A Theory of Indicative Bidding*

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Abstract

When selling a business by auction, sellers typically use indicative bids – non-binding preliminary bids – to select a limited number of bidders to conduct due diligence and submit binding bids. We show that if entry into the auction is costly, indicative bids can be informative: symmetric equilibrium exists in weakly-increasing strategies, with bidders "pooling" over a finite number of bids. The equilibrium helps the seller select high-value buyers with higher likelihood, although the highest-value bidders are not always selected. We characterize equilibrium play when the number of potential bidders is large, and show that both revenue and bidder surplus are higher than when entry is unrestricted.

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1 Introduction

Auctions with a fixed number of bidders have been the subject of a large and distinguished theoretical and empirical literature. However, in many applications, potential bidders have to acquire additional information at substantial cost before bidding. For example, in timber auctions, bidders perform "cruises" to obtain estimates of the volume and species composition of wood; in oil and gas lease sales, bidders invest in seismic surveys to learn more about the likelihood of finding hydrocarbons; and in takeover auctions, buyers conduct extensive due diligence to determine the value of the target. These costs are analogous to entry costs, in the sense that bidders are unwilling to bid without acquiring this information. The decision to incur these costs means that the number of actual bidders and the distribution of their values are endogenous. This fact has important implications for the design of optimal auctions. Since sellers bear some or all of the participation costs indirectly through lower bidder participation and bids, they have an incentive to restrict entry and select only those bidders most likely to have the highest willingness to pay. It also has important implications for implementation, which requires knowing the distribution of bidder values: when entry is costly, empirical researchers have to estimate this distribution taking into account the selection process for bidders.

This paper studies the effectiveness of using indicative bids to select bidders for an auction. This selection mechanism is commonly used in utility privatization, divestiture sales, and institutional real estate (see Kagel, Pevnitskaya and Ye (2008)). It is used extensively in takeover auctions, which rank second only to treasury auctions in the total value of assets sold each year. The value of mergers and acquisitions of U.S. companies over the past twenty years has ranged from \$400 billion to \$1.5 trillion dollars per year;[1] about half of these deals involve an auction, and nearly all of these employ some form of indicative bidding.[2] Therefore, understanding how indicative bids work in takeover auctions is important in its own right. It is also important for obtaining estimates of the distribution of bidder values in these auctions and conducting counterfactual analyses of alternative selling mechanisms.

Boone and Mulherin (2009) describe how sellers control entry into takeover auctions. Sellers contact many possible bidders; those who are interested are required to sign confidentiality and standstill agreements. These agreements commit bidders not to make public their interest or bids, nor to make unsolicited bids; in exchange, they are given access to non-public information on the target. (Thus, in the usual auction terminology, those buyers who signed such agreements make up the set of potential bidders.) Bidders then submit preliminary indications of interest, which include an estimate or range of estimates for the price they expect to be willing to pay for the target. The bidders who report the highest willingness to pay are invited to conduct extensive due

[1] Mergerstat Review.

[2] Using SEC filings, Boone and Mulherin (2007) and Gentry and Stroup (2015) study over a thousand takeovers of public companies announced between 1989 and 2009. They find that in roughly half the deals, the company was offered for sale to multiple competing buyers, and that in most of these deals indicative bids were used to determine which buyers could conduct due diligence and submit binding bids.

dilience and submit formal, binding bids.[3] Due diligence includes access to the data room where legal and accounting teams can inspect and verify the target's contracts and financials. Most of the buyers' costs occur at this stage of the process, and the costs can run into the millions of dollars. The indicative bids themselves are costless, in that they are never paid by the buyers; and they are non-binding, because they do not restrict in any way the real offers that a buyer may subsequently make in the auction. Despite this absence of commitment, the use of indicative bids in takeover auctions suggests that sellers find them informative.

We use a two-stage model to study this practice. We assume buyers observe noisy real-valued signals about their private values, and can learn their values perfectly by incurring an entry cost. The initial signals are independently distributed, but values can be correlated conditional on the signals, allowing for the presence of common value components. Our model nests all of the models that have been considered in both the theoreticaland empirical literatures on auctions with costly entry. Prior to entry, buyers are asked to simultaneously submit indicative bids or opt out. The seller commits to selecting the n buyers (typically two or three in practice) who send the highest indicative bids, with ties broken randomly. If fewer than n buyers submit indicative bids, then all of them enter the auction; if all buyers opt out, then no sale occurs. In the second stage, the selected bidders incur their entry costs, learn their values, and submit binding bids in a second-price auction. Ye (2007) shows that a fully separating equilibrium fails to exist in this model. We are interested in characterizing the symmetric equilibria that do exist.

Our central result is that indicative bids yield a partial sorting of buyers based on their signals (i.e., types) when the expected rents from the private information obtained in the second stage are small relative to entry costs. This condition is satisfied when most of the learning occurs prior to entry, or when the information obtained in the second stage is highly correlated across bidders. In this case, the buyers' incentives are sufficiently aligned with those of the seller that indicative bids are informative: the seller wants to restrict the number of buyers who incur the entry cost, and buyers want to avoid being selected and paying this cost if they are unlikely to win. Low-value buyers will try to separate themselves from high value buyers by submitting lower indicative bids. We show that a symmetric equilibrium is a finite partition of the space of buyers' types. Buyers with types in the same element of the partition submit the same indicative bid, and buyers in higher elements submit higher bids. Thus, the equilibrium helps the seller select high-value buyers with greater likelihood. We prove existence of a symmetric equilibrium (uniqueness can be shown in some special cases), and explore some comparative statics.

How well does the indicative bidding mechanism perform? A natural benchmark is an auction in which entry is unrestricted. Buyers decide on the basis of their private information whether or not to enter the auction, pay the entry cost, update their values and submit binding bids. Our main theoretical result is that indicative bidding yields greater revenue and greater buyer surplus

[3]Boone and Mulherin (2007) report that in their sample of 202 auctions, the average number of buyers contacted is 21, the average number that sign agreements is 7, and the average number that submit binding bids is 1.57. Gentry and Stroup (2015) report similar numbers for their sample. Unfortunately, neither reports the number of buyers submitting indicative bids.

than the unrestricted auction when the number of potential buyers is large.[4] Through numerical examples, we find that this result tends to hold even when the number of buyers is small. Thus, when entry is costly, the introduction of indicative bids does not involve the standard tradeoff in optimal auctions between revenues and efficiency.

Indicative bidding does better than unrestricted entry but, as Lu and Ye (2015) point out, it is hardly optimal within the class of two-stage mechanisms. The main alternative proposed in the literature is an entry rights auction (Fullerton and McAfee (1999), Ye (2007)). In characterizing the optimal two-stage mechanism, Lu and Ye (2015) show that it can be implemented under certain conditions using an all-pay auction for entry rights followed by a second-price auction with handicaps. However, in our view, an entry rights auction may not always be feasible, particularly in the case of takeover auctions, because it requires buyers to commit to paying substantial sums *before* they conduct due diligence. This would undermine the purpose of due diligence, and put management at risk of shareholder lawsuits if the asset turned out to be worth less than they anticipated. Even worse, since revenue is based in part on the expected, rather than actual, valuation of the asset, there is a risk of adverse selection among sellers: a rush of entry by sellers with worthless assets could crowd out sellers with legitimate ones and increase the need for due dilegence. If mechanisms requiring payments before bidders perform due diligence are ruled out, then it is not clear that one can do much better than indicative bidding followed by an auction.[5]

This paper contributes to the large literature on auctions with costly entry. The focus of the theoretical literature is on characterizing rational entry and bidding decisions for a given entry process and information environment, and on designing mechanisms that are more efficient or generate more revenue. One branch studies environments in which buyers have no private information prior to entry and pay a cost to learn their value. In this setting, the seller does not have to worry about selecting buyers, which is the central issue of our paper. Levin and Smith (1994) characterize the mixed strategy entry equilibrium in private value English auctions, and show that the failure of bidders to coordinate entry leads to outcomes that could be improved by capping entry at a fixed number. Cremer, Spiegel, and Zheng (2009) characterize the optimal mechanism, and show that the seller can use entry fees and subsidies to extract all buyers' surplus. In the absence of such payments, Bulow and Klemperer (2009) demonstrate that auctions with unrestricted entry are less efficient than sequential mechanisms but typically generate more revenue.[6] A second branch studies environments in which buyers know their value, or have a signal of their value, before paying an entry cost. The entry cost can be bidding costs as in Samuelson (1985) or additional information acquisition costs as in Ye (2007). This setting leads to selective entry, since bidders with high values

[4]We compare auctions without reserve prices, but in the unrestricted case, the optimal reserve price goes to zero as the number of buyers gets large, so when this number is large enough, indicative bidding outperforms the optimal standard auction.

[5]One could still improve on the mechanism we study by varying the number of bidders n advancing to the auction in response to the indicative bids received, or (as we show in one of our theorems) by partly subsidizing entry when the number of potential bidders is large.

[6]Other papers in this branch include McAfee and McMillan (1987), Burguet and Sakovics (1996), Menezes and Monteiro (2000), Tan (1992), Ye (2004) and Compte and Jehiel (2007).

are more likely to enter than bidders with low values. Samuelson (1985) showed that the entry equilibrium has a threshold property and that restricting the number of bidders could increase revenues. Ye (2007) showed that an entry rights auction can be used to induce efficient entry. Lu and Ye (2015) characterize the optimal two-stage mechanism. Our contribution to this literature is to model the use of "cheap talk" to resolve the coordination problem faced by buyers.

Recently an empirical literature on auctions with costly entry has developed. The focus of this literature is on identifying and estimating the joint distribution of bidder signals (or entry costs) and values in order to evaluate different entry/auction formats. Athey, Levin and Seira (2011) and Athey, Levin and Coey (2013) estimate a model in which loggers and mills in timber auctions pay an entry cost to learn their values. From an econometric perspective, the key simplication is that entry is not selective: costs and values are independently distributed. Bhattacharya, Roberts and Sweeting (2014), Li and Zheng (2009, 2012), and Roberts and Sweeting (2013a, 2013b) extend this model to account for selective entry by estimating parametric models of bidding for highway contracts and timber. In their models, bidders receive signals affiliated with their values before paying the cost to learn their values, and the (signal, value) pairs are independently distributed across bidders. Marmer, Shneyerov, and Xu (2013) provide non-parameteric tests to distinguish between the two kinds of entry models; Gentry and Li (2014) provide conditions under which the joint distribution of signals and values is identified. In a recent working paper, Gentry and Stroup (2015) apply the affiliated signal model to estimate the joint distribution of signals and values in takeover auctions from data on the numbers of potential and actual bidders and the deal premium. They use their estimates to examine whether takeover auctions generate higher prices than negotiations. On the normative side, our contribution to this literature is to provide an alternative entry format that should be considered, at least for auctions that meet our small rents condition (which is a testable restriction). On the positive side, our analysis informs empirical researchers studying takeover auctions as to how they can use the number (and values) of indicative bids to help identify the joint distribution of signals and values.

Our paper also contributes to the voluminous literature on "cheap talk" games. Our indicative bidding equilibria are similar to the "cheap talk" equilibria of Crawford and Sobel (1982): indicative bids are monotonic in buyers' initial information, but only a finite number of different bids are used in equilibrium, and different types of buyers "pool" on the same bid. In their seminal paper, Crawford and Sobel show that cheap talk can improve the ex ante payoffs of both parties when a biased sender has information relevant to the receiver's decision problem. Farrell and Gibbons (1989) and Matthews and Postlewaite (1989) similarly show that cheap talk can be informative prior to bilateral bargaining, and can therefore expand the set of equilibrium payoffs. Our contribution to this literature is to introduce a natural kind of receiver commitment into a cheap talk setting, which sharpens the predictions of the model. As Farrell and Gibbons (1989) have observed, in standard cheap-talk games, the receiver cannot commit to a choice of outcome as a function of the messages. Instead, the messages derive meaning only from the receiver's interpretation of them and the receiver must act optimally given that interpretation. In our setting, we assume the seller commits both to the rules of the auction (which is standard) *and* to how he will select entrants based on the indicative bids received. In particular, we assume the seller commits to selecting a fixed number of bidders, choosing those who send the highest indicative bids, and breaking ties randomly. This commitment to a monotone selection rule eliminates much of the multiplicity of equilibria that arises in cheap talk games. In particular, it rules out a "babbling" equilibrium, and any equilibrium where adverse off-equilibrium-path beliefs are used to deter unused messages.[7]

We should also mention two other related papers that feature entry models whose equilibria have similar structures to ours, but where voluntary delay rather than communication is used to screen and coordinate entrants. Levin and Peck (2003) do this in an oligopoly setting, where post-entry competition is symmetric and firms have private information about their entry costs; with multiple discrete opportunities to enter, a firm with intermediate-level costs will wait to see that his opponent does not enter for a certain number of periods before entering himself. McAdams (2015) considers second-price auctions with costly bidding and reserve prices, where bids can be submitted in discrete rounds and are made public after each round. In equilibrium, a bid deters any future bids, and a new set of bidder types become willing to bid (if nobody has bid prior) in each round. Like our model, both of these models admit a unique symmetric equilibrium in thresholds.

The paper proceeds as follows. Section 2 presents the model. Section 3 characterizes symmetric equilibrium and proves equilibrium existence. In Section 4, we use an example to show how symmetric equilibria are constructed and conduct comparative statics with respect to the size of the message space and the number of bidders who can advance. In Section 5, we evaluate the performance of the indicative bidding mechanism against the alternative of unrestricted entry, and establish general results for the case where N is large. Section 6 discusses extensions. Section 7 concludes. All proofs are in the appendix.

2 Model

We begin by describing the environment and the indicative bidding mechanism we consider and the assumptions we make.

Our model is based on the "private value updating" model of Lu and Ye (2015). There are $N \geq 3$ potential bidders, indexed by i. Bidder i's value of the asset is V_i. Initially, she does not know V_i, but observes a real-valued, private signal S_i of it, which we will refer to as her type. She learns V_i perfectly during due diligence. We assume that all buyers face the same cost of performing due diligence, denote this cost by c, and also refer to it as the entry cost.[8]

Let $V = (V_1, .., V_N)$ and $S = (S_1, .., S_N)$. We assume the joint distribution of (V, S) is continuous, and exchangeable with respect to bidder indices. We assume that $\{S_i\}$ are independent

[7]Navin Kartik and Joel Sobel (private communication) have similarly shown that imposing monotonicity on both the sender's and receiver's strategies in a standard cheap talk setting, combined with iterated weak dominance, uniquely selects the "most informative" cheap talk equilibrium.

[8]If the seller also incurs costs directly for each buyer who goes through this process, it would strengthen the seller's incentive to limit entry, but would not change bidder play within the mechanism we study.

across i, and that S_i is independent of $\{V_j\}_{j\neq i}$, but we allow for the possibility that $\{V_i\}$ are correlated conditional on S, i.e., that the new information learned during due diligence is correlated across bidders. (The literature on optimal mechanism design in this setting makes the assumption that (V_i, S_i) are independent across i, to rule out full surplus extraction à la Cremer and McLean (1988). While we will maintain the assumption of independent types $\{S_i\}$, since we study a particular mechanism, we have no need to assume that $\{V_i\}$ are independent given S.[9]) We assume that S_i has finite support $[\underline{S}, \overline{S}]$, and a continuous (marginal) distribution which admits a continuous density bounded below on its support. Under these assumptions, there is no loss of generality in assuming that S_i is distributed uniformly on $[0, 1]$, which we therefore assume.[10]

The indicative bidding mechanism we consider consists of a cheap talk stage and an auction stage. In the cheap talk stage, the seller asks potential buyers if they are interested in bidding for the asset, and if so, how much they are willing to pay. These bids are not binding, and are known as indicative bids. An indicative bid can be a precise number, although often the seller asks the bidder to report a range in which she believes her willingness to pay is likely to fall. We formalize this stage of the game by assuming that buyers simultaneously send messages to the seller. A message is denoted by m, and the set of messages available to each bidder is the set of integers $\{0, 1, ..., \overline{M}\}$, where "0" is "opt out", or decline to participate, and $\overline{M}$ is allowed to be either finite or infinite. Note that $\overline{M}$ is a parameter of choice for the seller: for example, the seller could simply ask each buyer to opt in or opt out, in which case $\overline{M} = 1$. The substantive restriction here is that the set of opt-in messages is bounded below (i.e., there is a lowest "opt-in" message), fully ordered, and countable. Given this restriction, there is no loss of generality in assuming that the message space consists of non-negative integers.[11]

The auction stage consists of an English auction. The seller selects bidders for the auction based on the messages received in the cheap talk stage. We assume the seller commits to a maximal number of bidders n and, if more than n bidders opt in, commits to selecting the bidders who sent the highest messages, breaking ties randomly. If all bidders opt out, then the game ends with no sale. If only one bidder opts in, then that bidder gets the asset at the reserve price. (For expositional and notational ease, we focus on the case where the reserve price is 0 and a lone entrant therefore acquires the asset for free; a positive reserve is easily incorporated into the model, and does not change our results.) In addition, the seller commits not to make public the messages that the bidders send to him, nor his response to those messages. Therefore, each bidder knows only

[9] Once (S_i, V_i) are assumed to be independent across i, one can use the normalization from Eső and Szentes (2007) and without loss of generality express V_i as $u(S_i, T_i)$, where u is increasing in both arguments, $\{T_i\}$ are independent across i, and T_i is independent of S_i and uniformly distributed on $[0, 1]$. The interpretation is that T_i is the new information that buyer i obtains about her value during due diligence. Our model is more general than this, primarily by allowing $\{T_i\}$ to be correlated across bidders. In addition, however, when $\{V_i\}$ are not independent, the expression of V_i as $u(S_i, T_i)$, with T_i independent of S_i, is no longer a normalization but a substantive assumption, which we have no need to impose.

[10] That is, if F_S denotes the marginal distribution of S_i, one could equivalently assume bidder i observed $F_S(S_i)$, which contains the same information as S_i but is distributed uniformly on $[0, 1]$.

[11] As we discuss in the appendix, symmetric equilibria fail to exist when the set of allowed messages is continuous, or more accurately, when for any two permitted messages, there is always another message between them.

whether or not she has advanced to the auction, and does not gain any additional information about which (if any) of the other bidders have also advanced. This nondisclosure commitment is important because if a bidder were to learn that she is facing maximal competition in the auction, it would sometimes be in her interest to drop out of the bidding to avoid incurring the entry cost.[12]

In an English auction, a bidder who advances has a (weakly) dominant strategy to bid her valuation, regardless of the message she sent in the first stage. Therefore, in what follows, we assume that conditional on advancing, bidder i bids V_i in the auction. Let $S_{-i,n}$ and $V_{-i,n}$ denote, respectively, the vectors of signals and values of bidder i's $n-1$ opponents in the auction. Then, conditional on being selected and performing due diligence, bidder i's ex post payoff before costs is V_i if she advances alone, and $\max\{0, V_i - \max\{V_{-i:n}\}\}$ if she advances against $n-1$ opponents. Given S_i, her interim expected payoff before costs from advancing against $n-1$ opponents with types $S_{-i,n}$ is therefore

$$u_n(S_i, S_{-i,n}) \quad \equiv \quad E_{(V_1,..,V_n|S_1,..,S_n)}(\max\{0, V_i - \max\{V_{-i,n}\}\})$$

and her expected payoff is

$$u_1(S_i) \quad \equiv \quad E_{V_i|S_i}(V_i)$$

if she advances alone.

We assume that expected auction payoffs depend on initial types in the expected way (or that types are ordered in the natural direction):

Assumption 1. *(a) $u_1(S_i)$ is continuous and strictly increasing in S_i.*

(b) For each $n > 1$, $u_n(S_i, S_{-i,n})$ is continuous in its n arguments; weakly increasing in S_i; weakly decreasing in $S_{-i,n}$; and strictly increasing in S_i if $S_i \geq \max\{S_{-i,n}\}$.

If $\{S_i, V_i\}$ are independent across i, then a sufficient condition for the monotonicity conditions to hold is for (S_i, V_i) to be affiliated. (The continuity conditions follow from continuity of the joint distribution of (S, V).) Finally, we assume that the entry cost c is neither "too large" nor "too small":

Assumption 2. *(a) $c < u_1(1)$.*

(b) For every $s_i \in [0, 1]$, $u_2(s_i, s_i) < c$.

Assumption 2(a) is simply the requirement that the game is non-trivial: that the entry cost is not so large as to completely preclude entry. Assumption 2(b) is more substantive: it requires the entry cost to be high enough that a bidder only wants to enter the auction against an opponent if her type (i.e., initial signal) is greater than her opponent's type. It is essentially a restriction on the information rents bidders can earn from due diligence, and we therefore refer to it as the "small

[12] More generally, nondisclosure strengthens a seller's bargaining position when only one bidder is seriously interested in buying the asset. Subramanian (2010) offers a couple of funny stories on how sellers keep bidders in the dark as to the number of competitors they face.

rents" assumption. For example, if $V_i = S_i + T_i$, with $\{T_i\}$ independent of $\{S_i\}$ representing a new signal bidder i learns during due diligence, then Assumption 2(b) would be satisfied if $\{T_i\}$ were sufficiently highly correlated across i, since bidders would compete away any rents associated with learning the realization of T_i.

Combined with our other assumptions, Assumption 2(b) implies that there is some $\varepsilon > 0$ such that $u_2(s_i, s_j) \geq c$ requires $s_i \geq s_j + \varepsilon$. This gives some intuition for Ye's (2007) result that an indicative bidding game cannot have a fully separating equilibrium: in any symmetric equilibrium, if her opponents were bidding "truthfully," a bidder would have an incentive to misrepresent her type downwards by at least ε, to avoid being selected in some scenarios where her expected payoff in the auction would be less than the participation cost. Thus, this ε is in a sense analogous to the bias between sender and receiver preferences in Crawford and Sobel (1982).[13] However, the fact that buyers do not always gain from participating in the auction is what ensures that buyer and seller incentives are sufficiently aligned for cheap talk to be informative. The seller wants to restrict the number of buyers (since he bears some of the participation cost indirectly through its effect on participation), without excluding the strongest buyers; while the buyers want to avoid being selected and paying the entry cost when they are unlikely to win. Thus, low types will try to separate themselves from high types by sending lower messages; and sellers will happily exclude them in favor of bidders with higher types. (In contrast, if too much idiosyncratic learning occurred during due diligence, then buyers would always want to enter the auction regardless of their types, and cheap talk would unravel. We discuss relaxing Assumption 2(b) in a later section.)

The indicative bidding game can be thought of as a cheap talk game with commitment. The messages of the bidders (senders) influence which action the seller (receiver) takes but, given that action, they do not affect the payoffs of the players. In the standard cheap talk game, the space of messages is unrestricted, and the receiver chooses an action that is his best response to the messages sent. By contrast, in the indicative bidding game (as is standard in auctions), the seller commits to the mechanism. In particular, he commits to a rule that selects bidders based on the messages they send, and ignores messages outside the set of allowed ones.[14]

3 Equilibrium Characterization and Existence

Next, we define strategies, payoffs, and equilibrium for this model, establish some properties of any symmetric equilibrium, and establish that a symmetric equilibrium always exists.

[13] We thank Navin Kartik for this observation. Note that the "bias" in our model goes in the opposite direction as in Crawford and Sobel: in the classic sender-receiver game, the sender wants to misrepresent his type upwards, while in our setting, a buyer would like to misrepresent her type downwards.

[14] Ex interim, once the messages are received, it would typically be in the seller's interest to allow more than n bidders to advance. However, on the equilibrium path, subject to the constraint of not advancing more than n bidders, the seller is selecting the bidders he would want to if he was not committed to a selection rule. Thus, as we discuss in Section 6.1, the equilibrium we find in the next section would still be an equilibrium if the seller had complete freedom to choose whichever bidders he wanted, rather than being committed to choosing those who submitted the highest indicative bids; but like in standard cheap talk games, we would also have less-informative equilibria in this case.

Given our model, the particular environment facing a seller is characterized by the number of bidders N, the joint distribution of signals and valuations (S, V), and the entry cost c. The particular indicative bidding mechanism faced by bidders is characterized by the maximal number of bidders who will advance to the auction n and the number of opt-in messages allowed $\overline{M}$. A pure strategy for bidder i consists of a message function

$$\tau_i \; : \; [0,1] \; \longrightarrow \; \{0, 1, .., \overline{M}\}$$

that maps the set of types to the set of messages; a mixed strategy is a mapping from $[0, 1]$ to probability distributions over $\{0, 1, ..., \overline{M}\}$. The support of a strategy is defined to be the set of messages played with positive probability by a positive measure of types.

Our objective is to characterize symmetric equilibria. Consequently, we need only define the expected payoff to bidder i when her $N-1$ opponents all play a common strategy. Let $v_\tau(m; s)$ denote the expected payoff to a bidder with type s if she sends message m and her opponents are all playing the strategy τ. (Since the game is symmetric, we do not index $v_\tau(\cdot, \cdot)$ by the identity of the bidder i.) A pure-strategy symmetric Bayesian-Nash equilibrium is then a strategy τ such that $v_\tau(\tau(s); s) \geq v_\tau(m'; s)$ for all $m' \in \{0, 1, .., \overline{M}\}$ and $s \in [0, 1]$.[15]

Under the assumptions stated above, we can establish two key properties which must hold in any symmetric equilibrium, and which allow us to characterize what any symmetric equilibrium must therefore "look like".

Lemma 1. *Given an environment and an indicative bidding mechanism, if τ is a symmetric equilibrium, then (i) τ is weakly increasing, and (ii) τ has support $\{0, 1, \ldots, M\}$ for some finite M.*

Thus, any symmetric equilibrium (should one exist) has the same structure as the cheap talk equilibria of Crawford and Sobel (1982): the type space $[0, 1]$ is partitioned into a finite number of subintervals $[0, \alpha_0]$, $[\alpha_0, \alpha_1]$, ..., $[\alpha_{M-1}, 1]$; bidders with types in the interior of each subinterval pool on the same message; and bidders with types at the boundary α_m between two subintervals send either message m or $m + 1$ or mix between the two.[16]

Lemma 1 says that even when $\overline{M}$ is infinite, only finitely many messages are used in equilibrium. Since we assume the seller is committed to advancing the bidders who sent the highest messages, it may seem counterintuitive that even a bidder with the highest type is not williing to separate herself by sending a higher, out-of-equilibrium message and advancing for certain. However, were she to make such a deviation, the increase in her probability of advancing would arise solely from breaking ties against opponents who are sending the highest equilibrium message M, and therefore have types within the highest subinterval $[\alpha_{M-1}, 1]$. As more messages get used, this interval gets sufficiently small that under Assumption 2(b), even the highest type's payoff against a randomly

[15] A mixed-strategy symmetric BNE is a strategy τ such that $v_\tau(m; s) \geq v_\tau(m'; s)$ for all $m' \in \{0, 1, .., \overline{M}\}$, all m on which $\tau(s)$ places positive probability, and all $s \in [0, 1]$.

[16] In the knife-edge case of indifference, a bidder with type $S_i = 1$ might send any message in the set $\{M, M + 1, \ldots, \overline{M}\}$, or any mixture among them.

selected opponent drawn from this interval would be negative. When this is the case, she will not want to separate herself by sending a higher message.

Lemma 1 gives only necessary, not sufficient, conditions for a strategy τ to be a symmetric equilibrium. It turns out that in addition to these conditions, two more turn out to be sufficient:

Lemma 2. *Let τ be a weakly-increasing strategy with support $\{0, 1, \ldots, M\}$ for some $M < \infty$. For $m = 0, 1, \ldots, M-1$, define α_m as the supremum of the set of bidder types who send message m with positive probability. Then τ is a symmetric equilibrium if and only if:*

1. *$v_\tau(m; \alpha_m) = v_\tau(m+1; \alpha_m)$ for $m = 0, 1, \ldots, M-1$, and*
2. *either $M = \overline{M}$ (the support of τ includes all available messages) or $v_\tau(M; 1) \geq v_\tau(M+1; 1)$, and $\tau(1)$ puts probability 1 on message M unless $v_\tau(M; 1) = v_\tau(M+1; 1)$.*

Necessity of both these conditions is straightforward: the first follows from continuity of each payoff function $v_\tau(\,\cdot\,; m)$, and the second is required for a bidder with type $S_i = 1$ to be playing a best-response. The significant part of Lemma 2 is that these conditions are sufficient: that if a partitional strategy is found satisfying indifference of the threshold types, and unused messages (if they exist) are not a profitable deviation for a bidder with the highest type, then the strategy in question is a symmetric equilibrium. This allows us to prove existence constructively, via an algorithm that finds exactly such a strategy, leading to the following result:

Theorem 1. *Fix an environment and an indicative bidding mechanism. A symmetric equilibrium exists. Further, given the environment and n, there is a number M^*, with $1 \leq M^* < \infty$, such that...*

- *if $\overline{M} \leq M^*$, a symmetric equilibrium exists in which all available messages are used with positive probability*
- *if $\overline{M} > M^*$, a symmetric equilibrium exists in which only the messages $\{0, 1, \ldots, M^*\}$ are used*

Thus, a symmetric equilibrium can always be found in which exactly $\min\{\overline{M}, M^*\}$ opt-in messages are used.

In two special cases, we can show that this is essentially the *unique* symmetric equilibrium. Maintaining the normalization that $S_i \sim U[0,1]$, if either...

1. $V_i = u(S_i)$, with u increasing and weakly convex, or
2. $n = 2$ and $V_i = \beta S_i + T_i$, with $\{T_i\}$ independent of $\{S_i\}$

then the equilibrium found in Theorem 1 is (up to the strategies of the indifferent types $\{\alpha_m\}$) the *only* symmetric equilibrium.[17] In the more general case, we are not certain whether uniqueness

[17] In these two cases, the additional properties that allow us to establish uniqueness of the symmetric equilibrium also lead to proofs of certain comparative statics. For example, in both cases, M^* is decreasing in c; in the latter case, M^* is also decreasing in N, and (if $\{T_i\}$ are independent across bidders) weakly increases when a mean-preserving spread is applied to the distribution of T_i.

holds, and if not, how much multiplicity to expect. However, in a sense, this can be thought of as an empirical problem. The constructive proof of Theorem 1 also offers a simple and computationally feasible way, given a particular environment and mechanism, to exhaustively search for *all* symmetric equilibria.

4 An Example

4.1 Preliminaries

In this section, we use an example to illustrate the construction of a symmetric equilibrium and some of its comparative statics. For the example, we let $V_i = S_i$, and therefore assume that $\{V_i\}$ are independently and identically distributed uniformly on $[0,1]$ and known perfectly prior to due diligence.[18] In this case, the payoffs to a bidder from advancing to the auction stage depends only on the number and distribution of opponents who send the *highest* message, as opponents who send lower messages have lower values and are certain to bid less and lose. This property helps to simplify the calculation of bidders' payoffs.

Suppose bidder i with type s_i is told that she has been selected and that her k highest opponents have types drawn (uniformly) from the interval $[a,b]$. For a given realization s^* of the highest opponent's type, bidder i's payoff is $\max\{0, s_i - s^*\} - c$. The CDF of this highest opponent type is $\Pr(s^* < s) = \left(\frac{s-a}{b-a}\right)^k$, and its density function is therefore $\frac{k}{b-a}\left(\frac{s-a}{b-a}\right)^{k-1}$. Thus, conditional on her own type s_i, advancing to the auction, and facing a set of opponents of whom k have types in $[a,b]$ and the remainder have types below a, we can write bidder i's expected payoff as

$$V(s_i, k, [a,b]) \quad = \quad \int_a^b \max\{0, s_i - s\} \frac{k}{b-a} \left(\frac{s-a}{b-a}\right)^{k-1} ds \; - \; c$$

Note that this payoff does not depend on the message bidder i herself sent, nor on the messages of opponents who did not advance.

As noted earlier, for any strategy τ satisfying the conditions of Lemma 1, we can associate with the strategy a series of thresholds $\alpha_0 < \alpha_1 < \ldots < \alpha_{M-1}$ separating the subintervals of types sending each message. Since $\alpha = (\alpha_0, \ldots, \alpha_{M-1})$ fully determines the strategy of all but a measure 0 of bidder types, a bidder's expected payoff depends on her opponents' strategy τ only through the thresholds α associated with it; hereafter, we will write expected payoffs as $v_\alpha(\cdot;\cdot)$ rather than $v_\tau(\cdot;\cdot)$ to emphasize this. For any opt-in message, the *unconditional* expected payoff to bidder i if her type is s_i, she sends message m, and her opponents are all playing a strategy τ described by a

[18] This is also equivalent to the case where $V_i = S_i + T_i$, where $\{T_i\}$ are learned during due diligence but are perfectly correlated across bidders, so that if two or more bidders enter, the rents from the realization of $\{T_i\}$ are fully competed away.

series of thresholds α is given by

$$
\begin{aligned}
v_\alpha(m; s_i) &= \alpha_0^{N-1}(s_i - c) \\
&+ \sum_{m'=1}^{m-1} \sum_{h=1}^{N-1} \binom{N-1}{h} (\alpha_{m'} - \alpha_{m'-1})^j (\alpha_{m'-1})^{N-1-h} V(s_i, \min\{n-1, h\}, [\alpha_{m'-1}, \alpha_{m'}]) \\
&+ \sum_{j=1}^{N-1} \binom{N-1}{j} (\alpha_m - \alpha_{m-1})^j (\alpha_{m-1})^{N-1-j} \min\left\{1, \tfrac{n}{j+1}\right\} V(s_i, \min\{n-1, j\}, [\alpha_{m-1}, \alpha_m]) \\
&+ \sum_{k=1}^{n-1} \sum_{j=0}^{N-2} \binom{N-1}{k} \binom{N-1-k}{j} (1-\alpha_m)^k (\alpha_m - \alpha_{m-1})^j (\alpha_{m-1})^{N-1-k-j} \min\left\{1, \tfrac{n-k}{1+j}\right\} V(s_i, k, [\alpha_m, 1])
\end{aligned} \tag{1}
$$

This expression groups the profiles of opponent messages under which bidder i advances into four terms depending on the highest message sent by her opponents.

- The first term covers profiles in which bidder i is the only buyer to opt-in. She advances for sure, pays c, and gets the asset at a price of 0.
- The second term covers events in which the highest opt-in message sent by any of i's opponents, m', is less than m, and the number of opponents sending this message is h. In this case, bidder i advances for sure. Her expected payoff conditional on advancing depends upon the number of opponents who send message m' *and* advance; this number is h if $h < n-1$ and $n-1$ if $h \geq n-1$.
- The third term covers events in which the highest opt-in message sent by i's opponents is m, and the number of opponents sending m is j. Bidder i advances for sure if $j < n$, and with probablity $n/(j+1)$ otherwise. Conditional on advancing, the number of opponents determining bidder i's payoff is either j or $n-1$, whichever is lower.
- The fourth term covers scenarios in which $k < n$ of bidder i's opponents send messages higher than m, and j more send message m. If $k + j < n$, then bidder i advances for sure; if $k + j \geq n$, the k bidders who sent messages above m advance for sure, and bidder i advances with probability $(n-k)/(j+1)$. If she does advance, her payoff is determined by the highest type among the k bidders sending messages above m, whose types are drawn from the interval $[\alpha_m, 1]$.

When $m = 1$, the second term in equation (1) vanishes and, when $m = M$, the last term vanishes. Finally, the payoff to opting out, $v_\alpha(0; s_i)$, is 0.

Equation (1) shows that $v_\alpha(m; s_i)$ depends on α_m and the lower thresholds $\{\alpha_0, \alpha_1, \ldots, \alpha_{m-1}\}$, but not on thresholds higher than α_m. In the event that bidder i advances to the second round after sending message m, *every* opponent with type above α_m, and who therefore sends a message higher than m, also advances. Consequently, bidder i can treat these opponents as random draws from the interval $[\alpha_m, 1]$, and the payoff from winning against them does not depend on how this interval is partitioned. This is not the true for opponents who send messages less than m. In this case, there is some probability that the opponent with the highest type may not advance if $n-1$ other oppenents send the same message as she does. The likelihood of this event depends on how the interval below α_{m-1} is partitioned. However, since bidder i is certain to advance in this event, the payoff associated with it is the same whether she sends message m or $m+1$. As a result, it drops out of the difference $v_\alpha(m+1; s_i) - v_\alpha(m; s_i)$, which then depends only on $(\alpha_{m-1}, \alpha_m, \alpha_{m+1})$.

4.2 Constructing The Equilibrium

As noted in Lemma 2, in a symmetric equilibrium, bidder i must be indifferent between sending message $m+1$ and m at $s_i = \alpha_m$ for each $m \in \{0, ..., M-1\}$. Thus, a symmetric equilibrium must satisfy the M indifference conditions

$$v_\alpha(m+1; \alpha_m) - v_\alpha(m; \alpha_m) \quad = \quad 0, \qquad m \in \{0, .., M-1\}$$

As noted above, each indifference condition depends only on α_{m-1}, α_m, and α_{m+1} (or on α_0 and α_1 for the indifference condition $v_\alpha(1; \alpha_0) = v_\alpha(0; \alpha_0) = 0$). Further, the difference $v_\alpha(m+1; \alpha_m) - v_\alpha(m; \alpha_m)$ satisfies a strict single-crossing property in α_{m-1}; so for a given choice of α_m and α_{m+1}, there is a unique value of α_{m-1} that satisfies $v_\alpha(m+1; \alpha_m) - v_\alpha(m; \alpha_m) = 0$. We exploit this fact to construct the symmetric equilibrium "from the top down".

Postponing (for now) the problem of calculating M^*, given Theorem 1, we expect to find an equilibrium with $M = \overline{M}$ if $\overline{M} < M^*$, and with $M = M^*$ otherwise. (In the former case, the second condition in Lemma 2 is automatically satisfied, so a solution to the M indifference conditions constitutes an equilibrium; in the latter case, we will still need to show separately that $\nu_\alpha(M+1; 1) \leq v_\alpha(M; 1)$.) Given this value of M, for $t \in [0, 1]$, define $\alpha_M(t) = 1$ and $\alpha_{M-1}(t) = 1-t$. Define $\alpha_{M-2}(t)$ as the unique value of α_{M-2} satisfying $v_\alpha(M; \alpha_{M-1}(t)) = v_\alpha(M-1; \alpha_{M-1}(t))$ given $\alpha_M = 1$ and $\alpha_{M-1} = 1-t$; as noted above, this value is uniquely defined. Similarly, define $\alpha_{M-3}(t)$ to satisfy $v_\alpha(M-1; \alpha_{M-2}(t)) = v_\alpha(M-2; \alpha_{M-2}(t))$ given the values of $\alpha_{M-1}(t)$ and $\alpha_{M-2}(t)$, and so on, until $\alpha_0(t)$ is defined by $v_\alpha(2; \alpha_1(t)) = v_\alpha(1; \alpha_1(t))$.

Now, letting $\alpha(t) = (\alpha_0(t), \alpha_1(t), \ldots, \alpha_{M-1}(t))$, note that at every value of t, $\alpha(t)$ satisfies the "top" $M-1$ indifference conditions by construction. Thus, if there is a value t^* of t at which the bottom indifference condition, $v_{\alpha(t^*)}(1; \alpha_0(t^*)) = v_{\alpha(t^*)}(0; \alpha_0(t^*)) = 0$, is satisfied, then the thresholds $\alpha(t^*)$ will be an equilibrium. The proof of existence, then, involves showing that $v_{\alpha(t)}(1; \alpha_0(t))$ is positive at $t = 0$, negative at large t, and continuous in between. Thus, a solution t^* exists to solve $v_{\alpha(t)}(1; \alpha_0(t)) = 0$, and therefore to satisfy the M indifference conditions.

The only remaining issue is the value of M^*. When $M = M^* < \overline{M}$, it's necessary for equilibrium that bidders with the highest type $S_i = 1$ be unwilling to deviate from message M (the highest message being used in equilibrium) to higher, unused messages, in order to advance to the auction for certain. By sending message M, bidder i is already assured of advancing except when at least n other bidders send message M as well, in which case every bidder advancing will have a type above α_{M-1}. The condition for unused messages to be unprofitable deviations, then,

$$v_\alpha(M+1;1) \quad \leq \quad v_\alpha(M;1)$$

turns out to be equivalent to the condition

$$V(1, n-1, [\alpha_{M-1}, 1]) \quad \leq \quad 0$$

i.e., that a bidder with type $s_i = 1$ does not make money, on average, from an auction fully stocked with competitors with types who send message M in equilibrium. Since the left-hand side of this latter expression is decreasing in α_{M-1}, this in turn requires that α_{M-1} be sufficiently close to 1. (If the interval $[\alpha_{M-1}, 1]$ were large, a bidder with type $S_i = 1$ would want to advance even if all her opponents were in this interval; if it is sufficiently small, however, she would not.) As M gets larger, the interval $[\alpha_{M-1}, 1]$ gets narrower; by defining M^* as the highest value of M for which the construction above works, this condition ends up being automatically satisfied for $M = M^*$.[19]

It is worth noting that this final necessary condition for equilibrium when $\overline{M} > M^*$ is analogous to the No Incentive To Separate (NITS) condition introduced by Chen, Kartin, and Sobel (2008).[20] In their model, it is an equilibrium refinement, and selects a unique equilibrium, the one with the maximal number of messages. In our setting, since we assume the seller is committed to advancing bidders monotonically based on messages, this is not a refinement, but a condition that must be satisfied in any equilibrium where some messages are not used.

4.3 Comparative Statics

Next, we use this example to illustrate how the symmetric equilibrium varies with the indicative bidding mechanism chosen. We set $c = 5$ and $N = 5$, maintain the assumption that $V_i = S_i$, but now let S_i be drawn uniformly on $[0, 100]$ rather than $[0, 1]$ (so that results are easier to read). Table 1 reports the equilibrium thresholds, revenues, bidder and total surplus for various values of $\overline{M}$ when $n = 2$. In this example, $M^* = 3$, so when $\overline{M} \geq 3$, the equilibrium uses only messages $\{0, 1, 2, 3\}$. (If we tried to construct an equilibrium with more messages, we would fail because all the thresholds could not fit into the type space $[0, 100]$.) When $\overline{M} < 3$, bidders use all of the available messages. In these cases, the highest type would like to separate but cannot do so because of the constraint that the seller imposes on the size of the message space. Note that the intervals are narrower at higher messages. In this sense, there is finer sorting at the top of the type space

[19] More formally, we define (in the proof of Theorem 1) M^* as the highest value of M for which $\alpha_0 < \alpha_1 < \ldots < \alpha_{M-2} < \alpha_{M-1}$ exist such that $\alpha_{M-1} = \alpha_M = 1$, the indifference condition $v_\alpha(m+1; \alpha_m) = v_\alpha(m; \alpha_m)$ holds at each $m = 1, 2, \ldots, M-1$; and $v_\alpha(1; \alpha_0) > 0$.

[20] In their setting, NITS is the condition that the *lowest*-type sender would not choose to reveal his type truthfully if he could, while in our model, the condition is on the highest type. As noted above, the direction in which the sender/bidder would like to misrepresent his type goes in opposite directions in the two models.

Table 1: $N = 5$, $V_i = S_i \sim U[0, 100]$, $c = 5$, $n = 2$, various $\overline{M}$

Opt-in messages available ($\overline{M}$):	1	2	3+
α_3	–	–	100.00
α_2	–	100.00	98.12
α_1	100.00	83.79	83.64
α_0	51.50	49.45	49.42
Revenue	53.67	57.21	57.26
Bidder Surplus	16.96	15.44	15.42
Total Surplus	70.63	72.65	72.68

than at the bottom.[21]

As $\overline{M}$ increases, the opt-in threshold α_0 decreases. As a result, both the probability that a bidder opts in, and the expected number of bidders opting in, increases with $\overline{M}$. In addition, as $\overline{M}$ increases and more messages are used, the bidders sort more effectively, and the selected bidders are more likely to be those with the highest types. Both of these effects favor higher revenue: greater participation means that the seller is more likely to sell the asset at a positive price (since this requires at least two bidders to opt in); and better selection implies that the seller is likely to sell for a higher price. Note, however, that bidder surplus goes the opposite direction, decreasing as $\overline{M}$ increases. Bidders benefit heavily from being the only one to opt in, which is more likely when $\overline{M}$ is lower; and they benefit from less effective sorting, since it increases the chance they do not face the toughest possible competition. Still, the increase in revenue appears to dominate the decrease in bidder surplus: in every example we've solved, total surplus is increasing in $\overline{M}$.[22]

In Table 2, we focus on the case where $\overline{M}$ is sufficiently large that $M = M^*$, and report the equilibrium partition and payoffs for values of n ranging from 2 to 5. As n increases, more bidders

Table 2: $N = 5$, $V_i = S_i \sim U[0, 100]$, $c = 5$, various n, $M = M^*$

Bidders advancing to second round (n):	2	3	4	5
α_3	100.00	–	–	–
α_2	98.12	100.00	100.00	–
α_1	83.64	91.04	94.26	100.00
α_0	49.42	53.93	54.82	54.93
Revenue	57.26	56.19	55.87	55.82
Bidder Surplus	15.42	14.27	13.99	13.96
Total Surplus	72.68	70.46	69.86	69.78

are likely to advance to the auction, making the auction more competitive. Bidders respond to

[21] Under the two special cases noted above in which uniqueness holds, this property holds as well when the number of available messages is not a binding constraint, i.e., when $\overline{M} \geq M^*$.

[22] We found the same result in numerical examples with private value updating, but these were limited to the case where $n = 2$ and $V_i = S_i + T_i$, with T_i independently distributed according to an exponential distribution.

lower expected payoffs from advancing by sending (weakly) lower messages. They are less likely to participate, and use fewer opt-in messages in equilibrium: in this example, the expected number of bidders opting in falls from 2.53 to 2.28 as n increases from 2 to 5, and the number of opt-in messages used falls from 3 to 1. The lower participation rate and less effective sorting have a negative impact on both revenue and bidders surplus: both are decreasing in n. Thus, both sides of the market benefit from reducing redundant entry costs.

The results of Tables 1 and 2 suggest that within this particular example, there is no tradeoff between efficiency and revenues in choosing the details of the indicative bidding mechanism used: both revenue and total surplus are maximized when bidders can send as many messages as they want (i.e., $\overline{M} = \infty$) and the number of bidders who can advance is limited to two (i.e., $n = 2$). We conjecture that this result holds more generally (we have not found any counterexamples), but have not been able to prove it. The main difficulty lies in quantifying the change in selection as the mechanism changes: since the equilibrium partitions are not ordered (one is not a refinement of another) as $\overline{M}$ or n changes, there are particular type profiles favoring either mechanism.[23] To prove the claim, one would need to show that expected revenues *averaged* across all type profiles are weakly increasing in $\overline{M}$ and decreasing in n, which we have been unable to prove. (With private value updating – i.e., when S_i is only a noisy signal of V_i – the tradeoff is more complicated, especially in regard to n, since even admitting the two bidders with the highest types does not ensure that the bidders with the highest valuations V_i advance.)

5 Welfare and Revenue

In this section, we evaluate the performance of the indicative bidding mechanism. The benchmark is an auction with unrestricted entry, where bidders choose (independently and simultaneously) whether to enter. The timing is the same as in our model: bidders learn their types S_i, decide simultaneously whether to enter, and then those who chose to enter incur the cost of due diligence, learn their values V_i, and submit binding bids.[24] We continue to assume a second-price auction, so bidding truthfully remains a dominant strategy. The symmetric equilibrium involves a cutoff strategy, in which bidders enter when their type S_i is above some threshold γ, which we refer to as the entry threshold.

Recall that $u_n(S_i, S_{-i,n})$ is defined as a bidder's expected payoff in an n-bidder auction as a

[23] For example, compare the mechanisms with $n = 2$ and $n = 3$ illustrated in Table 2. Suppose that bidder types are 90, 85, 65, 60, and 40. When $n = 2$, the bidders with the two highest types send message 2, while the next two send message 1; the top two advance, and revenue is 85. When $n = 3$, however, the top four bidders all send message 1; with probability $\frac{1}{2}$, either the highest or second-highest-type bidder is excluded, leading to revenue of only 65; so the mechanism with $n = 2$ yields higher expected revenue. On the other hand, if bidder types were 99, 92, 84, 40, and 30, then when $n = 2$, revenue would be either 92 or 84, with equal probability, while when $n = 3$, revenue would be 92 for sure; so the mechanism with $n = 3$ yields higher expected revenue.

[24] This is therefore the "selective entry model" considered by Marmer, Shneyerov and Xu (2013), but without their assumption that (V_i, S_i) are independent across bidders. The model can be thought of as a hybrid between the model of Levin and Smith (1994), in which bidders have no private information at the time of entry, and the model of Samuelson (1985), in which bidders know their valuations prior to entry and do not update at all post-entry.

function of his own type and his $n-1$ opponents' types. As in the last section, let

$$V(s_i, n-1, [\gamma, 1]) \quad \equiv \quad E\left\{u_n(s_i, S_{-i,n}) \mid S_{-i,n} \in [\gamma, 1]^{n-1}\right\} \; - \; c$$

denote the expected payoff when the $n-1$ opponents have types drawn randomly from the interval $[\gamma, 1]$, with $V(s_i, 0, [\gamma, 1]) \equiv u_1(s_i) - c$ being the payoff from advancing alone. The entry threshold is then uniquely determined by

$$0 \quad = \quad \sum_{j=0}^{N-1} \binom{N-1}{j} (1-\gamma)^j \gamma^{N-1-j} V(\gamma, j, [\gamma, 1])$$

since $\binom{N-1}{j}(1-\gamma)^j\gamma^{N-1-j}$ is the probability that exactly j of a bidder's $N-1$ opponents will have types above γ (and therefore enter the auction) and $V(\gamma, j, [\gamma, 1])$ is the bidder's payoff from entering in that case. (It's not hard to show that the right-hand side is strictly increasing in γ, and that this therefore defines a unique entry cutoff, which characterizes the unique symmetric equilibrium.)

The comparison between the indicative bidding mechanism and a standard auction with endogenous entry involves a clear tradeoff. On the one hand, by limiting the number of bidders who perform due diligence and make binding bids, the indicative bidding mechanism introduces the risk that the seller will not receive bids from the "right" bidders – those with the two highest valuations among those who opted in. This could happen due to tiebreaking eliminating a bidder with one of the two highest initial signal, or due to a bidder with a lower initial signal having an unexpectedly high valuation. On the other hand, because the indicative bidding mechanism reduces the "risk" to a marginal bidder who opts in – in the event competition is very strong, she won't be selected and won't incur the entry cost – it reduces the threshold at which bidders are willing to opt in, and therefore increases the likelihood that two or more bidders opt in and the seller earns positive revenue.

To see which of these two effects dominates, we use the following example. We let $V_i = S_i + T_i$, where $\{T_i\}$ are new signals learned upon entry and are independent of $\{S_i\}$. We let $c = 5$, $\{S_i\}$ be independently and identically distributed uniformly on $[0, 100]$, and $\{T_i\}$ be independently and identically distributed according to the exponential distribution with parameter $\lambda = 0.12$.[25] We consider the indicative bidding mechanism with $n = 2$ and $\overline{M}$ sufficiently large to not bind.

As noted above, α_0 is always less than γ, that is, the opt-in threshold for indicative bidding is always lower than the entry threshold for the auction with unrestricted entry. This implies that, for each N, the number of bidders opting in under indicative bidding first-order stochastically dominates the number of bidders entering under unrestricted entry. Hence, the probability of a sale and the probability of two or more bidders is higher under indicative bidding. The former is good for efficiency and the latter for revenue. On the other hand, since entry into the second round

[25] The exponential distribution was chosen because $T_i - T_j$ then follows a Laplace distribution, making $u_2(s_i, s_j)$ straightforward to calculate. In this case, $E \max\{0, T_i - T_j\} = \frac{1}{2\lambda} = 4\frac{1}{6} < c$, so the "small rents" assumption holds.

of the indicative bidding mechanism is capped at two bidders, the expected revenue conditional on at least two entrants is lower than it would be with unrestricted entry, as the seller may select the "wrong" bidders, reducing both efficiency and revenues. Nonetheless, the participation effect appears to consistently dominate the selection effect: for each N, revenue, bidder surplus and total surplus are all higher under indicative bidding.

Table 3: Indicative bidding ($n = 2$, $M = M^*$) versus unrestricted entry, various N

Indicative Bidding

Potential Bidders (N)	3	4	5	7	10	20	50	200
α_0 ("opt-in" threshold)	29.59	38.49	45.48	55.66	65.40	79.88	90.82	97.40
Revenue	50.50	59.52	65.63	73.42	79.98	88.77	95.20	99.50
Bidder Surplus	23.79	19.50	16.63	12.98	9.95	5.96	3.12	1.24
Total Surplus	74.29	79.02	82.25	86.40	89.92	94.73	98.32	100.74

Unrestricted Entry

Potential Bidders (N)	3	4	5	7	10	20	50	200
γ (entry threshold)	33.56	44.65	52.68	63.33	72.53	84.84	93.39	98.25
Revenue	50.04	58.24	63.58	70.15	75.65	83.22	89.27	93.28
Bidder Surplus	22.91	18.40	15.55	12.15	9.39	5.76	2.87	0.80
Total Surplus	72.95	76.63	79.12	82.30	85.03	88.98	92.14	94.08

The dominance of the indicative bidding mechansim depends on the message space being unconstrained. When N is low and $\overline{M} = 1$ – bidders are restricted to just opting in or opting out, rather than sorting themselves more finely through differential indicative bids – the selection effect (the risk of not advancing the right bidders) is exacerbated, and revenue is lower than under unrestricted entry. In all the numerical examples we've run, however, indicative bidding has consistently Pareto-dominated unrestricted entry when $\overline{M} \geq M^*$, i.e., when the number of messages available is not a binding constraint.

While we cannot prove that this result holds universally, we can prove it holds when N is sufficiently large. As N gets large, the risk under indicative bidding of selecting the "wrong" bidders shrinks, as only bidders with types close to 1 opt in, and the participation effect dominates. It's relatively easy to characterize limiting behavior in both the indicative bidding mechanism and the unrestricted auction as N grows, which allows us to compare the results for large N. When N is large, $M^* = 1$, and any symmetric equilibrium of the indicative bidding game has $M = 1$ (only messages 0 and 1 are used). As N grows, the cutoff for bidders opting in, α_0, approaches 1 at rate $\frac{1}{N}$, so in the limit, the probability distribution of the number of bidders opting in approaches a Poisson distribution, with all bidders opting in having types arbitrarily close to 1. In the unrestricted auction, the entry threshold γ similarly goes to 1 at rate $\frac{1}{N}$, and the number of bidders entering likewise approaches a Poisson distribution. We can easily characterize expected payoffs conditional on the number of entrants, allowing a clear comparison between the two mechanisms in the limit, and leading to the following results:

Theorem 2. *Fix an environment other than N. For any indicative bidding mechanism $(n, \overline{M})$, if N is sufficiently large, expected revenue and total surplus are both strictly higher than in an unrestricted auction.*

Note that as N grows, the optimal reserve price in the auction with unrestricted entry goes to 0, so we are comparing to the "optimal" standard auction.

The example above showed that not only did the indicative bidding mechanism outperform the unrestricted auction in expected revenue and total surplus, but also in bidder surplus. In the special case where V_i is additively separable into independent components learned before and after entry, this holds in general as well:

Theorem 3. *Suppose $V_i = u(S_i) + T_i$, with u strictly increasing and continuously differentiable and $\{T_i\}$ independent of $\{S_i\}$ (but not necessarily independent across i). Then for any indicative bidding mechanism $(n, \overline{M})$, if N is sufficiently large, bidder surplus is strictly higher than in an unrestricted auction.*

In the case of large N, we can also characterize which indicative bidding mechanisms perform best.

Theorem 4. *Fix an environment other than N. If N is sufficiently large...*

1. *The set of messages $\overline{M}$ doesn't matter, as $M = 1$ in equilibrium*
2. *Expected revenue and total surplus are both strictly decreasing in n*
3. *For any given indicative bidding mechanism $(n, \overline{M})$,*
    - *A positive reserve price would strictly reduce both expected revenue and total surplus*
    - *A small entry subsidy would strictly increase both expected revenue and total surplus*
4. *In the case of additive separability, bidder surplus is also strictly decreasing in n, strictly decreases if a positive reserve price is used, and strictly increases if a small entry subsidy is used*

Proofs of Theorems 2, 3, and 4 are given in the appendix, but much of the intuition can be gained rather quickly. In both the indicative bidding mechanism and the unrestricted auction, as N grows, total bidder surplus goes to 0,[26] so showing that total surplus is higher with indicative bidding, or decreases with n, or decreases with a positive reserve price, also establishes the same properties for revenue.

[26]Expected bidder surplus must be 0 at type $S_i = \alpha_0$ or $S_i = \gamma$, so if the effect of S_i on the distribution of V_i is bounded, a single bidder's ex ante expected surplus can be thought of as the area of a triangle whose base and height are both proportional to $\frac{1}{N}$, so the N bidders' combined surplus is proportional to $N\frac{1}{N^2} = \frac{1}{N}$.

The key to all of these results is that participation in the indicative bidding mechanism, while higher than participation in the unrestricted auction, is still below the socially optimal level. This is shown formally in the appendix, but can best be understood in the following way. In a standard English auction with private values, one bidder's entry decision imposes no net externality on the rest of the environment. (If the new entrant wins, her payoff is the difference between her own valuation and the second-highest, which is exactly her contribution to social surplus. If she loses and sets the price, her presence transfers surplus from the winner to the seller; if she loses and doesn't set the price, her presence has no effect.) However, in the indicative bidding mechanism, a bidder's entry decision can impose an externality, because her choice to opt in might lead to another bidder not advancing to the auction. This happens only when at least n bidders are already opting in, and therefore the auction would be "full" regardless. When N is large, all entrants have roughly the same type, so a marginal entrant's entry decision (when at least n others are opting in) does not effect the seller's or the other entrants' expected payoffs; the bidder who would have advanced otherwise but now does not, by the small rents assumption, was anticipating a negative payoff from the auction, and so the net externality caused by the entrant is *positive*. As a result, participation is below the socially optimal level, and any change that increases participation – lowering n, reducing the reserve price, or subsidizing entry – increases total surplus on the margin via participation. And none of these changes lower total surplus directly; so each of them increases surplus overall, and therefore increases revenue as well.

6 Extensions

6.1 Pure Cheap Talk

As is usual in the mechanism design literature, we have assumed the seller can credibly commit to the mechanism used, which includes how he responds to each message. Thus, the messages are not true cheap talk, as their "meaning" is built into the mechanism.

However, we can easily reverse this assumption. Suppose that bidders have a true "opt out" option, but beyond that, the indicative bids they submit (or the messages they send) are not binding on the seller in any way; the seller is committed to advancing a maximum of n bidders to the auction (and is committed to an English auction with no reserve price), but can select whichever bidders he wants out of those who opted in.

In this case (assuming $\overline{M}$ is large), a symmetric equilibrium exists with M opt-in messages used, for each $M = 1, 2, 3, \ldots, M^*$. Returning to Lemma 2, the second condition for equilibrium no longer needs to hold, since pessimistic off-equilibrium-path beliefs by the seller can be used to deter deviations away from any smaller set of equilibrium messages. Thus, for any M, we can find the thresholds $(\alpha_0, \ldots, \alpha_{M-1})$ satisfying the $M - 1$ indifference conditions, and use those to construct equilibrium strategies. Thus, if indicative bids are pure cheap talk, the equilibrium we found earlier is still an equilibrium, but we are now subject to the usual multiplicity problem common to cheap talk games.

6.2 Relaxing "Small Rents"

We have focussed on the case of "small rents," imposing the assumption that $u_2(s_i, s_j) < c$ for $s_j \geq s_i$. This is a key step in showing that $v_\tau(m+1; \alpha_m) - v_\tau(m; \alpha_m)$ is single-crossing in α_{m-1}, allowing us to construct equilibrium thresholds "down from the top" of the type space.

Based on numerical examples, "small" violations of this inequality do not have a discontinuous effect on equilibrium. But without this assumption, we cannot in general prove existence of a symmetric equilibrium. However, if we focus on the case of large N, we can characterize what must happen in any symmetric equilibrium, should one exist.

Let $u_n(1, \mathbf{1})$ denote the expected auction payoff to a bidder with type $S_i = 1$ facing $n-1$ opponents with type 1 as well. Since $u_n(1, \mathbf{1})$ is decreasing in n, define n^* as the unique solution to

$$u_{n^*}(1, \mathbf{1}) - c \quad > \quad 0 \quad \geq \quad u_{n^*+1}(1, \mathbf{1}) - c$$

Thus, n^* is the largest auction in which a bidder with type $S_i = 1$ would willingly compete, even if all his competitors were as strong as him. (The small rents assumption would imply that $n^* = 1$.) Consider the generic case where both inequalities hold strictly.

If $n > n^*$, then as N gets large, equilibrium is similar to the "small rents" case already analyzed. Very strong bidders opt in, but in the hope that the auction will not be "full"; conditional on $n-1$ other bidders entering, they would anticipate a negative expected payoff. Only messages 0 and 1 are used in equilibrium, and the opt-in threshold α_0 goes to 1 at rate $\frac{1}{N}$. Above n^*, for the same reasons as in Theorem 4 above, revenue and total surplus are decreasing in n.

If $n \leq n^*$, however, things change. A bidder with type $S_i \approx 1$ anticipates a positive expected payoff from advancing to the auction, even if the auction is "full" and even if all his competitors are also strong. Thus, if $\overline{M} = \infty$, no symmetric equilibrium can exist: the highest type of bidder would always want to deviate to a higher message. If $\overline{M}$ is finite, however, symmetric equilibrium may exist. Let ξ be the solution to

$$V(\xi, n-1, [\xi, 1]) \quad = \quad 0$$

so that a bidder with type $S_i = \xi$ is exactly indifferent to entering an auction against $n-1$ opponents with types above ξ. In the limit as N gets large, in any symmetric equilibrium, all bidders with types in the interval $[\xi, 1]$ pool on the highest available message $\overline{M}$. While we can't tell exactly what the lower types are doing, it doesn't matter for payoffs, as when N is sufficiently large, more than n bidders will have types above ξ with probability going to 1. Thus, there will be no chance of the object not selling; the only inefficiency comes from the fact that with ξ bounded away from 1, the bidders advancing will not be the strongest ones. For this reason, increasing ξ increases both revenue and total surplus, by improving selection; and as a result of this, below n^*, revenue and total surplus are *increasing* in n.

Thus, assuming symmetric equilibrium always exists, both revenue and total surplus are single-peaked in n, with an optimum of either $n = n^*$ or $n = n^* + 1$.

6.3 First Price Auctions

Finally, we consider an indicative bidding mechanism where the second-stage auction is a first price sealed-bid auction.

In a first-price auction, a bidder's bid depends both on his own valuation and on his beliefs about his opponents' valuations. This makes the analysis more complicated in a number of ways. While we do not have results about the existence of symmetric equilibrium in our general model, we can construct the equilibrium for the example of Section 4, in which $V_i = S_i \sim [0, 100]$, as an illustration of the fact that it may be possible more generally. In this example, a bidder will rationally update his beliefs about his opponents' types based on the new information that he has advanced to the auction, but he does not receive any other information. Thus, there is no loss in imagining that each bidder solves a static problem, choosing both his message m and his bid b at the beginning, just choosing b optimally for the beliefs that will prevail should he advance. As a result, we can treat the game as a static game and use standard mechanism design tools like the envelope theorem to characterize the equilibrium bid function.

A symmetric equilibrium will have the following properties:

1. The type space is partitioned into subintervals by thresholds $0 < \alpha_0 < \alpha_1 < \ldots < \alpha_{\overline{M}-1} < \alpha_{\overline{M}} = 100$, with types below α_0 opting out and types in the interval (α_{m-1}, α_m) sending message m.

2. On the equilibrium path, those bidders who advance bid $\beta(S_i)$, where $\beta(\alpha_0) = 0$ and β is strictly increasing and continuous on $[\alpha_0, 100]$.

It's not hard to show these are necessary conditions for a symmetric equilibrium. (If there were discontinuities in β or if $\beta(\alpha_0) > 0$, this would mean holes in the support of $\beta(S_i)$, which are impossible in a pay-as-bid auction with a symmetric equilibrium.) Along with these conditions, two more turn out to be necessary and sufficient for equilibrium: indifference of bidders with threshold types α_m between sending message m (and then bidding $\beta(\alpha_m)$ if selected) and sending message $m+1$ (and then bidding $\beta(\alpha_m)$); and an envelope theorem condition characterizing ex ante expected payoffs which constraints the bid function β. (This is formalized as Lemma 6 in the appendix.)

In the case of second-price auctions, we constructed the equilibrium from the top down: we guessed a value of α_{M-1}, calculated the other thresholds $\alpha_{M-2}, \alpha_{M-3}, \ldots, \alpha_0$ required to rationalize it (and each other), and then checked whether these thresholds satisfied the "terminal" condition $v_\tau(1; \alpha_0) = 0$, then adjusted the initial value of α_{M-1} until we found thresholds that did. In the case of first-price auctions, we work in the other direction: we will guess a value of α_0, then build up the equilibrium from there, finally checking whether a terminal condition holds at the *top* of the type space. This is because in addition to calculating message thresholds α_m, we also need to construct the equilibrium bid function β, and this is easier to do from the bottom of the type space.

The equilibrium construction is shown in the appendix. We should note that we do not have a theoretical proof that this construction will always work. However, we have tried it numerically for various values of c, N, and $\overline{M}$, and have never yet failed to find an equilibrium.

To facilitate comparisons with the second-price auction, Table 4 illustrates the equilibrium for the numerical example of Section 4 in which $c = 5$, $N = 5$, and $n = 2$. For each value of $\overline{M}$,

Table 4: $N = 5$, $V_i = S_i \sim U[0, 100]$, $c = 5$, $n = 2$, first-price auction, various $\overline{M}$

	Opt-in messages available ($\overline{M}$)						
	1	2	3	4	5	7	10
α_{10}	–	–	–	–	–	–	100.00
α_9	–	–	–	–	–	–	99.94
α_8	–	–	–	–	–	–	99.87
α_7	–	–	–	–	–	100.00	99.73
α_6	–	–	–	–	–	99.47	99.44
α_5	–	–	–	–	100.00	98.83	98.83
α_4	–	–	–	100.00	97.71	97.56	97.56
α_3	–	–	100.00	95.21	94.91	94.89	94.89
α_2	–	100.00	89.92	89.27	89.21	89.20	89.20
α_1	100.00	78.47	76.96	76.82	76.81	76.81	76.81
α_0	51.50	48.54	48.26	48.24	48.23	48.23	48.23
Revenue	53.67	57.40	57.69	57.71	57.71	57.71	57.71
Bidder Surplus	16.96	15.45	15.33	15.32	15.32	15.32	15.32
Total Surplus	70.63	72.84	73.02	73.03	73.03	73.03	73.03

there is a unique symmetric equilibrium in which all of the messages are used. The intervals are narrower at higher messages, so there is finer sorting at the top of the type space, particularly at higher values of $\overline{M}$. Even though bidders now use all of the available messages, the impact of the additional messages becomes minimal very quickly. The lower thresholds quickly asymptote, as do expected payoffs; the additional messages only serve to divide up the very top of the type space more finely. For example, when $\overline{M} = 10$, the top four messages are used only by bidders with types above 99.44, and 90% of bidders who opt in use message 1, 2 or 3. In fact, up to the precision shown in the table, revenue and bidder surplus do not change with $\overline{M}$ once $\overline{M}$ is above 4. Thus, even though there is no "natural" upper bound on the number of messages, this property turns out not to be payoff-relevant.

The above example illustrates an important difference between the equilibrium of first-price and second-price mechanisms. A bidder in a first-price auction always shades her bid by more than c below her value and since she pays her bid, she always earns a positive payoff from winning. Consequently, the small rents assumption no longer implies that each element of the equilibrium partition of types must have width at least ε, so there is therefore no natural upper bound on the number of messages used in equilibrium. Furthermore, the bidder with the highest possible type strictly prefers to advance to the auction even if his opponents have types very close to his own. This means that no matter how narrow the top interval gets, a bidder with the highest type would still choose to deviate if an unused high message was available so that she can be selected with probability 1. Hence, all messages are used and an equilibrium is only possible when $\overline{M} < \infty$.

As in the second-price mechanism, as $\overline{M}$ increases, expected revenue increases due to both greater participation and finer selection; bidder surplus decreases; and total surplus increases. Comparing Table 4 to Table 1, revenue and total surplus are marginally higher under the first-price than under the second-price mechanism, but this has nothing to do with the extra messages used, as it holds for $\overline{M}$ as low as 2. Instead, this appears to be driven by the equilibria shown in Table 4 having a slightly lower opt-in threshold α_0, and having thresholds which are more evenly distributed across the interval $[0, 100]$, leading to both more participation and better sorting. (Consistent with our earlier discussion, these same differences also lead to bidder surplus being lower in the first-price mechanism, but total surplus being higher.)

In the case of private value updating (where new information about V_i is learned during due diligence), equilibrium would be more complicated, because for a given realized valuation V_i, a bidder's second-stage bid will still depend on the message he sent at the first stage. (This is because he rationally updates his beliefs about his opponents' types conditional on the event that he himself advanced to the auction, and this updating depends on the message that he himself sent.) Even in the simple example where $V_i = S_i$, a bidder who deviated in the first stage would also optimally bid differently in the first stage, and we need to account for "two-stage deviations" like this in verifying that we have found an equilibrium.

7 Conclusion

We have developed a theory of indicative bidding. The theory establishes that, when entry is costly and expected rents from the private information obtained from entry are small relative to those costs, a seller can use indicative bids to "thin the field" and then hold an auction among a smaller number of buyers. The indicative bids are informative, and their use often leads to greater efficiency and higher revenue than an auction with unrestricted entry, particularly when the number of buyers is large. The theory explains the widespread use of indicative bidding in takeover auctions, where buyers need to conduct costly due diligenece prior to submitting binding bids.

The theory also provides an empirical framework for structural estimation of takeover auctions. The goal would be to use observable variation in the number of potential buyers, the number and bids of buyers submitting indicative bids, the number and bids of buyers submitting final bids, and the deal premium across heterogeneous auctions to estimate entry costs and the joint distribution of signals and values of buyers. Estimates of these model primitives can be used to quantify how much information buyers gain from due diligence (allowing one to directly test the "small rents" assumption) and to study how the deal premia would change if the target firm were sold via a negotiation rather than an auction. Gentry and Stroup (2015) develop an estimation strategy to address these issues but, due to limited data at the time, did not use information on indicative bids.[27] Our theory suggests that these bids provide useful information for identifying and estimating model primitives and should be incorporated into the econometric model.

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Appendix A. Proofs

A.1 Preliminaries

To begin, we prove one fact mentioned in the text, which we will make use of multiple times below.

Lemma 3. *There exists $\varepsilon > 0$ such that $u_2(s_i, s_j) \geq c$ requires $s_i > s_j + \varepsilon$.*

Proof. This follows from the assumptions that $u_2(\cdot, \cdot)$ is continuous in both arguments and $u_2(s, s) < c$ for every s. Suppose it were false. Then for any $\delta > 0$, one could find $s \in [\delta, 1]$ such that $u_2(s, s - \delta) \geq c$. Let $\delta_\ell = 1/\ell$, and let $\{s_\ell\}_{\ell=1,2,\ldots}$ be a sequence such that $u_2(s_\ell, s_\ell - \delta_\ell) \geq c$ for every ℓ. Since $s_\ell \in [0, 1]$ are bounded, $\{s_\ell\}_{\ell=1,2,\ldots}$ has a convergent subsequence; let $\{s_{j(k)}\}_{k=1,2,\ldots}$ be such a subsequence, and let $s_* = \lim s_{j(k)}$. Then $u_2(s_{j(k)}, s_{j(k)} - \delta_{j(k)}) \geq c$ for every k, with $\{s_{j(k)}\} \to s_*$ and $\{\delta_{j(k)}\} \to 0$; by continuity of u_2, $u_2(s^*, s^*) \geq c$, giving a contradiction. □

Throughout the rest of the appendix, ε will refer to the value in Lemma 3, i.e., a value such that $s_i \leq s_j + \varepsilon$ implies $u_2(s_i, s_j) < c$.

A.2 Proof of Lemma 1

Lemma 1 (in the text) says that any symmetric equilibrium must use a strategy which is weakly increasing and has support $\{0, 1, 2, \ldots, M\}$ for some $M < \infty$.

Weakly Increasing

Suppose τ is a symmetric equilibrium strategy, with $m \in \operatorname{supp} \tau(s)$ and $m' \in \operatorname{supp} \tau(s')$; we'll show that if $s < s'$ but $m > m'$, this leads to a contradiction.

As in the text, let $v_\tau(m; s)$ denote the expected payoff to bidder 1 if $S_1 = s$, he sends message m, and bidders 2 through N play the strategy τ. Let $\mathbf{m}$ denote a profile of messages sent by 1's opponents, and write v_τ as

$$v_\tau(m; s_1) \quad = \quad \sum_{\mathbf{m}} \Pr(\mathbf{m}) \Pr(adv|m, \mathbf{m}) V(s_1, \mathbf{m})$$

where $\Pr(\mathbf{m})$ is the probability (given τ) that 1's opponents send the messages $\mathbf{m}$, $\Pr(adv|m, \mathbf{m})$ is 1's probability of advancing to the auction if he sends message m and his opponents send message profile $\mathbf{m}$, and $V(s_1, \mathbf{m})$ is his expected payoff from advancing to the auction, given true type s_i, in expectation over all the different type profiles that would have generated $\mathbf{m}$. (What's significant here is that this last term $V(s_1, \mathbf{m})$ does not depend on m, the message sent by bidder 1: conditional on advancing *and on* $\mathbf{m}$, he faces the same distribution of opponent types regardless of the message he himself sends.)

Now, suppose $m > m'$. There are two cases to consider: either m and m' offer bidder 1 the same probability of advancing, or they do not. If they do not, then we can write

$$v_\tau(m; s_1) - v_\tau(m'; s_1) \quad = \quad \sum_{\mathbf{m}} \Pr(\mathbf{m}) \left[\Pr(adv|m, \mathbf{m}) - \Pr(adv|m', \mathbf{m})\right] V(s_1, \mathbf{m})$$

We consider two sub-cases: either for every $\mathbf{m}$ at which $\Pr(\mathbf{m}) [\Pr(adv|m, \mathbf{m}) - \Pr(adv|m', \mathbf{m})] > 0$, there is probability 1 that at least one of the other bidders advancing has type $S_j \geq s$; or at least one of these message profiles puts positive probability on $\max\{S_j\} < s$ among the other bidders j who advance.

- In the first case, the "small rents" assumption guarantees that $V(s, \mathbf{m}) < 0$ for every $\mathbf{m}$ in the sum, meaning $v_\tau(m; s) - v_\tau(m'; s) < 0$, contradicting the assumption that m is a best-response for a bidder with type $S_1 = s$.
- In the second case, the monotonicity assumption (Assumption 1(b)) guarantees that $V(s', \mathbf{m}) \geq V(s, \mathbf{m})$ for every $\mathbf{m}$, with strict inequality holding for some of them, so $v_\tau(m; s') - v_\tau(m'; s') > v_\tau(m; s) - v_\tau(m'; s)$; this means that either the first expression is strictly positive or the second is strictly negative, contradicting either m' being a best-response when $S_1 = s'$ or m being a best-response when $S_1 = s$.

This ensures that τ must satisfy monotonicity among messages that give different probabilities of advancing. What's left is to rule out non-monotonicities among multiple messages giving the same probability of advancing. That is, we need to rule out the possibility that m and m' are such that $\Pr(adv|m, \mathbf{m}) = \Pr(adv|m', \mathbf{m})$ for every $\mathbf{m}$ with $\Pr(\mathbf{m}) > 0$, since that would allow $m \in \operatorname{supp} \tau(s)$ and $m' \in \operatorname{supp} \tau(s')$ without contradicting monotonicity of V.

So suppose that were the case, that is, $\Pr(adv|m, \mathbf{m}) = \Pr(adv|m', \mathbf{m})$ for every $\mathbf{m}$ with $\Pr(\mathbf{m}) > 0$. If bidders with types s and s' were both playing strategies giving this probability of advancing, then all bidders with types $S_i \in (s, s')$ would also have to play messages giving this probability of advancing with probability 1, since otherwise this would violate the type of monotonicity we showed above. This means at least a measure $s' - s$ of types send messages giving this same probability of advancing. Let $\underline{m}$ denote the lowest such message, and let $\overline{m}$ denote a message giving the same probability of advancing but such that a positive measure of bidders play messages in $\{\underline{m}, \ldots, \overline{m}\}$ with positive probability. (If a positive measure of bidders only play messages giving this probability of advancing, such a message $\overline{m}$ must exist.) It's easy to show that $\overline{m}$ and $\underline{m}$ can't give the same probability of advancing, yielding a contradiction; this proves that symmetric equilibrium must be monotonic.

Finite Support

Now that we have monotonicity, define α_m as the supremum of the set of types s such that $\tau(s)$ puts positive probability on message m or lower (or as 0 if this set is empty). This means that $\alpha_{m-1} = \alpha_m$ if $m \notin \operatorname{supp} \tau$, and if $\alpha_{m-1} < \alpha_m$ then τ has types in (α_{m-1}, α_m) sending message m for sure.

Now, consider two messages m and $m' > m$, both of which are in the support of τ, and which have no message between them in the support of τ. For δ small enough, this means $\tau(\alpha_m - \delta) = m$ and $\tau(\alpha_m + \delta) = m'$, which requires $v_\tau(m; \alpha_m - \delta) \geq v_\tau(m'; \alpha_m - \delta)$ and $v_\tau(m; \alpha_m + \delta) \leq v_\tau(m'; \alpha_m + \delta)$. Since v_τ is a weighted sum of u_k terms, which are each continuous in a bidder's own type, they're continuous, and therefore $v_\tau(m; \alpha_m) = v_\tau(m'; \alpha_m)$.

As noted before,

$$v_\tau(m'; \alpha_m) - v_\tau(m; \alpha_m) \quad = \quad \sum_{\mathbf{m}} \Pr(\mathbf{m}) \left[\Pr(adv|m', \mathbf{m}) - \Pr(adv|m, \mathbf{m})\right] V(\alpha_m, \mathbf{m})$$

Now, all the message profiles $\mathbf{m}$ that show up in the sum have at least n opponents sending message m or higher, since if fewer than that did, then $\Pr(adv|m', \mathbf{m}) = \Pr(adv|m, \mathbf{m}) = 1$. This means that there are two types of terms in the sum:

- Profiles $\mathbf{m}$ that involve at least one opponent sending message m' or higher, and therefore have at least one opponent with type α_m or higher advancing, which by the small rents assumption have $V(\alpha_m, \mathbf{m}) < 0$
- Profiles $\mathbf{m}$ that involve at least n opponents sending message m and none sending a higher message, which therefore have $n-1$ opponents with types in $[\alpha_{m-1}, \alpha_m]$

Since the overall sum must be equal to 0, the latter terms must be positive (since the former are negative). This requires $u_2(\alpha_m, \alpha_{m-1}) > c$; by Lemma 3, this requires $\alpha_m - \alpha_{m-1} > \varepsilon$.

So if m and $m' > m$ are both played in equilibrium, there must be an interval of types of width at least ε who send message m. This applies to every message in the support of τ other than the highest and 0, so the maximal number of messages in the support of τ is $2 + \lceil \frac{1}{\varepsilon} \rceil$.

Support $\{0, 1, 2, \ldots, M\}$

First, we'll show that if messages m and m' are consecutive messages in the support of τ, there can't be any other message in between them not in the support of τ. (For this reason, no symmetric equilibrium exists when the set of allowed messages is continuous: such an equilibrium would require both that a finite number of messages be used and that no unused messages exist between messages used in equilibrium, which can't both hold.)

In the sum $v_\tau(m'; \alpha_m) - v_\tau(m; \alpha_m)$, separate the opponent message profiles $\mathbf{m}$ into two sets: let M_1 denote all the profiles where at least n opponents sent message m' or higher, and M_2 the profiles where fewer than n sent message m' or higher. Write

$$\begin{aligned}
v_\tau(m';\alpha_m) - v_\tau(m;\alpha_m) &= \sum_{\mathbf{m}\in M_1} \Pr(\mathbf{m}) \left[\Pr(adv|m',\mathbf{m}) - \Pr(adv|m,\mathbf{m})\right] V(\alpha_m,\mathbf{m}) \\
&\quad + \sum_{\mathbf{m}\in M_2} \Pr(\mathbf{m}) \left[\Pr(adv|m',\mathbf{m}) - \Pr(adv|m,\mathbf{m})\right] V(\alpha_m,\mathbf{m}) \\
&= \sum_{\mathbf{m}\in M_1} \Pr(\mathbf{m}) \left[\Pr(adv|m',\mathbf{m}) - 0\right] V(\alpha_m,\mathbf{m}) \\
&\quad + \sum_{\mathbf{m}\in M_2} \Pr(\mathbf{m}) \left[1 - \Pr(adv|m,\mathbf{m})\right] V(\alpha_m,\mathbf{m})
\end{aligned}$$

Now, by small rents, $V(\alpha_m,\mathbf{m}) < 0$ for every $\mathbf{m} \in M_1$, so the first sum is negative; which means since the entire expression must be zero, the second sum must be positive.

But the second sum is exactly the benefit that a bidder with type α_m would get if he deviated from message m to a message in between m and m', since he would no longer be rationed against other bidders sending message m, but would still not be selected when at least n others sent m' or higher. So for a bidder with type close to α_m, and therefore close to indifferent between m and m', another message in between them would give a strictly higher payoff than either one; so no such message can exist.

Next, I show that the lowest opt-in message must be used. Suppose not. Consider the lowest type sending any opt-in message. Since any opponent he faces will have type higher than him, small rents implies his payoff in the auction is negative whenever anyone else opts in. So if there was an opt-in message lower than the one he's using, he would deviate to it, since he would still advance when everyone else opted out but would be less likely to be chosen when others opted in.

Finally, if message 1 is being used, and used by an interval of width at least ε, then there is a positive probability that all of bidder 1's opponents send message 1; so if bidder 1 has the lowest type among the types who opt in, he must be getting a positive payoff from the probability all his opponents opt out, so 0 must be in the support of τ as well. □

A.3 Proof of Lemma 2

Lemma 2 says that if τ satisfies the conditions of Lemma 1, then the following are necessary and sufficient for τ to be a symmetric equilibrium:

1. $v_\tau(m;\alpha_m) = v_\tau(m+1;\alpha_m)$ for $m = 0, 1, \ldots, M-1$, and
2. either $M = \overline{M}$ (the support of τ includes all available messages) or $v_\tau(M;1) \geq v_\tau(M+1;1)$, and $\tau(1)$ puts probability 1 on message M unless $v_\tau(M;1) = v_\tau(M+1;1)$.

Proof of Necessity

Necessity of these conditions is easy to show.

- We noted above that $v_\tau(m+1; s_1) - v_\tau(m; s_1)$ must equal 0 at $s_1 = \alpha_m$, since it must be weakly negative for $s_1 \in (\alpha_{m-1}, \alpha_m)$ and weakly positive for $s_1 \in (\alpha_m, \alpha_{m+1})$ and $v_\tau(m+1; \cdot)$ and $v_\tau(m; \cdot)$ are both continuous.

- If the second condition is violated, then $\overline{M} > M$ and $v_\tau(M+1; 1) > v_\tau(M; 1)$. By continuity, $v_\tau(M+1; s_i) > v_\tau(M; s_i)$ for s_i sufficiently close to 1, contradicting the equilibrium requirement that all bidders with types $s_i \in (\alpha_{M-1}, 1)$ send message M.

- Finally, if $\overline{M} = M$ or $v_\tau(M+1; 1) < v_\tau(M; 1)$, then $\operatorname{supp} \tau(1)$ must put probability 1 on message M, since in the first case no higher messages exist and in the second case they give lower payoff than M.

Thus, what is left to do is to prove sufficiency – i.e., that if τ satisfies these conditions, it constitutes a symmetric equilibrium.

Proof of Sufficiency

So now suppose τ exists satsifying these conditions; we need to show that when one's opponents play τ, playing $\tau(s_1)$ is a best-response for a bidder with type s_1.

The key thing to note is that for $m' > m$, $v_\tau(m'; s_1) - v_\tau(m; s_1)$ is weakly increasing in s_1, since as noted above

$$v_\tau(m'; s_1) - v_\tau(m; s_1) \quad = \quad \textstyle\sum_{\mathbf{m}} \Pr(\mathbf{m}) \left[\Pr(adv|m', \mathbf{m}) - \Pr(adv|m, \mathbf{m})\right] V(s_1, \mathbf{m})$$

Since $m' > m$, $\Pr(adv|m', \mathbf{m}) - \Pr(adv|m, \mathbf{m}) \geq 0$; and since $V(s_1, \mathbf{m})$ is an expected value of $u_k(s_1, S_{-i,k})$ over the range of opponent types who would have generated message profile $\mathbf{m}$, $V(s_1, \mathbf{m})$ is weakly increasing in s_1.

This means, then, that $v_\tau(m+1; \alpha_m) = v_\tau(m; \alpha_m)$ immediately implies $v_\tau(m+1; s_1) \geq v_\tau(m; s_1)$ for every $s_1 > \alpha_m$, and likewise $v_\tau(m+1; s_1) \leq v_\tau(m; s_1)$ for every $s_1 < \alpha_m$. (In fact, both of these can be shown to hold strictly, but that's not important for this proof.)

Using this, then, we can show that for every bidder type $s_1 \in [0, 1)$, $\tau(s_1)$ selects a best-response:

- Suppose that for $s_1 = \alpha_m$, some message lower than m gave a strictly higher payoff than m and $m+1$. Let m' be the highest message below m giving a strictly higher payoff than m; then $v_\tau(m'; \alpha_m) > v_\tau(m'+1; \alpha_m)$. Since $v_\tau(m'; \alpha_{m'}) = v_\tau(m'+1; \alpha_{m'})$ and $\alpha_m > \alpha_{m'}$, this violates the monotonicity of $v_\tau(m'+1; s_1) - v_\tau(m'; s_1)$ shown above.

  A similar contradiction follows if a bidder with type α_m strictly preferred a message $m' > m+1$ ($m' \in \operatorname{supp} \tau$) to m and $m+1$.

- Suppose that for some $s_1 \in (\alpha_{m-1}, \alpha_m)$, some message $m' < m$ gave a higher payoff than m. By monotonicity of $v_\tau(m; s) - v_\tau(m'; s)$, this would also have to hold for $S_1 = \alpha_{m-1}$. If $m' = m - 1$, this means $v_\tau(m-1; \alpha_{m-1}) > v_\tau(m; \alpha_{m-1})$, contradicting the fact that this same statement has to hold with equality; if $m' < m-1$, this means a bidder with type α_{m-1} strictly prefers a message $m' < m-1$ to either $m-1$ or m, which was ruled out above.

  Similarly, if a higher message $m' > m$ gave a higher payoff than m, monotonicity of $v_\tau(m'; s) - v_\tau(m; s)$ would require this to also hold for $s_1 = \alpha_m$, which similarly generates a contradiction.

- All that's left, then, is deviations to unused messages, which Lemma 1 implies must be above M. (Note that since no opponent is sending a message above M with positive probability, all messages above M give the same expected payoff.) The second condition in Lemma 2 implies that such messages are either unavailable or no better than M even when $S_1 = 1$; monotonicity of $v_\tau(M+1; s) - v_\tau(M; s)$ ensures they're no better than M for all other types. The final part of the second condition in Lemma 2 ensures that bidders with type $S_1 = 1$ are playing a best-response as well.

Thus, the conditions of Lemma 2 ensure that τ is a best-response to one's opponents playing τ, and therefore that "everyone plays τ" is an equilibrium. □

A.4 Other Preliminaries for Existence Proof

Above, we somewhat informally defined the difference $v_\tau(m+1; s_1) - v_\tau(m; s_1)$, and noted that in equilibrium, at $s_1 = \alpha_m$, it must be equal to zero. Here, we will calculate $v_\tau(m+1; \alpha_m) - v_\tau(m; \alpha_m)$ more explicitly as a function of τ, and show that it has two important features. First, given the environment, it is a function only of $\{\alpha_{m-1}, \alpha_m, \alpha_{m+1}\}$; and second, it is strictly single-crossing in α_{m-1}. This means that for given values of α_m and α_{m+1}, there is a unique value of α_{m-1} that is consistent with equilibrium, which will allow us to construct the equilibrium "downwards" from the top of the type space.

Fixing an environment and a number of messages M, let $\mathcal{A}$ be the set of possible thresholds $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{M-1})$ with $0 \leq \alpha_0 \leq \alpha_1 \leq \ldots \leq \alpha_{M-1} \leq 1$. Define a function Δ from $\mathcal{A} \times \{1, 2, \ldots, M-1\}$ to the reals as

$$\begin{aligned} \Delta(\alpha, m) &\equiv v_\tau(m+1; \alpha_m) - v_\tau(m; \alpha_m) \\ &= \sum_{\mathbf{m}} \Pr(\mathbf{m}) \left[\Pr(adv|m+1, \mathbf{m}) - \Pr(adv|m, \mathbf{m}) \right] V(\alpha_m, \mathbf{m}) \end{aligned}$$

(Note that the analysis below will not hold for $m = 0$, as $v_\tau(\alpha_0, 1) - v_\tau(\alpha_0, 0)$ takes a different form; hence, $\Delta(\alpha, m)$ is defined only for $m \geq 1$.)

In order for $\Pr(adv|m+1, \mathbf{m}) - \Pr(adv|m, \mathbf{m}) \neq 0$, it must be that $\Pr(adv|m, \mathbf{m}) < 1$, which means that in every opponent message profile $\mathbf{m}$ which shows up in the sum, there will be at least n opponents sending message m or higher. Similarly, $\Pr(adv|m+1, \mathbf{m}) - \Pr(adv|m, \mathbf{m}) \neq 0$ requires

$\Pr(adv|m+1,\mathbf{m}) > 0$, which means that at $\mathbf{m}$, fewer than n opponents are sending messages above $m+1$. Thus, if we define $\mathcal{M}_{q,r}$ as the set of message profiles at which q opponents send message $m+1$ and r send messages higher than that, we can write Δ as

$$\Delta(\alpha, m) = \sum_{r=0}^{n-1} \sum_{q=0}^{N-1-r} \left(\sum_{\mathbf{m} \in \mathcal{M}_{q,r}} \Pr(\mathbf{m}) \left[\Pr(adv|m+1,\mathbf{m}) - \Pr(adv|m,\mathbf{m}) \right] V(\alpha_m, \mathbf{m}) \right)$$

and define $\Delta^{q,r}(\alpha, m)$ as the inner sum, i.e., the term in large parentheses.

We will calculate $\Delta^{q,r}$ separately for the two cases $q+r < n$ and $q+r \geq n$.

First case: $q+r<n$.

If $q+r<n$, then fewer than n opponents are sending message $m+1$ or higher. This means that $\Pr(adv|m+1,\mathbf{m}) = 1$; for a message profile $\mathbf{m}$ to show up in the sum, then, it must be that $p \geq n-q-r$ opponents send message m, since otherwise $\Pr(adv|m,\mathbf{m}) = 1$ as well. If $p \geq n-q-r$ opponents send message m, then $\Pr(adv|m,\mathbf{m}) = \frac{n-q-r}{p+1}$, since counting bidder 1, there are $p+1$ bidders sending message m competing for the $n-q-r$ spots in the auction not taken by bidders sending even higher messages. Regardless of p, if bidder 1 advances, he will face the $q+r$ opponents who sent message $m+1$ or higher, plus $n-1-(q+r)$ of the opponents who sent message m.

Putting it all together, then, for $q+r<n$,

$$\begin{aligned}
\Delta^{q,r}(\alpha, m) &= \sum_{\mathbf{m} \in \mathcal{M}_{q,r}} \Pr(\mathbf{m}) \left[\Pr(adv|m+1,\mathbf{m}) - \Pr(adv|m,\mathbf{m}) \right] V(\alpha_m, \mathbf{m}) \\
&= \sum_{p=n-q-r}^{N-1-q-r} \binom{N-1}{r} \binom{N-1-r}{q} \binom{N-1-r-q}{p} w^{N-1-p-q-r} x^p y^q z^r \left[1 - \frac{n-q-r}{p+1} \right] V^{q,r}
\end{aligned}$$

where $z = 1-\alpha_{m+1}$ is the probability (under τ) that a bidder sends a message higher than $m+1$; $y = \alpha_{m+1} - \alpha_m$ the probability a bidder sends message $m+1$; $x = \alpha_m - \alpha_{m-1}$ the probability a bidder sends message m; $w = \alpha_{m-1}$ the probability he sends a message lower than m; and $V^{q,r}$ the expected payoff to a bidder with type $S_1 = \alpha_m$ from advancing to an auction against r opponents with types above α_{m+1}, q opponents with types in $[\alpha_m, \alpha_{m+1}]$, and $n-1-q-r$ opponents with types in $[\alpha_{m-1}, \alpha_m]$.

Second case: $q+r \geq n$.

If $q+r \geq n$, at least n opponents are sending messages $m+1$ or higher, so $\Pr(adv|m,\mathbf{m}) = 0$, and $\Pr(adv|m+1,\mathbf{m}) = \frac{n-r}{q+1}$, since r spots in the auction are taken by opponents sending messages above $m+1$, leaving $n-r$ for the $q+1$ bidders sending $m+1$. If bidder 1 advances, he faces the r bidders who sent messages above $k+1$, plus $n-1-r$ opponents who sent message $m+1$. How many of his opponents sent message m, versus lower messages, is irrelevant, since he never advances when he sends m. So

$$\begin{aligned}\Delta^{q,r}(\alpha,m) &= \sum_{\mathbf{m}\in\mathcal{M}_{q,r}} \Pr(\mathbf{m})\left[\Pr(adv|m+1,\mathbf{m}) - \Pr(adv|m,\mathbf{m})\right] V(\alpha_m,\mathbf{m}) \\ &= \binom{N-1}{r}\binom{N-1-r}{q}(w+x)^{N-1-q-r}y^q z^r \frac{n-r}{q+1} V^{n-1-r,r}\end{aligned}$$

Next, we establish several key properties of our function Δ.

Lemma 4. *Fixing an environment,*

1. *$\Delta(\alpha,m)$ is a function only of α_{m-1}, α_m, and α_{m+1}, so we can write it as $\Delta(\alpha_{m-1},\alpha_m,\alpha_{m+1})$*
2. *$\Delta(\alpha_{m-1},\alpha_m,\alpha_{m+1})$ is continuous in each of its arguments*
3. *There is $\varepsilon > 0$ such that when $\alpha_m - \alpha_{m-1} \in (0,\varepsilon)$, $\Delta(\alpha_{m-1},\alpha_m,\alpha_{m+1}) < 0$*
4. *On $[0,\alpha_m)$, $\Delta(\alpha_{m-1},\alpha_m,\alpha_{m+1})$ is strictly single-crossing from above in α_{m-1}*

Together, these tell us that for given values of α_m and α_{m+1}, either $\Delta(0,\alpha_m,\alpha_{m+1}) \leq 0$ – in which case there is no value of $\alpha_{m-1} > 0$ that would satisfy $\Delta(\alpha_{m-1},\alpha_m,\alpha_{m+1}) = 0$ – or else $\Delta(0,\alpha_m,\alpha_{m+1}) > 0$, in which case there is a unique value of α_{m-1} satisfying $\Delta(\alpha_{m-1},\alpha_m,\alpha_{m+1}) = 0$.

Proof of Lemma 4. Parts 1 and 2 of Lemma 4 follow directly from the expressions for $\Delta^{q,r}(\alpha,m)$: for each q and r, $\Delta^{q,r}(\alpha,m)$ is a function only of α_{m-1}, α_m, and α_{m+1}, and is continuous in each of them, so Δ inherits these properties. For part 3, note from Lemma 4 that $u_2(s_i,s_j) > c$ requires $s_i > s_j + \varepsilon$. Then $V^{q,r} < 0$ for $q+r > 0$, and $V^{0,0} < 0$ whenever $\alpha_m - \alpha_{m-1} < \varepsilon$, and so $\Delta = \sum_{q,r}\Delta^{q,r} < 0$ if $\alpha_m - \alpha_{m-1} < \varepsilon$. ($\alpha_m - \alpha_{m-1} > 0$ is needed here because when $\alpha_{m-1} = \alpha_m = \alpha_{m+1}$, $\Delta(\alpha_{m-1},\alpha_m,\alpha_{m+1}) = 0$.) Part 4 of the lemma – single-crossing – requires a lot of additional algebra, and will be done separately at the end of the appendix. □

With Δ defined, and given Lemma 4, define $\alpha^*(\alpha_m,\alpha_{m+1})$ as the solution to $\Delta(x,\alpha_m,\alpha_{m+1}) = 0$ on $[0,\alpha_{m-1})$, or as 0 when no such solution exists, and note that since Δ is continuous, α^* is continuous in both its arguments. There are two other useful facts we'll need:

Lemma 5. *For any environment and α,*

1. *If $\alpha_1 - \alpha_0 \geq \varepsilon$ and $\alpha_0 \approx 0$, $v_\tau(1;\alpha_0) < 0$*
2. *If $\Delta(0,\alpha_0,\alpha_1) > 0$, then $v_\tau(1;\alpha_0) > 0$*

Proof of Lemma 5. To show the first part, note that if $\alpha_0 \approx 0$, a bidder opting in has virtually no chance of entering the auction alone, and if $\alpha_1 - \alpha_0 \geq \varepsilon$, he has at least probability $\varepsilon^{N-1}\frac{n}{N}$ of being selected; a bidder with type α_0 is assured that any opponents have types at least as high as him, ensuring (via the small rents assumption) a strictly negative payoff from entering against any competition, making $v_\tau(1;\alpha_0)$ negative.

For the second part, note that

$$\begin{aligned}
\Delta(0,\alpha_0,\alpha_1) &= \sum\sum_{q+r<n} \Delta^{q,r}(0,\alpha_0,\alpha_1) + \sum\sum_{r<n,\, q+r\geq n} \Delta^{q,r}(0,\alpha_0,\alpha_1) \\
&= \sum\sum_{q+r<n} \left(\sum_{p=n-q-r}^{N-1-q-r} \binom{N-1}{r}\binom{N-1-r}{q}\binom{N-1-r-q}{p} w^{N-1-p-q-r}x^p y^q z^r \left[1-\frac{n-q-r}{p+1}\right] V^{q,r}\right) \\
&\quad + \sum\sum_{r<n,\, q+r\geq n} \left(\binom{N-1}{r}\binom{N-1-r}{q}(w+x)^{N-1-q-r}y^q z^r \frac{n-r}{q+1} V^{n-1-r,r}\right)
\end{aligned}$$

where now $w=0$, $x=\alpha_0$, $y=\alpha_1-\alpha_0$, and $z=1-\alpha_1$. Since $w=0$, all but the $p=N-1-q-r$ term of the innermost sum on the first line vanish (and $w+x$ becomes x), and so

$$\begin{aligned}
\Delta(0,\alpha_0,\alpha_1) &= \sum\sum_{q+r<n} \left(P(q,r)\left[1-\frac{n-q-r}{N-q-r}\right] V^{n-1-q-r,q,r}\right) \\
&\quad + \sum\sum_{r<n,\, q+r\geq n} \left(P(q,r)\frac{n-r}{q+1} V^{0,n-1-r,r}\right)
\end{aligned}$$

where $P(q,r)$ is shorthand for $\binom{N-1}{r}\binom{N-1-r}{q}x^{N-1-q-r}y^q z^r$ and where we now write $V^{p,q,r}$ as the expected payoff in the auction given type α_0 when facing p opponents with types in $[0,\alpha_0]$, q in $[\alpha_0,\alpha_1]$, and r in $[\alpha_1,1]$. On the other hand, we can calculate $v_\tau(\alpha_0,1)$ as

$$v_\tau(1;\alpha_0) = \sum\sum_{q+r<n} P(q,r)V^{0,q,r} + \sum\sum_{r<n,\, q+r\geq n} \left(P(q,r)\frac{n-r}{q+1} V^{0,n-1-r,r}\right)$$

Note that the second double-sum in $\Delta(0,\alpha_0,\alpha_1)$ and $v_\tau(\alpha_0,1)$ are exactly the same, and the two expressions therefore differ in just two ways: the latter replaces $1-\frac{n-q-r}{N-q-r}$ with 1, and $V^{n-1-q-r,q,r}$ with $V^{0,q,r}$, in each summand in the first double-sum. Now, fewer opponents certainly leads to a higher expected auction payoff, so this second change is an increase in value; we'll show that when $\Delta(0,\alpha_0,\alpha_1)\geq 0$, the first change must be an overall increase as well.

To show this, note first that out of all the $V^{n-1-q-r,q,r}$ and $V^{0,n-1-r,r}$ terms, the only one that isn't strictly negative (by the small-rents assumption) is $V^{n-1,0,0}$. This means the second double-sum is strictly negative, which means that if $\Delta(0,\alpha_0,\alpha_1)\geq 0$, the first sum must be strictly positive. Rewrite the first sum as

$$\sum\sum_{q+r<n} \left(P(q,r)\left[\frac{N-n}{N-q-r}\right] V^{n-1-q-r,q,r}\right)$$

and note that if this is strictly positive, then $\sum\sum P(q,r)V^{n-1-q-r,q,r}$ is strictly larger: it consists of multiplying each term by $\frac{N-q-r}{N-n}>1$, with the term corresponding to $q=r=0$ (which is the only positive term) therefore being multiplied by a strictly larger factor than all the other terms.

So going from $\Delta(0, \alpha_0, \alpha_1)$ to $v_\tau(1; \alpha_0)$ involves two changes: the first multiplies a bunch of terms (which collectively are positive) by factors greater than 1, with the only positive term getting multiplied by a larger factor than the others; the second replaces a bunch of terms $V^{n-1-q-r,q,r}$ with larger (less negative) $V^{0,q,r}$ terms. Thus, if $\Delta(0, \alpha_0, \alpha_1) > 0$, then $v_\tau(1; \alpha_0) > \Delta(0, \alpha_0, \alpha_1)$, giving the result. □

A.5 Proof of Theorem 1 (equilibrium construction)

Finally, we can launch into the actual construction. First, we define M^*.

1. For a given $M > 0$, define $\alpha^{(M)} = (\alpha_0^{(M)}, \alpha_1^{(M)}, \ldots, \alpha_M^{(M)})$ as follows:

   - Let $\alpha_{M-1}^{(M)} = \alpha_M^{(M)} = 1$
   - For $m = M-2, M-3, \ldots, 0$, define α_m recursively as $\alpha_m^{(M)} = \alpha^*(\alpha_{m+1}^{(M)}, \alpha_{m+2}^{(M)})$.
     (If at any point $\alpha_m = 0$, then stop and say the construction failed.)

2. Then define the pure strategy τ_M by $\tau_M(s_1) = 0$ for $s_1 \leq \alpha_0$ and, for $m = 1, 2, \ldots, M$, $\tau_M(s_1) = m$ for $s_1 \in (\alpha_{m-1}^{(M)}, \alpha_m^{(M)}]$.

   If $\alpha^{(M)}$ are all well-defined and greater than 0 and $v_{\tau_M}(1; \alpha_0^{(M)}) > 0$, we will say the construction succeeded; if some $\alpha_m^{(M)} = 0$ (meaning $\Delta(x, \alpha_{m+1}^{(M)}, \alpha_{m+2}^{(M)}) = 0$ had no solution) or $v_{\tau_M}(1; \alpha_0^{(M)}) \leq 0$, we will say the construction failed.

3. Define M^* as the largest value of M at which the construction succeeded.

   - Note that the construction must succeed at $M = 1$ (since at $\alpha_0 = \alpha_1 = 1$, $v_\tau(1; \alpha_0) = E(V_1 | S_1 = 1) - c > 0$ by assumption)
   - Also note that it must fail at M sufficiently large (since as noted above, $v_\tau(m; \alpha_m) = v_\tau(m+1; \alpha_m)$ requires $\alpha_m - \alpha_{m-1} \geq \varepsilon$, and therefore when $M > 2 + \frac{1}{\varepsilon}$ the construction can't succeed)
   - Finally, note that it must succed at every $M \leq M^*$.
     For $M < M^*$, by construction, $\alpha_{M-m}^{(M)} = \alpha_{M^*-m}^{(M^*)}$; so $\alpha^{(M)}$ is well-defined, with $\alpha_1^{(M)} = \alpha_{M^*-M+1}^{(M^*)}$ and $\alpha_0^{(M)} = \alpha_{M^*-M}^{(M^*)}$.
     Since the construction works at M^*, $\alpha^*(\alpha_{M^*-M}^{(M^*)}, \alpha_{M^*-M+1}^{(M^*)}) = \alpha_{M^*-M-1}^{(M^*)} > 0$, and therefore $\Delta(0, \alpha_{M^*-M}^{(M^*)}, \alpha_{M^*-M+1}^{(M^*)}) > 0$, or $\Delta(0, \alpha_0^{(M)}, \alpha_1^{(M)}) > 0$; by part 2 of Lemma 5, then, $v_{\tau_M}(1; \alpha_0^{(M)}) > 0$.

Next, for a given $M \leq M^*$, do the following:

1. For $t \in [0, 1]$, define $\alpha_M(t) = 1$ and $\alpha_{M-1}(t) = 1 - t$.

2. For $m = M-2, M-3, \ldots, 0$, define $\alpha_m(t)$ recursively as $\alpha_m(t) = \alpha^*(\alpha_{m+1}(t), \alpha_{m+2}(t))$.

3. We will let v_α denote v_τ for the strategy τ defined by the thresholds α in the obvious way. Define $v(t) = v_{\alpha(t)}(\alpha_0(t), 1)$.

4. Note that since α^* is continuous, each $\alpha_m(t)$ is continuous in t, at least until it hits 0

5. If we let $\bar{t}$ denote the value of t at which $\alpha_0(t)$ hits 0, then all of the $\alpha_m(t)$ are continuous on $[0, \bar{t}]$, and therefore $v(t)$ is continuous on $[0, \bar{t}]$ as well

6. Note that $v(0) > 0$.

   (If $M = M^*$, then $v(0) = v_{\tau_{M^*}}(1; \alpha_0^{(M^*)}) > 0$ by the original construction of M^*; if $M < M^*$, then in the original construction of M^*, $\alpha^*(\alpha_0(0), \alpha_1(0)) > 0$, or $\Delta(0, \alpha_0(0), \alpha_1(0)) > 0$, implying $v_{\alpha(0)}(1; \alpha_0(0)) > 0$ by the second part of Lemma 5.)

7. Note that $v(\bar{t}) < 0$.

   This is because $\alpha_0(\bar{t}) = 0$, and since $\Delta(\alpha_0(t), \alpha_1(t), \alpha_2(t)) = 0$, $\alpha_1(t) - \alpha_0(t) \geq \varepsilon$ for every t. So the first part of Lemma 5 holds at $\bar{t}$, meaning $v_{\alpha(\bar{t})}(1; \alpha_0(\bar{t})) < 0$.

8. Since $v(t)$ is continuous on $[0, \bar{t}]$, positive at 0, and negative at $\bar{t}$, it crosses 0; let t^* be the lowest value at which $v(t) = 0$, and define τ based on the thresholds $\alpha(t^*)$.

9. The recursive definition of $\alpha_m(t)$ ensures that $\Delta(\alpha_{m-1}(t^*), \alpha_m(t^*), \alpha_{m+1}(t^*)) = 0$ for $m = 1, 2, \ldots, M-1$, or that at τ defined by $\alpha(t^*)$, $v_\tau(m; \alpha_m(t^*)) = v_\tau(m+1; \alpha_m(t^*))$; the definition of t^* ensures $v_\tau(1; \alpha_0(t^*)) = 0$; so the indifference conditions all hold.

10. If $M = \overline{M}$, the second sufficient condition in Lemma 2 holds vacuously; so for $\overline{M} \leq M^*$, we have constructed an equilibrium when $M = \overline{M}$.

11. If $\overline{M} > M^*$, we need to show that at $M = M^*$, $v_\tau(M+1; 1) \leq v_\tau(M; 1)$. We do this below, completing the proof that we've found an equilibrium either way.

All that's left is to show that at $M = M^*$ and τ defined by the thresholds $\alpha(t^*)$, $v_\tau(M+1; 1) \leq v_\tau(M; 1)$. Consider a bidder with type $S_i = 1$ who considers sending an unused message above M instead of M. If we calculate $v_\tau(M+1; 1) - v_\tau(M; 1)$, it's the same expression for the difference as before, but with $\alpha_m = \alpha_{m+1} = 1$, and therefore $y = z = 0$; as a result, $\Delta^{q,r} = 0$ for $q + r > 0$, and therefore

$$v_\tau(M+1; 1) - v_\tau(M; 1) \quad = \quad \sum_{p=n}^{N-1} \binom{N-1}{p} w^{N-1-p} x^p \left[1 - \frac{n}{p+1}\right] V^{0,0}$$

Thus, $v_\tau(M+1; 1) - v_\tau(M; 1)$ has the same sign as

$$V^{0,0} \quad = \quad E\left\{u_n(1, S_{-i,n}) \mid S_{-i,n} \in [\alpha_{M-1}, 1]^{n-1}\right\} - c \quad = \quad V(1, n-1, [\alpha_{M-1}, 1])$$

the payoff to a bidder with type 1 facing $n-1$ opponents with types in $[\alpha_{M-1}, 1]$. (In words: unless at least n of my opponents sent message M, I would have advanced for sure anyway by sending message M; so the only time message $M+1$ gives me a different outcome from message M, it's when I'll face an auction full of opponents who sent message M, and therefore have types above α_{M-1}.)

Now, if $V(1,n-1,[0,1]) \leq 0$, then by monotonicity, $V(1,n-1,[\alpha_{M-1},1]) \leq 0$, and therefore $v_\tau(M+1;1) \leq v_\tau(M;1)$ and we're done. So suppose $V(1,n-1,[0,1]) > 0$, and define $\hat{S}$ as the solution to

$$V(1,n-1,[\hat{S},1]) \quad = \quad 0$$

(Continuity, monotonicity, and small rents ensure that there's a unique solution, strictly less than 1.) Again by monotonicity, $v_\tau(M+1;1) \leq v_\tau(M;1)$ if and only if $\alpha_{M-1} \geq \hat{S}$. Also, since $\Delta(\hat{S},1,1) = 0$ by definition, $\hat{S} = \alpha^*(1,1)$.

Now, recall that we defined our thresholds based on $\alpha_M(t) = 1$, $\alpha_{M-1}(t) = 1-t$, and $\alpha_m(t) = \alpha^*(\alpha_{m+1}(t), \alpha_{m+2}(t))$, and defined t^* as the lowest value of t at which $v_\tau(1;\alpha_0(t)) = 0$; so for $t < t^*$, $v_\tau(1;\alpha_0(t)) > 0$. Define $\hat{t}$ as $1-\hat{S}$, so that $\hat{S} = 1-\hat{t} = \alpha_{M-1}(\hat{t})$. Noting that $\alpha_{M-1}(\hat{t}) = \alpha^*(1,1) = \alpha_{M-2}(0)$, it's easy to show inductively that $\alpha_m(\hat{t}) = \alpha_{m-1}(0)$.

We'll show that $t^* \leq \hat{t}$, which ensures that $\alpha_{M-1} \geq \hat{S}$, which in turn ensures that $V^{0,0} = V(1,n-1,[\alpha_{M-1},1]) \leq 0$, and therefore that $v_\tau(M+1;1) \leq v_\tau(M;1)$. Suppose (toward contradiction) that $t^* > \hat{t}$. Since $\alpha_0(t)$ is continuous at least until t^*, this means that $\alpha_0(\hat{t})$ exists, and that $v_\tau(1;\alpha_0(\hat{t})) > 0$. But $\alpha_0(\hat{t}) = \alpha^*(\alpha_1(\hat{t}),\alpha_2(\hat{t})) = \alpha^*(\alpha_0(0),\alpha_1(0))$; if $v_\tau(1;\alpha_0(\hat{t})) > 0$, then $v_\tau(1;\alpha^*(\alpha_0(0),\alpha_1(0))) > 0$, contradicting the definition of M^* as the largest M for which this is possible.

Thus, when $M = M^*$, $t^* \leq \hat{t}$, which means that $\alpha_{M-1} \geq \hat{S}$, and therefore that $v_\tau(M+1;1) \leq v_\tau(M;1)$, completing the proof. □

A.6 Proof of Theorems 2 and 4

We begin by characterizing equilibrium as N gets large.

For N large enough, $M = 1$ in every symmetric equilibrium

Let ε be the value defined in Lemma 3. Suppose an equilibrium existed with $M > 1$. Then there would be an interval of width $\alpha_1 - \alpha_0 \geq \varepsilon$ of types who sent message 1, which would in turn imply that $\alpha_0 < 1-\varepsilon$.

Let $-\underline{u} = \max_{s\in[0,1]} \{u_2(s,s) - c\}$ be the largest (i.e., least negative) payoff a bidder can get from entering against an equally-strong opponent; and let $\overline{V} = E(V_i|S_i = 1)$.

Consider a bidder with type $S_i = \alpha_0$. If he opts in, he has a α_0^{N-1} chance of being the only one to enter, in which case his payoff will be $E(V_i|S_i = \alpha_0) - c < E(V_i|S_i = 1) - c = \overline{V} - c$; and a probability $(N-1)\alpha_0^{N-2}(1-\alpha_0)$ of advancing against one other opponent with type above α_0 and therefore earning a payoff at most $-\underline{u}$. Thus, even ignoring all the scenarios in which multiple other bidders opt in and he may advance and earn a negative expected payoff, our bidder's payoff from opting in is at most

$$\begin{aligned} v_\tau(1;\alpha_0) &\leq \alpha_0^{N-1}\left(\overline{V}-c\right)-(N-1)\alpha_0^{N-2}(1-\alpha_0)\underline{u} \\ &\leq \alpha_0^{N-2}\left[\alpha_0\left(\overline{V}-c\right)-(N-1)\varepsilon\underline{u}\right] \\ &\leq \alpha_0^{N-2}\varepsilon\underline{u}\left[\frac{\overline{V}-c}{\varepsilon\underline{u}}-(N-1)\right] \end{aligned}$$

since $\alpha_0 \leq 1$. Thus, for $N > 1+\frac{\overline{V}-c}{\varepsilon\underline{u}}$, $v_\tau(1;\alpha_0)$ would have to be strictly negative in a symmetric equilibrium with $M > 1$, and therefore no such equilibrium can exist.

As N grows, $\alpha_0 \to 1$ but $\alpha_0^N \not\to 0$ or 1

As noted above,

$$\begin{aligned} v_\tau(1;\alpha_0) &\leq \alpha_0^{N-1}\left(\overline{V}-c\right)-(N-1)\alpha_0^{N-2}(1-\alpha_0)\underline{u} \\ &\leq \alpha_0^{N-2}\left[\left(\overline{V}-c\right)-(N-1)(1-\alpha_0)\underline{u}\right] \end{aligned}$$

If α_0 does not go to 1, or converges at a rate slower than $\frac{1}{N}$, then $(N-1)(1-\alpha_0) \to +\infty$, in which case $v_\tau(1;\alpha_0) < 0$, which is impossible.

On the other hand, we can show that

$$v_\tau(1;\alpha_0) \;\geq\; \alpha_0^{N-1}\left(E(V_i|S_i=\alpha_0)-c\right)+(1-\alpha_0^{N-1})(-c)$$

since $E(V_i|S_i=\alpha_0)-c$ is the bidder's payoff if all his opponents opt out and $-c$ is a lower bound on his payoff if they don't. If α_0 converges to 1 faster than $\frac{1}{N}$, then $(N-1)(1-\alpha_0) \to 0$, which means $\alpha_0^{N-1} \to 1$. (If the expected number of opponents opting in goes to 0, then the probability that none of them enter goes to 1.) In that case, $v_\tau(1;\alpha_0) \to E(V_i|S_i=1)-c > 0$, again a contradiction. So α_0 must go to 1 at rate $\frac{1}{N}$, making α_0^N converge to an interior limit.

This means as N grows, the number of bidders opting in approaches a Poisson random variable, with all entrants having types arbitrarily close to 1. We can calculate the Poisson parameter by noting that, even away from the limit, α_0 satisfies

$$0 \;=\; \sum_{k=0}^{n-1}\Pr(k:N-1)V(\alpha_0,k,[\alpha_0,1])+\sum_{k=n}^{N-1}\Pr(k:N-1)\frac{n}{k+1}V(\alpha_0,n-1,[\alpha_0,1])$$

where $\Pr(k:N-1)$ is the probability that exactly k of a bidder's $N-1$ opponents opt in and (as before) $V(\alpha_0,k,[\alpha_0,1])$ is the expected surplus (net of costs) a bidder with type α_0 earns in an auction with k opponents with types drawn randomly from $[\alpha_0,1]$. As N grows and $\alpha_0 \to 1$,

$$V(\alpha_0,n-1,[\alpha_0,1]) \quad\to\quad u_{k+1}-c \quad\equiv\quad u_{k+1}(1,\mathbf{1})-c$$

and $\Pr(k : N-1)$ approaches the probability defined by the Poisson distribution; so in the limit, the equilibrium participation level λ_n is the solution to

$$0 = \sum_{k=0}^{n-1} \frac{\lambda^k e^{-\lambda}}{k!}(u_{k+1} - c) + \sum_{k=n}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} \frac{n}{k+1}(u_n - c)$$

To show that this has a unique solution, define the right-hand side as $V_n(\lambda)$, and note that

$$\begin{aligned}
\frac{\partial}{\partial \lambda} V_n(\lambda) &= \sum_{k=1}^{n-1} \frac{k\lambda^{k-1} e^{-\lambda}}{k!}(u_{k+1} - c) - \sum_{k=0}^{n-1} \frac{\lambda^k e^{-\lambda}}{k!}(u_{k+1} - c) \\
&\quad + \sum_{k=n}^{\infty} \frac{k\lambda^{k-1} e^{-\lambda}}{k!} \frac{n}{k+1}(u_n - c) - \sum_{k=n}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} \frac{n}{k+1}(u_n - c) \\
&= \sum_{k'=0}^{n-2} \frac{\lambda^{k'} e^{-\lambda}}{k'!}(u_{k'+2} - c) + \sum_{k'=n-1}^{\infty} \frac{\lambda^{k'} e^{-\lambda}}{k'!} \frac{n}{k'+2}(u_n - c) - V_n(\lambda)
\end{aligned}$$

Now, the two sums are negative, since $u_k < c$ for $k > 1$ under small rents; so this says that whenever V_n is positive, it's strictly decreasing in λ, so it's strictly single-crossing from above; so if a solution exists, it's unique. $V_n(0) = u_1 - c > 0$, and it's not hard to show that V_n is negative when λ gets large, so a solution does indeed exist.

Limit Welfare

When $M = 1$, total surplus generated in equilibrium can be written as

$$W(n, N) = \sum_{k=0}^{n-1} \Pr(k : N)(w_k(\alpha_0) - kc) + \sum_{k=n}^{N} \Pr(k : N)(w_n(\alpha_0) - nc)$$

where $\Pr(k : N)$ is the probability that exactly k out of N bidders opt in and $w_k(\alpha_0)$ is the expected total surplus (gross of entry costs) generated by an auction with k bidders with types $s_i \in [\alpha_0, 1]$. If we let

$$w_k = \lim_{\alpha_0 \to 1} w_k(\alpha_0) = E\{\max\{V_1, \ldots, V_k\} \mid S_1 = \ldots = S_k = 1\}$$

then as N gets large and $\alpha_0 \to 1$, $w_k(\alpha_0) \to w_k$. As N gets large, $\Pr(k : N)$ goes to the Poisson distribution, so

$$W_n \equiv \lim_{N \to \infty} W(n, N) = \sum_{k=0}^{n-1} \frac{\lambda_n^k e^{-\lambda_n}}{k!}(w_k - kc) + \sum_{k=n}^{\infty} \frac{\lambda_n^k e^{-\lambda_n}}{k!}(w_n - nc)$$

where λ_n is the equilibrium Poisson parameter.

Now, fixing n, let

$$W_n(\lambda) \equiv \sum_{k=0}^{n-1} \frac{\lambda^k e^{-\lambda}}{k!}(w_k - kc) + \sum_{k=n}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!}(w_n - nc)$$

so that $W_n = W_n(\lambda_n)$ but $W_n(\cdot)$ is also defined at non-equilibrium levels of participation. Next, we calculate

$$\begin{aligned}
\frac{\partial}{\partial\lambda} W_n(\lambda) &= \sum_{k=1}^{n-1} \frac{k\lambda^{k-1} e^{-\lambda}}{k!}(w_k - kc) - \sum_{k=0}^{n-1} \frac{\lambda^k e^{-\lambda}}{k!}(w_k - kc) \\
&\quad + \sum_{k=n}^{\infty} \frac{k\lambda^{k-1} e^{-\lambda}}{k!}(w_n - nc) - \sum_{k=n}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!}(w_n - nc) \\
&= \sum_{k'=0}^{n-2} \frac{\lambda^{k'} e^{-\lambda}}{k'!}(w_{k'+1} - (k'+1)c) - \sum_{k=0}^{n-1} \frac{\lambda^k e^{-\lambda}}{k!}(w_k - kc) \\
&\quad + \sum_{k'=n-1}^{\infty} \frac{\lambda^{k'} e^{-\lambda}}{k'!}(w_n - nc) - \sum_{k=n}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!}(w_n - nc)
\end{aligned}$$

($\frac{k\lambda^{k-1}e^{-\lambda}}{k!} = \frac{\lambda^{k-1}e^{-\lambda}}{(k-1)!}$; the first and third sums then simply substitute k' for $k-1$, changing the range over which the sum is taken accordingly.) Simplifying, then,

$$\frac{\partial}{\partial\lambda} W_n(\lambda) = \sum_{k=0}^{n-1} \frac{\lambda^k e^{-\lambda}}{k!}(w_{k+1} - w_k - c)$$

The next cool thing to notice: $w_{k+1} - w_k = u_{k+1}$! This is because in a private-values auction, a single bidder's expected payoff is exactly his contribution to total surplus – the expected difference between the highest valuation when he's there and the highest valuation when he's not. So

$$\frac{\partial}{\partial\lambda} W_n(\lambda) = \sum_{k=0}^{n-1} \frac{\lambda^k e^{-\lambda}}{k!}(u_{k+1} - c)$$

Note that this is exactly the first term in the expression for $V_n(\lambda)$, and that the second term in $V_n(\lambda)$ is negative, so $\frac{\partial}{\partial\lambda} W_n(\lambda) > V_n(\lambda)$. But we know that for $\lambda \leq \lambda_n$, $V_n(\lambda) \geq 0$, and therefore $\frac{\partial}{\partial\lambda} W_n(\lambda) > 0$. So for participation levels at and below the equilibrium level, W is strictly increasing in participation.

Limit Bidder Surplus is 0

Since λ_n is (by virtue of being the Poisson parameter) the expected number of bidders opting in, if we write α_0 as $\alpha_0(N)$ to emphasize its dependence on N, as N gets large,

$$N(1-\alpha_0(N)) \quad \to \quad \lambda_n$$

For N large, then, $1-\alpha_0(N) \approx \lambda_n/N$, or

$$\alpha_0(N) \quad \approx \quad 1-\frac{\lambda_n}{N}$$

Suppressing the suffix on λ and the dependence of α_0 on N, we know a bidder's expected payoff is

$$v_\tau(1;s_i) \quad = \quad \sum_{k=0}^{n-1}\frac{\lambda^k e^{-\lambda}}{k!}V(s_i,k,[\alpha_0,1]) + \sum_{k=n}^{\infty}\frac{\lambda^k e^{-\lambda}}{k!}\frac{n}{k+1}V(s_i,n-1,[\alpha_0,1])$$

Since $v_\tau(1;\alpha_0)=0$ in equilibrium, we can subtract that off and write this as

$$\begin{aligned} v_\tau(1;s_i) \quad &= \quad \sum_{k=0}^{n-1}\frac{\lambda^k e^{-\lambda}}{k!}\left(V(s_i,k,[\alpha_0,1]) - V(\alpha_0,k,[\alpha_0,1])\right) \\ &\quad + \sum_{k=n}^{\infty}\frac{\lambda^k e^{-\lambda}}{k!}\frac{n}{k+1}\left(V(s_i,n-1,[\alpha_0,1]) - V(\alpha_0,n-1,[\alpha_0,1])\right) \\ &\leq \quad \sum_{k=0}^{n-1}\frac{\lambda^k e^{-\lambda}}{k!}\left(V(1,k,\{\alpha_0\}) - V(\alpha_0,k,\{1\})\right) \\ &\quad + \sum_{k=n}^{\infty}\frac{\lambda^k e^{-\lambda}}{k!}\frac{n}{k+1}\left(V(1,n-1,\{\alpha_0\}) - V(\alpha_0,n-1,\{1\})\right) \end{aligned}$$

Now, a single bidder's ex ante expected surplus is

$$u \quad = \quad \int_0^{\alpha_0} v_\tau(0;s_i)ds_i + \int_{\alpha_0}^1 v_\tau(1;s_i)ds_i \quad = \quad \int_{\alpha_0}^1 v_\tau(1;s_i)ds_i$$

Multiplying by N, then, total bidder surplus is

$$\begin{aligned} N \cdot u &= N \int_{\alpha_0}^{1} v_\tau(1; s_i) ds_i \\ &\leq N(1-\alpha_0) \left[\sum_{k=0}^{n-1} \frac{\lambda^k e^{-\lambda}}{k!} \left(V(1,k,\{\alpha_0\}) - V(\alpha_0,k,\{1\}) \right) \right. \\ &\quad \left. + \sum_{k=n}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} \frac{n}{k+1} \left(V(1,n-1,\{\alpha_0\}) - V(\alpha_0,n-1,\{1\}) \right) \right] \end{aligned}$$

As $N \to \infty$, $N(1-\alpha_0) \to \lambda_n$; and by continuity, as $\alpha_0 \to 1$, each of the difference terms $V(1,k,\{\alpha_0\}) - V(\alpha_0,k,\{1\})$ in both summands go to 0, so the entire expression in square brackets goes to 0, so total bidder surplus goes to zero.

Revenue and Total Surplus Results (Theorem 2)

We've shown that as N grows, total surplus approaches a limit $W_n(\lambda_n)$, where

$$W_n(\lambda) = \sum_{k=0}^{n-1} \frac{\lambda^k e^{-\lambda}}{k!}(w_k - kc) + \sum_{k=n}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!}(w_n - nc)$$

and that $W_n(\lambda)$ is strictly increasing in λ on $[0, \lambda_n]$. We also noted earlier that $u_k = w_k - w_{k-1}$, and by the small rents assumption, $u_k < c$ unless $k = 1$, which means $w_k - kc$ is decreasing in k. This means that fixing λ, $W_n(\lambda)$ is decreasing in n, since decreasing n by one replaces a bunch of $w_n - nc$ terms with $w_{n-1} - (n-1)c$, which is larger (less negative).

Next, note that λ_n is decreasing in n. This is because we can rewrite the expression defining λ_n as

$$0 = \sum_{k=0}^{\infty} \frac{\lambda_n^k e^{-\lambda_n}}{k!} h(k)$$

where

$$h(k) = \min\left\{1, \frac{n}{k+1}\right\} (u_{\min\{n,k+1\}} - c)$$

For $n > 1$, an increase in n weakly decreases all of the $h(k)$ terms, since they're all negative except for $h(0)$ and adding either an opponent, or a greater chance of being selected when opposed, lowers a bidder's expected payoff; we showed earlier that this expression is decreasing in λ until after it becomes negative, so lowering the $h(k)$ terms requires lowering λ to compensate.

All together, these imply that for $n' < n$,

$$W_n = W_n(\lambda_n) < W_{n'}(\lambda_n) < W_{n'}(\lambda_{n'}) = W_{n'}$$

i.e., limit total surplus is strictly decreasing in n. Limit bidder surplus is 0 at every n, so limit revenue equals limit total surplus, and is therefore strictly decreasing in n. Finally, note that the unrestricted auction case is simply the case of $n = \infty$ (every bidder who opts in advances), and so "decreasing in n" implies "higher at finite n than infinite n" implies "higher with indicative bidding than without".

Reserve Prices and Subsidies (Theorem 4)

Adding a reserve price simply lowers a bidder's expected payoff $V(s_i, k, [\alpha_0, 1])$ in each state of the world where he advances, lowering each limit u_k. By the arguments above, this decreases λ. Adding a reserve weakly decreases total surplus independent of λ, since it may prevent sales; and it decreases total surplus via λ, since surplus is increasing in λ at and below the equilibrium level. So adding a reserve lowers limit total surplus; it doesn't change the fact that limit bidder surplus is 0, so it lowers limit revenue as well.

A bidder subsidy works in reverse; as long as it is small enough that (i) the small rents assumption continues to hold for the modified auction payoffs, and (ii) the resulting change in λ is small enough that it remains in the range where welfare is increasing, it increases limit total surplus. Once again, limit bidder surplus remains 0, so limit revenue increases.

A.7 Proof of Theorem 3

We now turn to the additively-separable case $V_i = u(S_i) + T_i$, where the joint distribution of $\{T_i\}$ is non-degenerate. First, we calculate the derivative of $V(s_i, k-1, [\alpha_0, 1])$ with respect to s_i, which will help us to calculate expected bidder surplus more explicitly. Now,

$$V(s_i, k-1, [\alpha_0, 1]) \quad = \quad E_{T_i, \{T_j\}, \{S_j\} | \{S_j\} \in [\alpha_0, 1]} \max\{0, u(s_i) + T_i - \max\{u(S_j) + T_j\}\} \; - \; c$$

For given values of s_i, $\{S_j\}$, T_i, and $\{T_j\}$, this maximum has derivative $u'(s_i)$ with respect to s_i if $u(s_i) + T_i \geq \max\{u(S_j) + T_j\}$, and derivative 0 if not; so taking the expectation over $\{S_j\}$ and all the T, V' is exactly $u'(s_i)$ times the probability, given s_i, that bidder i wins the auction. If the joint distribution of all the T is nondegenerate – in the sense of having continuous, bounded density – then as $\alpha_0 \to 1$, the variation in $\{T\}$ will swamp differences among $\{S\}$, and by symmetry, this probability will simply be $\frac{1}{k}$. We can therefore write

$$\begin{aligned}
v_\tau(1;s_i) &= \sum_{k=0}^{n-1} \frac{\lambda^k e^{-\lambda}}{k!} \left(V(s_i,k,[\alpha_0,1]) - V(\alpha_0,k,[\alpha_0,1])\right) \\
&\quad + \sum_{k=n}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} \frac{n}{k+1} \left(V(s_i,n-1,[\alpha_0,1]) - V(\alpha_0,n-1,[\alpha_0,1])\right) \\
&= \sum_{k=0}^{n-1} \frac{\lambda^k e^{-\lambda}}{k!} \int_{\alpha_0}^{s_i} V'(s,k,[\alpha_0,1])ds + \sum_{k=n}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} \frac{n}{k+1} \int_{\alpha_0}^{s_i} V'(s,n-1,[\alpha_0,1])ds \\
&= \sum_{k=0}^{n-1} \frac{\lambda^k e^{-\lambda}}{k!} \int_{\alpha_0}^{s_i} \frac{u'(s)}{k+1} ds + \sum_{k=n}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} \frac{n}{k+1} \int_{\alpha_0}^{s_i} \frac{u'(s)}{n} ds \\
&= \sum_{k=0}^{n-1} \frac{\lambda^k e^{-\lambda}}{(k+1)!} (u(s_i) - u(\alpha_0)) + \sum_{k=n}^{\infty} \frac{\lambda^k e^{-\lambda}}{(k+1)!} (u(s_i) - u(\alpha_0)) \\
&= \frac{1}{\lambda} \sum_{k=0}^{\infty} \frac{\lambda^{k+1} e^{-\lambda}}{(k+1)!} (u(s_i) - u(\alpha_0)) \\
&= \frac{1}{\lambda} (1 - e^{-\lambda})(u(s_i) - u(\alpha_0))
\end{aligned}$$

If we want to calculate ex ante bidder surplus for all bidders, then, it's

$$N \cdot E_{S_i} \max\{0, v_\tau(1;S_i)\} = N \int_{\alpha_0}^{1} \frac{1}{\lambda}(1 - e^{-\lambda})(u(s_i) - u(\alpha_0)) ds_i$$

As $\alpha_0 \to 1$, $u(s_i) - u(\alpha_0) \approx (s_i - \alpha_0)u'(s_i) \approx (s_i - \alpha_0)u'(1)$, so

$$\begin{aligned}
N \cdot E_{S_i} \max\{0, v_\tau(1;S_i)\} &\approx N \int_{\alpha_0}^{1} \frac{1}{\lambda}(1 - e^{-\lambda})(s_i - \alpha_0)u'(1) ds_i \\
&= \frac{N}{\lambda}(1 - e^{-\lambda}) \frac{1}{2}(1 - \alpha_0)^2 u'(1) \\
&\approx \frac{N}{\lambda}(1 - e^{-\lambda}) \frac{1}{2} \left(\frac{\lambda}{N}\right)^2 u'(1) \\
&= \frac{\lambda}{2N}(1 - e^{-\lambda}) u'(1)
\end{aligned}$$

We already knew this was 0 in the limit; but for finite but large N (such that the approximation is valid but the term is not yet 0), this is strictly increasing in λ, and independent of n other than through λ. This means that bidder surplus is strictly decreasing in n, and higher with indicative bidding than without, for large finite N.

(The proof is different when $V_i = S_i$, since without nondegenerate T_i, $V'(\cdot, k, \cdot)$ is $u'(s_i)$ when $s_i \geq \max\{S_j\}$ and 0 otherwise, rather than being approximately $\frac{u'(s_i)}{k+1}$ either way. The proof for that case was in a previous version of this paper, the result is the same.)

A.8 Constructing Equilibrium for the First Price Auction

Here, we show the construction of symmetric equilibrium in the indicative bidding game when the auction is a first-price auction, $n = 2$, and $V_i = S_i \sim U[0,1]$. (The example in the text uses $V_i = S_i \sim U[0,100]$ simply to make the results easier to read.)

First, we establish necessary and sufficient conditions for symmetric equilibrium. We focus on pure strategy equilibria; as in the case of second-price auctions, allowing mixed strategies only expands the set of equilibrium strategies for a measure zero of bidders, and does not change the set of equilibrium payoffs.

A strategy will consist of a mapping

$$\tau \; : \; [0,1] \; \longrightarrow \; \{0, 1, 2, \ldots, \overline{M}\}$$

from types to messages, and a mapping

$$\beta \; : \; [0,1] \; \longrightarrow \; \Re_+$$

from types to second-stage bids. (As noted in the text, a bidder learns nothing between the first and second stage other than whether or not he or she advanced to the auction, so there is nothing else to condition his bid on and no loss in assuming a bidder chooses his message and bid simultaneously.) We will let

$$v_{\tau,\beta}(m, b; s_i)$$

denote a bidder's expected payoff given type $S_i = s_i$ if he sends message m and bids b if selected and his opponents are playing the strategy (τ, β); equilibrium consists of a strategy (τ, β) such that

$$v_{\tau,\beta}(\tau(s_i), \beta(s_i); s_i) \quad \geq \quad v_{\tau,\beta}(m, b; s_i)$$

for every $s_i \in [0,1]$, every $m \in \{0, 1, \ldots, \overline{M}\}$, and every $b \in \Re_+$.

It is straightforward to show that if (τ, β) is a symmetric equilibrium, τ must be weakly increasing; as before, define α_m as the supremum of the set of types playing messages m or lower with positive probability. We can then write the probability of advancing to the auction, given τ

and one's own message m, as

$$
\begin{aligned}
L_\tau(m) \quad = \quad & (\alpha_{m-1})^{N-1} + \sum_{j=1}^{N-1} \tbinom{N-1}{j} (\alpha_m - \alpha_{m-1})^j (\alpha_{m-1})^{N-1-j} \left(\tfrac{2}{j+1}\right) \\
& + \sum_{j=0}^{N-2} (N-1) \tbinom{N-2}{j} (1-\alpha_m)(\alpha_m - \alpha_{m-1})^j (\alpha_{m-1})^{N-j-2} \left(\tfrac{1}{j+1}\right)
\end{aligned}
$$

(The first term is the probability all the other bidders send messages below m. In the second term, j denotes the number of opponents sending message m, with the rest of the opponents sending messages below m, leading to a $\frac{2}{j+1}$ chance of advancing. In the third term, j again denotes the number of opponents sending message m, with one opponent sending a message above m and the rest sending messages below m, giving a $\frac{1}{j+1}$ chance of advancing.)

It is also straightforward to show that β must be strictly increasing, continuous, and satisfy $\beta(\alpha_0) = 0$. (If $\beta(\alpha_0) > 0$ or β were discontinuous, this would correspond to a gap in the support of $\beta(S_i)$, which is impossible in a symmetric equilibrium of a pay-as-you-bid auction.) Given these properties, for $b \in [\beta(\alpha_{m-1}), \beta(\alpha_m)]$, we can write the probability of winning (i.e., the probability of advancing to the second stage and then outbidding one's opponent) by sending message m and then bidding b as

$$
Q_{\tau,\beta}(m,b) \quad = \quad (\alpha_{m-1})^{N-1} + \sum_{j=1}^{N-1} \tbinom{N-1}{j} (\alpha_m - \alpha_{m-1})^j (\alpha_{m-1})^{N-1-j} \left(\tfrac{2}{j+1}\right) \left(\tfrac{\beta^{-1}(b) - \alpha_{m-1}}{\alpha_m - \alpha_{m-1}}\right).
$$

The first term is the probability that all of bidder i's opponents sent messages below m, in which case bidder i will advance for sure and also be the high bidder for sure; the second term is the probability that bidder i will advance against an opponent who also sent message j (whose types is therefore uniformly distributed on $[\alpha_{m-1}, \alpha_m]$) times the probability of winning the auction against such an opponent. (Note that this expression only holds for "equilibrium combinations" of message and bid; for $m \neq \tau(\beta^{-1}(b))$, $Q_{\tau,\beta}(m,b)$ would take a different form.)

If we let $Q(s_i) = Q_{\tau,\beta}(\tau(s_i), \beta(s_i))$ denote a bidder's equilibrium probability of winning the auction, a standard envelope theorem argument establishes that the expected payoff to a bidder with type $S_i = s_i$ must be

$$
v_{\tau,\beta}(\tau(s_i), \beta(s_i); s_i) \quad = \quad \int_0^{s_i} Q(s) ds
$$

Since we can also decompose the equilibrium expected payoff as

$$
v_{\tau,\beta}(\tau(s_i), \beta(s_i); s_i) \quad = \quad Q(s_i)(s_i - \beta(s_i)) - L(\tau(s_i))c
$$

we can equate these two and find

$$\int_0^{s_i} Q(s)ds \quad = \quad Q(s_i)(s_i - \beta(s_i)) - L(\tau(s_i))c$$

from which we can calculate

$$\beta(s_i) \quad = \quad s_i - \frac{1}{Q(s_i)} \left(\int_0^{s_i} Q(s)ds + L(\tau(s_i))c \right)$$

This gives us the following necessary conditions for equilibrium, which also turn out to be sufficient:

Lemma 6. *Fix an indicative bidding mechanism $(n, \overline{M})$ with $n = 2$ and $\overline{M} < \infty$. A strategy (τ, β) is a symmetric equilibrium if and only if the following all hold:*

- *τ is weakly increasing, and characterized by thresholds $0 < \alpha_0 < \alpha_1 < \ldots < \alpha_{\overline{M}-1} < 1$*
- *$v_{\tau,\beta}(m, \beta(\alpha_m); \alpha_m) = v_{\tau,\beta}(m+1, \beta(\alpha_m); \alpha_m)$ for each $m = 0, 1, \ldots, \overline{M} - 1$*
- *β is strictly increasing and continuous and $\beta(\alpha_0) = 0$*
- *$\beta(s_i) = s_i - \frac{1}{Q(s_i)} \left(\int_0^{s_i} Q(s)ds + L(\tau(s_i))c \right)$*

The arguments for necessity have already been discussed. For sufficiency, as is usual in mechanism design problems, the envelope condition defining β is equivalent to incentive compatibility, which rules out profitable deviations to other types' equilibrium strategies. Ruling out deviations to off-equilibrium combinations of (m, b) is tedious but mechanical. A full proof is available upon request.

Finally, we can construct a strategy (τ, β) satisfying the conditions of Lemma 6 as follows. (We suppress the dependence of $v_{\tau,\beta}(m, b; s_i)$, $Q_{\tau,\beta}(m, b)$, and $L_\tau(m)$ on (τ, β).)

1. *Guess a value of α_0.* Since a bidder can do no better than $V_i - c = S_i - c$ by entering, α_0 must be greater than c, so we start with a value of $\alpha_0 \in [c, 1]$.

2. *Calculate the required value of α_1.* Since $v(0, \beta(\alpha_0); \alpha_0) = v(1, \beta(\alpha_0); \alpha_0)$ and $\beta(\alpha_0) = 0$, we need

$$0 \quad = (\alpha_0 - 0)Q(1, 0) - L(1)c$$

Now, $Q(1, 0) = \alpha_0^{N-1}$, since a bidder bidding 0 in the second stage will only win the auction if all his opponents opt out; this means

$$0 \quad = \alpha_0^N - L(1)c$$

While it's not immediately obvious, $L(1)$ depends only on α_0 and α_1 and is strictly increasing in α_1, and this therefore uniquely pins down the value of α_1 given the value of α_0.

3. *Calculate the bid function on* $[\alpha_0, \alpha_1]$. Once α_0 and α_1 are known, we can calculate $Q(s)$ on $[\alpha_0, \alpha_1]$, and therefore calculate $\beta(s_i)$ on $[\alpha_0, \alpha_1]$ via the envelope theorem condition.

4. *Calculate the required value of* α_2. Knowing the value of $\beta(\alpha_1)$, the indifference condition $v(1, \beta(\alpha_1); \alpha_1) = v(2, \beta(\alpha_1); \alpha_1)$ becomes

   $$Q(1, \beta(\alpha_1))(\alpha_1 - \beta(\alpha_1)) - L(1)c \quad = \quad Q(2, \beta(\alpha_1))(\alpha_1 - \beta(\alpha_1)) - L(2)c$$

   We already noted that $L(1)$ depends only on α_0 and α_1, as do $Q(1, \beta(\alpha_1))$ and $Q(2, \beta(\alpha_1))$. (While sending message 2 will make bidder i more likely to advance against an opponent who sent message 2, he will never win in those cases, so $Q(2, \beta(\alpha_1))$ does not depend on α_2.) $L(2)$ does depend on α_2, and is strictly increasing, so this indifference condition uniquely pins down α_2 given what we already know.

5. *Calculate the bid function on* $[\alpha_1, \alpha_2]$. Once α_2 is known, $Q(s)$ can be calculated for $s \in [\alpha_1, \alpha_2]$; from this, we can calculate $\beta(s_i)$ for $s_i \in [\alpha_1, \alpha_2]$.

6. *Continue to iterate in this way.* Once $\beta(\alpha_2)$ is known, the indifference condition at α_2 determines α_3; once α_3 is known, the envelope theorem lets us recover $\beta(s_i)$ for $s_i \in [\alpha_2, \alpha_3]$; and so on.

7. *Check the terminal condition at the top.* Eventually, the indifference condition at $\alpha_{\overline{M}-1}$ will determine a required value of $\alpha_{\overline{M}}$. If this is equal to 1, we've found an equilibrium; if not, we adjust our initial guess at α_0 and try again.

As we noted in the text, we do not have a proof that this construction will always succeed. (In particular, given $\{\alpha_0, \alpha_1, \ldots, \alpha_m\}$ and $\beta(s_i)$ up to α_m, we have not found a way to ensure that a value of α_{m+1} exists satisfying the indifference condition $v(m, \beta(\alpha_m); \alpha_m) = v(m+1, \beta(\alpha_m); \alpha_m)$.) That said, the construction has succeeded for every value of c, N, and $\overline{M}$ we've tried.

Appendix B. Proof $\Delta(\cdot, \alpha_m, \alpha_{m+1})$ is Strictly Single Crossing

Earlier, we deferred the proof of Lemma 4 part 4, i.e., that $\Delta(\alpha_{m-1}, \alpha_m, \alpha_{m+1})$ is strictly single-crossing from above in α_{m-1} on $[0, \alpha_m)$. We prove that now. We will do this by showing that $\Delta(\alpha_{m-1}, \alpha_m, \alpha_{m+1}) = 0$ implies $\frac{\partial}{\partial \alpha_{m-1}} \Delta(\alpha_{m-1}, \alpha_m, \alpha_{m+1}) < 0$.

We begin by writing

$$\Delta(\alpha_{m-1}, \alpha_m, \alpha_{m+1}) = \sum_{q+r<n}\sum C^{q,r} V^{q,r} + \sum_{r<n, q+r \geq n}\sum D^{q,r} V^{n-1-r,r}$$

where

$$C^{q,r} = \sum_{p=n-q-r}^{N-1-q-r} \binom{N-1}{r}\binom{N-1-r}{q}\binom{N-1-q-r}{p} w^{N-1-p-q-r} x^p y^q z^r \left[1 - \frac{n-q-r}{p+1}\right]$$

$$D^{q,r} = \binom{N-1}{r}\binom{N-1-r}{q}(w+x)^{N-1-q-r} y^q z^r \left[\frac{n-r}{q+1}\right]$$

and $w = \alpha_{m-1}$, $x = \alpha_m - \alpha_{m-1}$, $y = \alpha_{m+1} - \alpha_m$, $z = 1 - \alpha_{m+1}$, and $V^{q,r}$ denotes the expected payoff in an auction given type α_m and r opponents with types above α_{m+1}, q opponents with types in $[\alpha_m, \alpha_{m+1}]$, and $n-1-q-r$ opponents with types in $[\alpha_{m-1}, \alpha_m]$.

Now, differentiate with respect to α_{m-1}, which we will indicate with subscript-α_{m-1}:

$$\begin{aligned}\Delta_{\alpha_{m-1}}(\alpha_{m-1}, \alpha_m, \alpha_{m+1}) &= \sum_{q+r<n}\sum C^{q,r}_{\alpha_{m-1}} V^{q,r} + \sum_{q+r<n}\sum C^{q,r} V^{q,r}_{\alpha_{m-1}} \\ &\quad + \sum_{r<n, q+r \geq n}\sum D^{q,r}_{\alpha_{m-1}} V^{n-1-r,r} + \sum_{r<n, q+r \geq n}\sum D^{q,r} V^{n-1-r,r}_{\alpha_{m-1}}\end{aligned}$$

First, note that $D^{q,r}$ does not depend on α_{m-1} at all, because its only dependence on either w or x is through $w + x = 1 - \alpha_m$. Second, note that $V^{n-1-r,r}$ does not depend on α_{m-1} either, since it's the payoff in an auction where all the opponents have types in either $[\alpha_m, \alpha_{m+1}]$ or $[\alpha_{m+1}, 1]$, which doesn't depend on α_{m-1} at all. Third, note that for $q + r < n$, $V^{q,r}$ is weakly decreasing in α_{m-1}, since it is the payoff in an auction where $n-1-q-r$ opponents have types randomly drawn from $[\alpha_{m-1}, \alpha_m]$, and stronger opponents mean lower payoffs. This means that

$$\Delta_{\alpha_{m-1}}(\alpha_{m-1}, \alpha_m, \alpha_{m+1}) \leq \sum_{q+r<n}\sum C^{q,r}_{\alpha_{m-1}} V^{q,r}$$

where $C^{q,r}$ depends on α_{m-1} through both w (positively) and x (negatively).

Since our interest is in signing this when $\Delta = 0$, we calculate

$$\Delta_{\alpha_{m-1}} - \frac{C^{0,0}_{\alpha_{m-1}}}{C^{0,0}}\Delta \leq \sum\sum_{q+r<n} C^{q,r}_{\alpha_{m-1}} V^{q,r} - \frac{C^{0,0}_{\alpha_{m-1}}}{C^{0,0}} \sum\sum_{q+r<n} C^{q,r} V^{q,r} - \frac{C^{0,0}_{\alpha_{m-1}}}{C^{0,0}} \sum\sum_{r<n,\, q+r\geq n} D^{q,r} V^{q,r}$$

$$= \sum\sum_{0<q+r<n} \left[\frac{C^{q,r}_{\alpha_{m-1}}}{C^{q,r}} - \frac{C^{0,0}_{\alpha_{m-1}}}{C^{0,0}}\right] C^{q,r} V^{q,r} - \frac{C^{0,0}_{\alpha_{m-1}}}{C^{0,0}} \sum\sum_{r<n,\, q+r\geq n} D^{q,r} V^{q,r}$$

By small rents, $V^{q,r} < 0$ for $q + r > 0$. Also note that $C^{q,r}$ and $D^{q,r}$ are all weakly positive, since they're basically sums of probabilities. We'll show that $C^{0,0}_{\alpha_{m-1}} \leq 0$, establishing that

$$\Delta_{\alpha_{m-1}} - \frac{C^{0,0}_{\alpha_{m-1}}}{C^{0,0}}\Delta \leq \sum\sum_{0<q+r<n} \left[\frac{C^{q,r}_{\alpha_{m-1}}}{C^{q,r}} - \frac{C^{0,0}_{\alpha_{m-1}}}{C^{0,0}}\right] C^{q,r} V^{q,r}$$

and then we'll show that the difference in square brackets is positive, ensuring the right-hand side is negative; thus guaranteeing that when $\Delta = 0$, $\Delta_{\alpha_{m-1}} < 0$.

Define $Z(q,r) = \binom{N-1}{r}\binom{N-1-r}{q} y^q z^r$, and write $C^{q,r}$ as

$$C^{q,r} = Z(q,r) \sum_{p=n-q-r}^{N-1-q-r} \binom{N-1-q-r}{p} w^{N-1-p-q-r} x^p \left[1 - \frac{n-q-r}{p+1}\right]$$

Note that $Z(q,r)$ does not depend on α_{m-1}, and $C^{q,r}$ depends on q and r only through $q+r$; letting $e \equiv q + r$, then,

$$C^{q,r} = Z(q,r) \sum_{p=n-e}^{N-1-e} \binom{N-1-e}{p} w^{N-1-e-p} x^p \left[1 - \frac{n-e}{p+1}\right]$$

Now, α_{m-1} effects $C^{q,r}$ through both w and x: specifically, since $w = \alpha_{m-1}$ and $x = \alpha_m - \alpha_{m-1}$, $dw = d\alpha_{m-1}$ and $dx = -d\alpha_{m-1}$, so

$$C^{q,r}_{\alpha_{m-1}} = Z(q,r) \sum_{p=n-e}^{N-2-e} \frac{(N-1-e)!}{(N-1-e-p)!p!}(N-1-e-p) w^{N-2-e-p} x^p \left[1 - \frac{n-e}{p+1}\right]$$

$$-Z(q,r) \sum_{p=n-e}^{N-1-e} \frac{(N-1-e)!}{(N-1-e-p)!p!} w^{N-1-e-p} p x^{p-1} \left[1 - \frac{n-e}{p+1}\right]$$

(Note that the first sum only goes up to $N - 2 - e$, because the $p = N - 1 - e$ term had no w and therefore vanishes when we differentiate; on the other hand, since $e = q + r < n$, every term had a positive x power, so the entire sum survives in the second row.) Next, cancelling $N-1-e-p$ from the numerator and denominator in the first row, cancelling p from the numerator and denominator in the second row, and reindexing the sum by $p' = p - 1$,

$$C^{q,r}_{\alpha_{m-1}} = Z(q,r)\sum_{p=n-e}^{N-2-e} \frac{(N-1-e)!}{(N-2-e-p)!p!} w^{N-2-e-p} x^p \left[1 - \frac{n-e}{p+1}\right]$$

$$-Z(q,r)\sum_{p'=n-e-1}^{N-2-e} \frac{(N-1-e)!}{(N-2-e-p')!p'!} w^{N-2-e-p'} x^{p'} \left[1 - \frac{n-e}{p'+2}\right]$$

If we separate the $n-e-1$ term from the second sum, and combine corresponding terms otherwise,

$$\begin{aligned}
C^{q,r}_{\alpha_{m-1}} &= Z(q,r)\sum_{p=n-e}^{N-2-e} \frac{(N-1-e)!}{(N-2-e-p)!p!} w^{N-2-e-p} x^p \left[\frac{n-e}{p+2} - \frac{n-e}{p+1}\right] \\
&\quad -Z(q,r)\sum_{p=n-e-1}^{n-e-1} \frac{(N-1-e)!}{(N-2-e-p)!p!} w^{N-2-e-p} x^p \left[1 - \frac{n-e}{p+2}\right] \\
&= Z(q,r)\sum_{p=n-e-1}^{N-2-e} \frac{(N-1-e)!}{(N-2-e-p)!p!} w^{N-2-e-p} x^p \left[\frac{n-e}{p+2} - \frac{n-e}{p+1}\right] \\
&= Z(q,r)\sum_{p=n-e-1}^{N-2-e} \frac{(N-1-e)!}{(N-2-e-p)!p!} w^{N-2-e-p} x^p \left[-\frac{n-e}{(p+1)(p+2)}\right] \\
&= -Z(q,r)\frac{n-e}{N-e}\sum_{p=n-e-1}^{N-2-e} \frac{(N-e)!}{(N-2-e-p)!(p+2)!} w^{N-2-e-p} x^p \\
&= -Z(q,r)\frac{n-e}{N-e}\frac{1}{x^2}\sum_{p=n-e-1}^{N-2-e} \frac{(N-e)!}{(N-2-e-p)!(p+2)!} w^{N-2-e-p} x^{p+2} \\
&= -Z(q,r)\frac{n-e}{N-e}\frac{1}{x^2}\sum_{p=n-e-1}^{N-2-e} \binom{N-e}{p+2} w^{N-2-e-p} x^{p+2} \\
&= -Z(q,r)\frac{n-e}{N-e}\frac{1}{x^2}\sum_{p=n-e+1}^{N-e} \binom{N-e}{p} w^{N-e-p} x^{p}
\end{aligned}$$

Note that this is negative for any $e < n$, including $e = q + r = 0$, so $C^{0,0}_{\alpha_{m-1}} < 0$, as noted above.

What's left to show is that $\left[\frac{C^{q,r}_{\alpha_{m-1}}}{C^{q,r}} - \frac{C^{0,0}_{\alpha_{m-1}}}{C^{0,0}}\right]$ is positive. Before we do that, rewrite

$$\begin{aligned}
C^{q,r} &= Z(q,r)\sum_{p=n-e}^{N-1-e}\binom{N-1-e}{p}w^{N-1-e-p}x^p\left[1-\frac{n-e}{p+1}\right] \\
&= Z(q,r)\left[\sum_{p=n-e}^{N-1-e}\binom{N-1-e}{p}w^{N-1-e-p}x^p - \frac{n-e}{N-e}\sum_{p=n-e}^{N-1-e}\binom{N-e}{p+1}w^{N-1-e-p}x^p\right] \\
&= Z(q,r)\left[\sum_{p=n-e}^{N-1-e}\binom{N-1-e}{p}w^{N-1-e-p}x^p - \frac{n-e}{N-e}\frac{1}{x}\sum_{p'=n-e+1}^{N-e}\binom{N-e}{p'}w^{N-e-p'}x^{p'}\right] \\
&= Z(q,r)\sum_{p=n-e}^{N-1-e}\binom{N-1-e}{p}w^{N-1-e-p}x^p + xC^{q,r}_{\alpha_{m-1}}
\end{aligned}$$

Then, calculate

$$\begin{aligned}
&C^{0,0}C^{q,r}_{\alpha_{m-1}} - C^{q,r}C^{0,0}_{\alpha_{m-1}} \\
&= \left(Z(0,0)\sum_{p=n}^{N-1}\binom{N-1}{p}w^{N-1-p}x^p\right)\left(-Z(q,r)\frac{n-e}{N-e}\frac{1}{x^2}\sum_{p=n-e+1}^{N-e}\binom{N-e}{p}w^{N-e-p}x^p\right) \\
&\quad -xC^{0,0}_{\alpha_{m-1}}C^{q,r}_{\alpha_{m-1}} \\
&\quad -\left(Z(q,r)\sum_{p=n-e}^{N-1-e}\binom{N-1-e}{p}w^{N-1-e-p}x^p\right)\left(-Z(0,0)\frac{n}{N}\frac{1}{x^2}\sum_{p=n+1}^{N}\binom{N}{p}w^{N-p}x^p\right) \\
&\quad +xC^{q,r}_{\alpha_{m-1}}C^{0,0}_{\alpha_{m-1}} \\
&= \frac{-Z(0,0)Z(q,r)}{x^2}\left[\left(\sum_{i=n}^{N-1}\binom{N-1}{i}w^{N-1-i}x^i\right)\left(\frac{n-e}{N-e}\sum_{j=n-e+1}^{N-e}\binom{N-e}{j}w^{N-e-j}x^j\right)\right. \\
&\quad \left.-\left(\sum_{j=n-e}^{N-1-e}\binom{N-1-e}{j}w^{N-1-e-j}x^j\right)\left(\frac{n}{N}\sum_{i=n+1}^{N}\binom{N}{i}w^{N-i}x^i\right)\right]
\end{aligned}$$

Reindexing the last three sums,

$$C^{0,0}C^{q,r}_{\alpha_{m-1}} - C^{q,r}C^{0,0}_{\alpha_{m-1}}$$

$$= \frac{-Z(0,0)Z(q,r)}{x^2}\left[\left(\sum_{i=n}^{N-1}\binom{N-1}{i}w^{N-1-i}x^i\right)\left(\frac{n-e}{N-e}\sum_{j'=n}^{N-1}\binom{N-e}{j'+1-e}w^{N-1-j'}x^{j'+1-e}\right)\right.$$

$$\left.-\left(\sum_{j'=n}^{N-1}\binom{N-1-e}{j'-e}w^{N-1-j'}x^{j'-e}\right)\left(\frac{n}{N}\sum_{i'=n}^{N-1}\binom{N}{i'+1}w^{N-1-i'}x^{i'+1}\right)\right]$$

$$= \frac{-Z(0,0)Z(q,r)}{x^2}\left[\frac{1}{x^{e-1}}\frac{n-e}{N-e}\sum_{i=n}^{N-1}\sum_{j=n}^{N-1}\binom{N-1}{i}\binom{N-e}{j+1-e}w^{N-1-i}x^iw^{N-1-j}x^j\right.$$

$$\left.-\frac{1}{x^{e-1}}\frac{n}{N}\sum_{i=n}^{N-1}\sum_{j=n}^{N-1}\binom{N}{i+1}\binom{N-1-e}{j-e}w^{N-1-i}x^iw^{N-1-j}x^j\right]$$

$$= \frac{-Z(0,0)Z(q,r)}{x^{e+1}}\sum_{i=n}^{N-1}\sum_{j=n}^{N-1}\left[\frac{n-e}{N-e}\binom{N-1}{i}\binom{N-e}{j+1-e} - \frac{n}{N}\binom{N}{i+1}\binom{N-1-e}{j-e}\right]w^{2N-2-i-j}x^{i+j}$$

Next, we rewrite the double-sum, pairing the (i,j) term with the (j,i) term and separating the "diagonal" $(i=j)$ terms:

$$C^{0,0}C^{q,r}_{\alpha_{m-1}} - C^{q,r}C^{0,0}_{\alpha_{m-1}}$$

$$= \frac{-Z(0,0)Z(q,r)}{x^{e+1}}\left(\sum_{i=n}^{N-1}\left[\tfrac{n-e}{N-e}\tbinom{N-1}{i}\tbinom{N-e}{i+1-e} - \tfrac{n}{N}\tbinom{N}{i+1}\tbinom{N-1-e}{i-e}\right]w^{2N-2-2i}x^{2i}\right.$$

$$\left.+\sum\sum_{i<j}\left[\tfrac{n-e}{N-e}\tbinom{N-1}{i}\tbinom{N-e}{j+1-e} + \tfrac{n-e}{N-e}\tbinom{N-1}{j}\tbinom{N-e}{i+1-e} - \tfrac{n}{N}\tbinom{N}{i+1}\tbinom{N-1-e}{j-e} - \tfrac{n}{N}\tbinom{N}{j+1}\tbinom{N-1-e}{i-e}\right]w^{2N-2-i-j}x^{i+j}\right)$$

Next, we show the summand in the first sum is negative:

$$\frac{n-e}{N-e}\binom{N-1}{i}\binom{N-e}{i+1-e}-\frac{n}{N}\binom{N}{i+1}\binom{N-1-e}{i-e}$$

$$\begin{aligned}
&= \frac{n-e}{N-e}\frac{(N-1)!}{(N-1-i)!i!}\frac{(N-e)!}{(N-i-1)!(i+1-e)!}-\frac{n}{N}\frac{N!}{(N-i-1)!(i+1)!}\frac{(N-1-e)!}{(N-1-i)!(i-e)!} \\
&= \frac{(N-1)!}{(N-1-i)!i!}\frac{(N-e-1)!}{(N-i-1)!(i-e)!}\left[(n-e)\frac{1}{i+1-e}-n\frac{1}{i+1}\right] \\
&\propto (n-e)(i+1)-n(i+1-e) \\
&= ni-ei+n-e-ni-n+ne \\
&= -ei-e+ne
\end{aligned}$$

which is negative since $i \geq n$ (the sum runs from $i=n$ to $i=N-1$).

Next, we show the summand in the second sum is negative, which takes a bit more work:

$$\tfrac{n-e}{N-e}\tbinom{N-1}{i}\tbinom{N-e}{j+1-e}+\tfrac{n-e}{N-e}\tbinom{N-1}{j}\tbinom{N-e}{i+1-e}-\tfrac{n}{N}\tbinom{N}{i+1}\tbinom{N-1-e}{j-e}-\tfrac{n}{N}\tbinom{N}{j+1}\tbinom{N-1-e}{i-e}$$

$$\begin{aligned}
&= \frac{n-e}{N-e}\frac{(N-1)!}{(N-1-i)!i!}\frac{(N-e)!}{(N-1-j)!(j+1-e)!}+\frac{n-e}{N-e}\frac{(N-1)!}{(N-1-j)!j!}\frac{(N-e)!}{(N-1-i)!(i+1-e)!} \\
&\quad -\frac{n}{N}\frac{N!}{(N-i-1)!(i+1)!}\frac{(N-1-e)!}{(N-1-j)!(j-e)!}-\frac{n}{N}\frac{N!}{(N-j-1)!(j+1)!}\frac{(N-1-e)!}{(N-1-i)!(i-e)!} \\
&= \frac{(N-1)!}{(N-1-i)!}\frac{(N-1-e)!}{(N-1-j)!}\left[(n-e)\frac{1}{i!}\frac{1}{(j+1-e)!}+(n-e)\frac{1}{j!}\frac{1}{(i+1-e)!}\right. \\
&\quad \left.-n\frac{1}{(i+1)!}\frac{1}{(j-e)!}-n\frac{1}{(j+1)!}\frac{1}{(i-e)!}\right]
\end{aligned}$$

Dividing by $\frac{(N-1)!}{(N-1-i)!}\frac{(N-1-e)!}{(N-1-j)!}$ and multiplying by $(i+1)!(j+1)!$,

$$\tfrac{n-e}{N-e}\tbinom{N-1}{i}\tbinom{N-e}{j+1-e} + \tfrac{n-e}{N-e}\tbinom{N-1}{j}\tbinom{N-e}{i+1-e} - \tfrac{n}{N}\tbinom{N}{i+1}\tbinom{N-1-e}{j-e} - \tfrac{n}{N}\tbinom{N}{j+1}\tbinom{N-1-e}{i-e}$$

$$\begin{aligned}
&\propto\quad (n-e)\frac{(i+1)!}{i!}\frac{(j+1)!}{(j+1-e)!} + (n-e)\frac{(j+1)!}{j!}\frac{(i+1)!}{(i+1-e)!} - n\frac{(j+1)!}{(j-e)!} - n\frac{(i+1)!}{(i-e)!} \\
&=\quad (n-e)(i+1)\frac{(j+1)!}{(j+1-e)!} - n\frac{(j+1)!}{(j-e)!} + (n-e)(j+1)\frac{(i+1)!}{(i+1-e)!} - n\frac{(i+1)!}{(i-e)!} \\
&=\quad \frac{(j+1)!}{(j-e+1)!}\left[(n-e)(i+1) - n(j-e+1)\right] + \frac{(i+1)!}{(i+1-e)!}\left[(n-e)(j+1) - n(i-e+1)\right] \\
&=\quad \frac{(j+1)!}{(j-e+1)!}\left[ni - ei + n - e - nj + ne - n\right] + \frac{(i+1)!}{(i+1-e)!}\left[nj - ej + n - e - ni + ne - n\right] \\
&=\quad \frac{(j+1)!}{(j-e+1)!}\left[n(i-j) - e(i+1-n)\right] + \frac{(i+1)!}{(i+1-e)!}\left[n(j-i) - e(j+1-n)\right] \\
&=\quad n(i-j)\left(\frac{(j+1)!}{(j-e+1)!} - \frac{(i+1)!}{(i+1-e)!}\right) - \frac{(j+1)!}{(j-e+1)!}e(i+1-n) - \frac{(i+1)!}{(i+1-e)!}e(j+1-n)
\end{aligned}$$

This, it turns out, is the sum of three negative terms. If $i > j$, then $\frac{(j+1)!}{(j-e+1)!} < \frac{(i+1)!}{(i+1-e)!}$, and if $i < j$, then $\frac{(j+1)!}{(j-e+1)!} > \frac{(i+1)!}{(i+1-e)!}$; either way, the first term is negative. And since both sums run from $i, j = n$ upwards, the last two terms are both negative as well.

So all the summands in both sums are negative, so the big double sum is negative; so $C^{0,0}C^{q,r}_{\alpha_{m-1}} - C^{q,r}C^{0,0}_{\alpha_{m-1}}$, which is $\frac{-Z(0,0)Z(q,r)}{x^{e+1}}$ times that big double sum, is positive. That means $\frac{C^{q,r}_{\alpha_{m-1}}}{C^{q,r}} - \frac{C^{0,0}_{\alpha_{m-1}}}{C^{0,0}}$ is positive, which was all that was left in proving $\Delta_{\alpha_{m-1}} < 0$ when $\Delta = 0$. Thus, Δ is strictly single-crossing from above in its first term, as claimed. $\square$