* $\theta < \tilde{\theta}$ *and* $\theta > \tilde{\theta}$. *For* $\theta \leq \tilde{\theta}$, *we write*

$$\begin{aligned} b(\tilde{\theta}) - b(\theta) &= \Delta(\tilde{\theta}) - \Delta(\theta) - \int_{\theta}^{\tilde{\theta}} \Delta(y)dy, \text{ or} \\ b(\theta) &= \pi(\tilde{\theta};\tilde{\theta}) + \Delta(\theta) + \int_{\theta}^{\tilde{\theta}} \Delta(y)dy, \end{aligned}$$

and similarly for $\theta > \tilde{\theta}$ *we can write*

$$b(\theta) = \pi(\tilde{\theta};\tilde{\theta}) + \Delta(\theta) - \int_{\tilde{\theta}}^{\theta} \Delta(y)dy.$$

Then argue that (P) must bind at $\tilde{\theta}$ *so that* $\pi(\tilde{\theta};\tilde{\theta}) = 0$.

(iii) Using the partial integration facts provided and the expressions for $b(\theta)$, *we now*

write the maximization problem as

$$\max_q \int_0^{\tilde{\theta}} [S(q_1) + S(q_2) - \theta q_1(\theta) - (1-\theta) q_2(\theta) - (q_1(\theta) - q_2(\theta)) \frac{F(\theta)}{f(\theta)}] f(\theta) d\theta +$$
$$+ \int_{\tilde{\theta}}^1 [S(q_1) + S(q_2) - \theta q_1(\theta) - (1-\theta) q_2(\theta) + (q_1(\theta) - q_2(\theta)) \frac{1-F(\theta)}{f(\theta)}] f(\theta) d\theta,$$

leading to first-order conditions for $\theta \leq \tilde{\theta}$:

$$S'(q_1(\theta)) = \theta + \frac{F(\theta)}{f(\theta)}, S'(q_2(\theta)) = 1 - \theta - \frac{F(\theta)}{f(\theta)},$$

and for $\theta > \tilde{\theta}$:

$$S'(q_1(\theta)) = \theta - \frac{1-F(\theta)}{f(\theta)}, S'(q_2(\theta)) = 1 - \theta + \frac{1-F(\theta)}{f(\theta)},$$

which when plugged in, shows

$$q_1(\theta) = \begin{cases} 1-\theta, \theta \leq \tilde{\theta} \\ \frac{3}{2} - \theta, \theta > \tilde{\theta} \end{cases}, q_2(\theta) = \begin{cases} \frac{1}{2} + \theta, \theta \leq \tilde{\theta} \\ \theta, \theta > \tilde{\theta} \end{cases}.$$

Finally, one needs to find $\tilde{\theta}$ and check the monotonicity constraint (M). By definition, $\Delta(\grave{\theta}) = 0$, *which occurs at either* $\theta \in \{\frac{1}{4}, \frac{3}{4}\}$. *However, setting* $\tilde{\theta}$ *equal to either shows that the functions above cannot be the optimal functions as they violate (M) on* $[\frac{1}{4}, \frac{3}{4}]$. *Hence, (M) must bind in this region: we must have* $\Delta(\theta) = 0$ *for all* $\theta \in [\frac{1}{4}, \frac{3}{4}]$. *Then we have*

$$q_1^*(\theta) = \begin{cases} 1-\theta, \theta \leq \frac{1}{4} \\ \frac{3}{4}, \frac{1}{4} \leq \theta \leq \frac{3}{4} \\ \frac{3}{2} - \theta, \theta \geq \frac{3}{4} \end{cases}, q_2^*(\theta) = \begin{cases} \frac{1}{2} + \theta, \theta \leq \frac{1}{4} \\ \frac{3}{4}, \frac{1}{4} \leq \theta \leq \frac{3}{4} \\ \theta, \theta \geq \frac{3}{4} \end{cases}.$$

One notes that $q_1^*(0)$ *and* $q_2^*(1)$ *are not distorted from the first best, as these are the best types for those goods. Also, only the extreme types* ($\theta < .25, \theta > .75$) *receive information rents at all, as only these types' production will depend on their costs. There is an extreme tradeoff between efficiency and truthful revelation in this model, as types are not unambiguous. One notes that it is possible to give no information rents to any type by setting quantities equal and constant across all types; this would, however, not be efficient, as it requires the extreme types, who are very bad at producing one good but very good at the other, to produce the same quantity of both. On the other hand, the middling types cannot be induced to reveal their type correctly in an efficient contract that is monotone in costs, as the incentives for good and bad types work against each other in this region. Setting quantities equal for the middle types is less costly in efficiency* *terms than getting them to reveal their type truthfully.*