

Lecture 11: Indivisible good problems - Auctions

1 The first-price (sealed-bid) auction

1.1 Definitions

One seller has an indivisible good; there are n potential buyers, whose valuations for the good are their private information. We will consider the case where agent's valuations are independently distributed (*private values*). Instead of looking for the optimal selling mechanism, here we look at some specific mechanisms that are used in practice. In the first-price sealed-bid auction, buyers simultaneously submit bids; the highest bid wins the object, and the winning bidder is required to pay the amount of his bid to the seller. (No one else makes a payment). Implicit in this formulation is that the seller's valuation for the good is below the lowest buyer's valuation, so that he always wants to sell.

- the set of players: $\mathcal{I} = \{1, \dots, n\}$.
- action space (space of bids b_i): $\mathcal{A}_i = \mathbb{R}_0^+$
- payoff functions:

$$u_i(b_i, \theta_i) = \begin{cases} \theta_i - b_i & \text{for the winning bidder} \\ 0 & \text{else} \end{cases}$$

- the information structure: $\theta_i \in \Theta = [\theta_L, \theta_H]$ is the agent's private information, $\theta \sim F(\theta)$ for all i .
- game tree \mathcal{T} : b_i are submitted simultaneously, payoffs realize, and the game ends.

A Bayes-Nash equilibrium is a tuple of functions (b_1^*, \dots, b_n^*) , $b_i^* : \Theta \rightarrow \mathcal{A}_i$. A symmetric BNE is a function b^* such that $b_i = b^*, \forall i$, is a BNE.

1.2 The direct revelation mechanism associated with the first-price auction

Like any game, the first-price auction has an associated direct revelation mechanism that yields the same equilibrium outcomes as the auction. If this mechanism is symmetric, it is a pair of functions (q, b) , $q : \Theta^n \rightarrow \mathcal{I}$, $b : \Theta \rightarrow \mathbb{R}_0^+$, such that players make announcements $\hat{\theta}_i \in \Theta$; given a vector of announcements θ , player $q(\hat{\theta})$ is selected to receive the good and pay $b(\hat{\theta}_i)$. The allocation rule $q(\hat{\theta})$ is such that for any $\hat{\theta}$, $q(\hat{\theta}) = \{i | b(\hat{\theta}_i) \geq b(\hat{\theta}_j), \forall j\}$. (Ties are broken uniformly).

Claim 1 (*Revelation principle*) A symmetric mechanism (q, b) induces truthful revelation if $b = b^*$.

1.3 Equilibrium bid functions

From above, we know that deriving the equilibrium b^* is equivalent to characterizing the function b that induces direct revelation (together with q as specified above). Player i 's expected probability of winning given $\hat{\theta}_i$ is defined as $p(\hat{\theta}_i) = P(b(\hat{\theta})_i = \max_{j \in \mathcal{I}} b(\hat{\theta}_j))$. We can again write player's utility as

$$U^b(\hat{\theta}_i, \theta_i) = p(\hat{\theta}_i)(\theta_i - b(\hat{\theta}_i)).$$

Proposition 1 *The unique symmetric monotone equilibrium bid function is*

$$b^*(\theta) = \theta - \int_{\theta_L}^{\theta} \left[\frac{F(y)}{F(\theta)} \right]^{n-1} dy.$$

Proof: A mechanism induces truthful revelation if the BNE of the mechanism $(\hat{\theta}_1^*, \dots, \hat{\theta}_n^*) = (\theta_1, \dots, \theta_n)$ for all profiles $(\theta_1, \dots, \theta_n)$.

We first show that if (q, b) is a truthful mechanism, then $b = b^*$ as defined above. We then show that given b^* , $\hat{\theta}_i = \theta_i$ is always a unique best response to $\hat{\theta}_{-i} = \theta_{-i}$.

Step 1: Necessity. Suppose (q, b) is a truthful and monotone mechanism. This implies

$$\begin{aligned} U_1^b(\hat{\theta}_i, \theta_i)|_{\hat{\theta}_i = \theta_i} &= 0. \\ p(\hat{\theta}_i) &= (F(\hat{\theta}_i))^{n-1}. \end{aligned} \tag{R'_\theta}$$

Rewriting R'_θ in the usual manner gives the above.

Step 2: Sufficiency. Suppose $b = b^*$, and $\hat{\theta}_{-i} = \theta_{-i}$. We show that then $U^b(\hat{\theta}_i, \theta_i)$ is strictly increasing in its first argument for $\hat{\theta}_i < \theta_i$, and strictly decreasing in its first argument for $\hat{\theta}_i > \theta_i$.

Hence, the mechanism (q, b^*) is the only symmetric monotone mechanism inducing truthful revelation. But then there cannot be another symmetric monotone BNE of the auction game.

2 The second-price (sealed-bid) auction

2.1 Definitions

One seller has an indivisible good; there are n potential buyers, whose valuations for the good are their private information. We will consider the case where agent's valuations are independently distributed (*private values*). Instead of looking for the optimal selling mechanism, we look at some specific mechanisms that are used in practice. In the second-price sealed-bid auction, buyers simultaneously submit bids; the highest bid wins the object, and the winning bidder is required to pay the amount of the second-highest bid.

- the set of players: $\mathcal{I} = \{1, \dots, n\}$.
- action space (space of bids b_i): $\mathcal{A}_i = \mathbb{R}_0^+$

- payoff functions:

$$u_i(b_i, b_{-i}, \theta_i) = \begin{cases} \theta_i - b_{(n-1)} & \text{for the winning bidder} \\ 0 & \text{else} \end{cases},$$

where $b_{(n-1)}$ is the second-highest bid.

- the information structure: $\theta_i \in \Theta = [\theta_L, \theta_H]$ is the agent's private information, $\theta \sim F(\theta)$ for all i .
- game tree \mathcal{T} : b_i are submitted simultaneously, payoffs realize, and the game ends.

A Bayes-Nash equilibrium is a tuple of functions $(b_1^*, \dots, b_n^*), b_i^* : \Theta \rightarrow \mathcal{A}_i$.

2.2 Equilibrium bid functions

Proposition 2 *The unique BNE in which no weakly dominated strategies are played is $b_i^*(\theta_i) = \theta_i, \forall i$.*