

Approximate Asymptotic P Values for Structural-Change Tests

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Numerical approximations to the asymptotic distributions of recently proposed tests for structural change are presented. This enables easy yet accurate calculation of asymptotic p values. A GAUSS program is available to perform the computations.

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Recently, Andrews (1993) found the asymptotic distribution of a wide class of tests for structural change in econometric models. In a related article, Andrews and Ploberger (1994) developed an analogous class of tests with stronger optimality properties. The asymptotic distributions of the tests are nonstandard and depend on two parameters, the number of parameters tested and the range of the sample that is examined for the break date.

Although a selected set of asymptotic critical values has been tabulated, the nonstandard nature of these distributions means that p values cannot be calculated from previously published information. This is a disadvantage in applications because applied economists are frequently more interested in p values than in classical Neyman–Pearson significance tests.

This article presents computationally convenient approximations $p^*(x)$ to the asymptotic p -value functions $p(x)$ for the Andrews and Andrews–Ploberger asymptotic distributions. The approximation methods proposed here may find use in a wide range of nonstandard statistical contexts.

Previous attempts to estimate p -value functions for nonstandard test statistics in econometrics were made by Hansen (1992) and MacKinnon (1994). Hansen (1992) set $p^*(x)$ to be a simple polynomial

$$\alpha_v(x|\theta) = \theta_0 + \theta_1 x + \dots + \theta_v x^v \quad (1)$$

and fitted the coefficients by a least squares polynomial regression of upper percentiles on quantiles. MacKinnon (1994) improved the approach by setting $p^*(x) = \Psi(\alpha_v(x|\theta))$, where $\Psi(\cdot)$ is a leading distribution function of interest (in his case, the standard normal). He fitted the coefficients by a least squares polynomial regression of $\Psi^{-1}(p)$ (where p are upper percentiles) on quantiles.

The methods presented in this article extend this literature. Similarly to MacKinnon (1994), I set $p^*(x) = \Psi(\alpha_v(x|\theta)|\eta)$, where $\alpha_v(x|\theta)$ is a polynomial and $\Psi(z|\eta)$ is a leading distribution of interest. One difference is that I allow the distribution to depend on an unknown parameter η . To fit the approximation, I use a weighted loss function over the p -value space. I find that my approximations are extremely accurate, even though my models are quite parsimonious.

In independent and complementary work, Adda and Gonzalo (1995) used the seminonparametric approach of Gal-

lant and Nychka (1987) to approximate the asymptotic distribution of the Dickey–Fuller test. Although their approximating p -value function is different, their method to fit the coefficients is quite similar to mine.

Section 1 reviews the tests and distribution theory of Andrews (1993) and Andrews and Ploberger (1994). Section 2 presents the methodology used to approximate the p -value function. Section 3 presents the approximations. A GAUSS program that computes the test statistics and asymptotic p values can be downloaded from the *JBES* web site and from <http://fmwww.bc.edu/EC-V/Hansen.fac.html>.

1. TESTS FOR STRUCTURAL CHANGE

An $m \times 1$ parameter β , describing some aspect of a time series x_t , takes the value β_1 for $t < k$ and the value β_2 for $t \geq k$, where $m \leq k \leq n - m$. Let $F_n(k)$ denote a Wald, Lagrange multiplier (LM), or likelihood ratio statistic of the hypothesis of no structural change ($\beta_1 = \beta_2$) for given k . When k (the date of structural change) is known only to lie in the range $[k_1, k_2]$, the Quandt or “sup” test statistic is

$$\text{Sup}F_n = \sup_{k_1 \leq k \leq k_2} F_n(k).$$

The Andrews and Ploberger (1994) “Exp” and “Ave” tests are

$$\text{Exp}F_n = \ln \left(\frac{1}{k_2 - k_1 + 1} \sum_{t=k_1}^{k_2} \exp \left(\frac{1}{2} F_n(k) \right) \right),$$

and

$$\text{Ave}F_n = \frac{1}{k_2 - k_1 + 1} \sum_{t=k_1}^{k_2} F_n(k).$$

As shown by Andrews (1993) and Andrews and Ploberger (1994), under a wide set of regularity conditions, these statistics have the asymptotic null distributions

$$\text{Sup}F_n \rightarrow_d \text{Sup}F(\pi_0) = \sup_{\pi_1 \leq \tau \leq \pi_2} F(\tau), \quad (2)$$

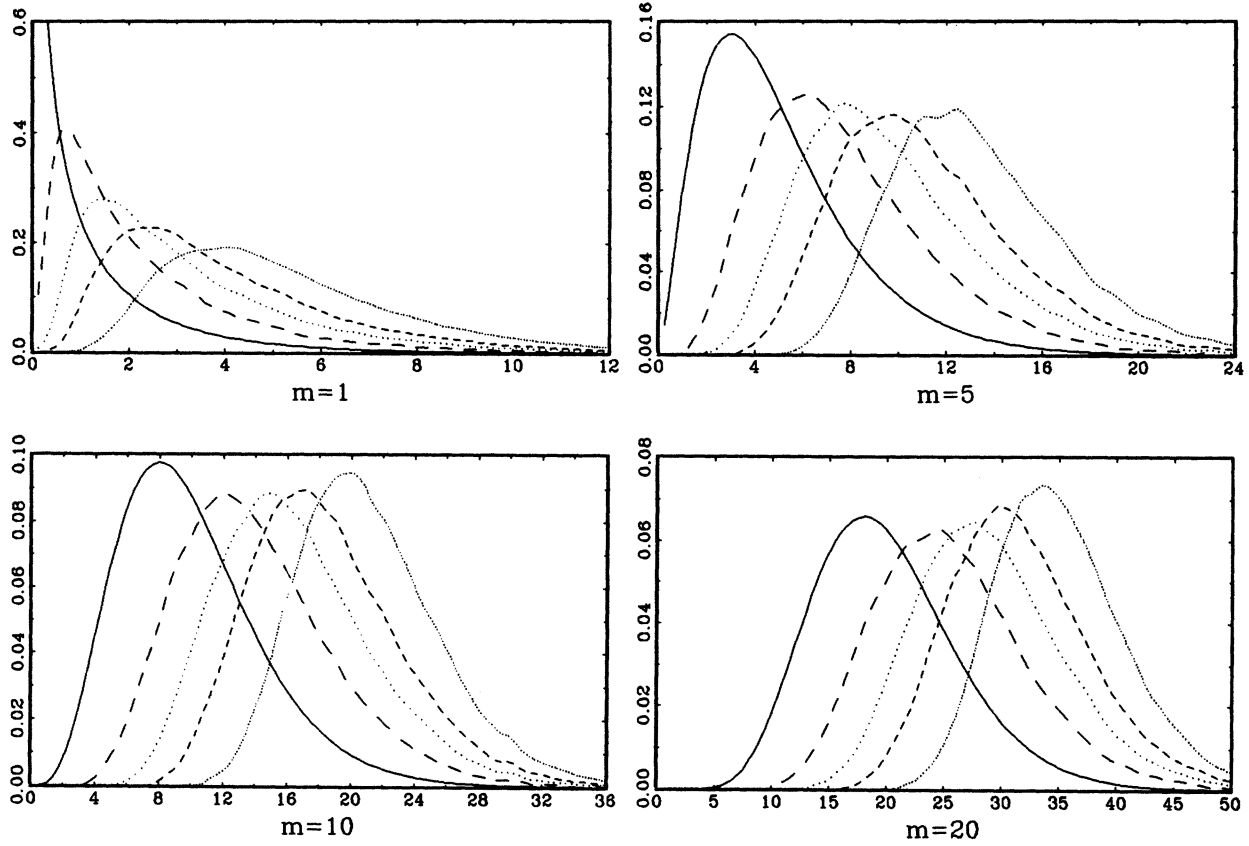


Figure 1. SupF Distribution.

$$\text{Exp}F_n \rightarrow_d \text{Exp}F(\pi_0)$$

$$= \ln \left(\frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} \exp \left(\frac{1}{2} F(\tau) \right) d\tau \right), \quad (3)$$

and

$$\text{Ave}F_n \rightarrow_d \text{Ave}F(\pi_0) = \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} F(\tau) d\tau, \quad (4)$$

where

$$F(\tau) = \frac{(W(\tau) - \tau W(1))'(W(\tau) - \tau W(1))}{\tau(1 - \tau)}, \quad (5)$$

$W(\tau)$ is an $m \times 1$ -vector Brownian motion, $\pi_1 = k_1/n$, and $\pi_2 = k_2/n$. These distributions are nonstandard. In addition to m , the distributions depend on π_1 and π_2 through the single index

$$\pi_0 = \frac{1}{1 + \sqrt{\lambda_0}}, \quad (6)$$

where

$$\lambda_0 = \frac{\pi_2(1 - \pi_1)}{\pi_1(1 - \pi_2)}. \quad (7)$$

Note that, when the range $[k_1, k_2]$ is symmetric in the sample, $\pi_0 = \pi_1 = 1 - \pi_2$.

2. METHODOLOGY

Let T_n denote one of the three tests $\text{Sup}F_n$, $\text{Exp}F_n$, or $\text{Ave}F_n$ for some π_0 , and let T denote the associated asymptotic distribution [e.g., $\text{Sup}F(\pi_0)$, $\text{Exp}F(\pi_0)$, or $\text{Ave}F(\pi_0)$].

Let $p(x) = P(T > x)$ denote the p -value function of T . Define the inverse function of $p(x) : Q(q) = p^{-1}(q)$ that satisfies $q = p(Q(q))$. Note that $Q(1 - q)$ is the quantile function of the distribution. For simplicity, I shall refer to $Q(q)$ as the quantile function.

Although $p(x)$ may be (in principle) calculable, it may be computationally burdensome in applications, so I desire a parametric approximation, valid at least for small p values. In the following sections I describe how I obtain such an approximation.

2.1 Approximating P -Value Functions

I need a parametric function $p(x|\theta)$ that can be made close to the true function $p(x)$ by appropriate selection of the parameter θ . In principle, I would like my functional choice $p(x|\theta)$ to have the standard properties of a distribution function (bounded between 0 and 1 and monotonically decreasing in x), although these properties are not essential if the function gives good approximations.

A general approach is to pick a flexible function class with known approximation properties. Let $\alpha_v(x|\theta)$ be the v th-order polynomial in x defined in (1). By the Stone-Weierstrass theorem, any bounded continuous function $f(x)$ can be arbitrarily well approximated on a compact set by $\alpha_v(x|\theta)$ for a suitable choice of θ . It thus makes sense to consider setting $p(x|\theta) = \alpha_v(x|\theta)$, which is the approach of Hansen (1992). An improvement suggested by MacKinnon (1994) is to set $p(x|\theta) = 1 - \Psi(\alpha_v(x|\theta))$, where $\Psi(\cdot)$ is a distribution function of leading interest. This retains the approximation properties of the polynomial but may

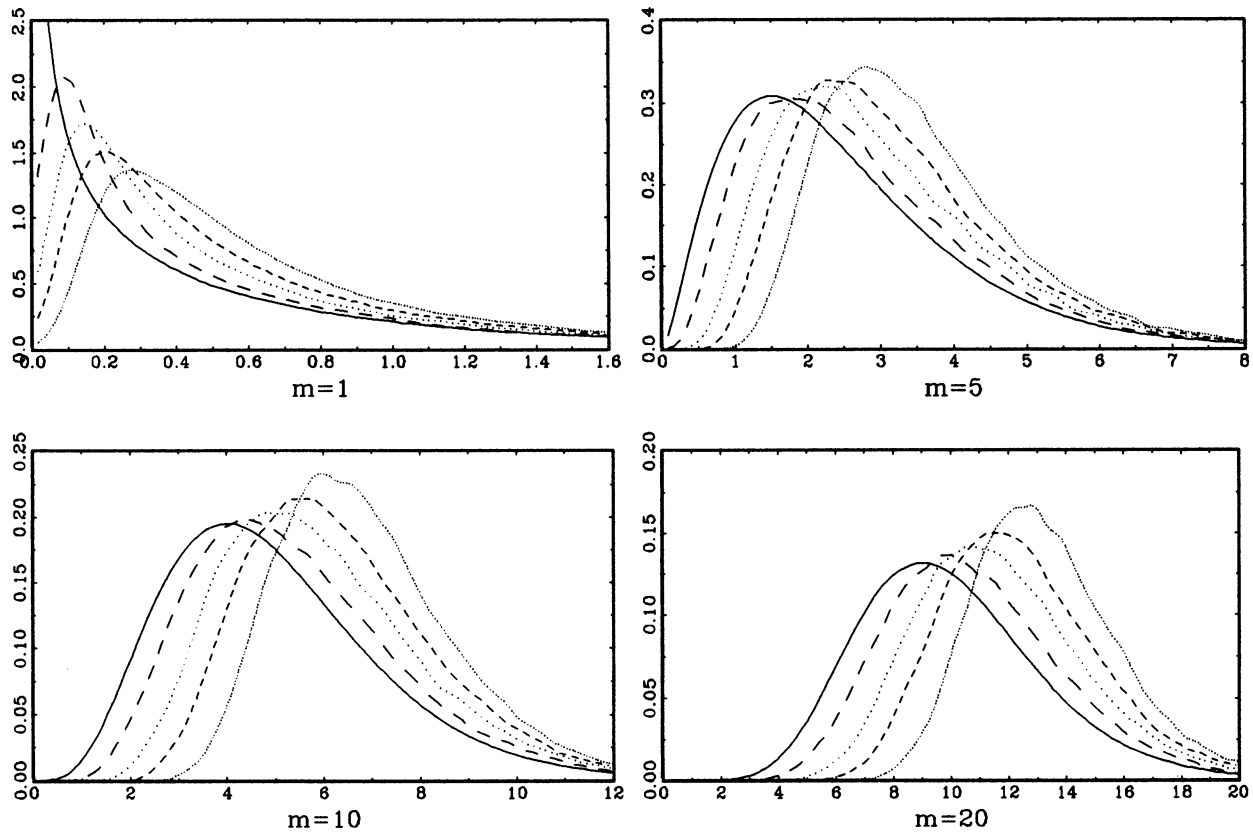


Figure 2. ExpF Distribution.

be more parsimonious, at least when $\Psi(\cdot)$ is close to the true distribution function. I extend this idea one step further and allow Ψ to depend on an unknown parameter η —namely, $\Psi(\cdot|\eta)$ —so that my approximating p -value function is $p(x|\theta) = 1 - \Psi(\alpha_v(x|\theta)|\eta)$.

In my specific applications, I set $\Psi(\cdot|\eta) = \chi^2(\eta)$, the chi-squared distribution with η df, although other distribution functions could be selected in appropriate contexts. In summary, my approximating function is

$$p(x|\theta) = 1 - \chi^2(\theta_0 + \theta_1 x + \dots + \theta_v x^v|\eta), \quad (8)$$

where

$$\chi^2(z|\eta) = \int_0^z \frac{y^{\eta/2-1} e^{-y/2}}{\Gamma(\eta/2) 2^{\eta/2}} dy$$

is the cumulative chi-squared distribution and $\theta = (\theta_0, \theta_1, \dots, \theta_v, \eta)$.

Why this particular choice? The asymptotic theory of Section 1 shows that when $\pi_0 = \frac{1}{2}$, the $\text{Sup}F(\pi_0)$, $\text{Exp}F(\pi_0)$, and $\text{Ave}F(\pi_0)$ distributions simplify to the χ_m^2 distribution. By continuity, their distributions will be close to the χ_m^2 for π_0 close to $\frac{1}{2}$. For other values of π_0 , we can get a sense of the distributions through numerical plots. Figures 1, 2, and 3 display estimated plots of the density functions of the $\text{Sup}F(\pi_0)$, $\text{Exp}F(\pi_0)$, and $\text{Ave}F(\pi_0)$ distributions, respectively, for $m = 1, 5, 10$, and 20 , and several values of π_0 . The densities appear to resemble those of the chi-square but with shifts in location and spread. It therefore seems reasonable to use the chi-squared distribution as our “leading case” distribution.

2.2 Loss Function

Given the function $p(x|\theta)$ of (8), I need to select θ to make $p(x|\theta)$ as close as possible to the true p -value function $p(x)$. Because the objects of interest are the p values themselves, I wish to make the difference $|p(x|\theta) - p(x)|$ small. This is equivalent to making $|p(Q(q)|\theta) - q|$ small. In principle, I want all errors, not just the “average” error, to be small. The natural metric to measure the statement “all errors are small” is the uniform metric

$$d_\infty(\theta) = \max_{0 \leq q \leq 1} |p(Q(q)|\theta) - q|. \quad (9)$$

The uniform metric is difficult to implement numerically. A close relative is the L^r norm

$$d_r(\theta) = \left[\int_0^1 |p(Q(q)|\theta) - q|^r dq \right]^{1/r} \quad (10)$$

for r large. (I use $r = 8$ in the work that follows.) Metric (10) seems inappropriate, however, because it weights all quantiles equally. It seems reasonable to believe that I am more concerned with precision in p values when the p values are small. This desire can be incorporated by including a weight function in (10):

$$d_r(\theta) = \left[\int_0^1 |p(Q(q)|\theta) - q|^r w(q) dq \right]^{1/r}, \quad (11)$$

where $w(q) \geq 0$. Beyond the fact that $w(q)$ should be decreasing in q , it is not clear exactly what shape it should

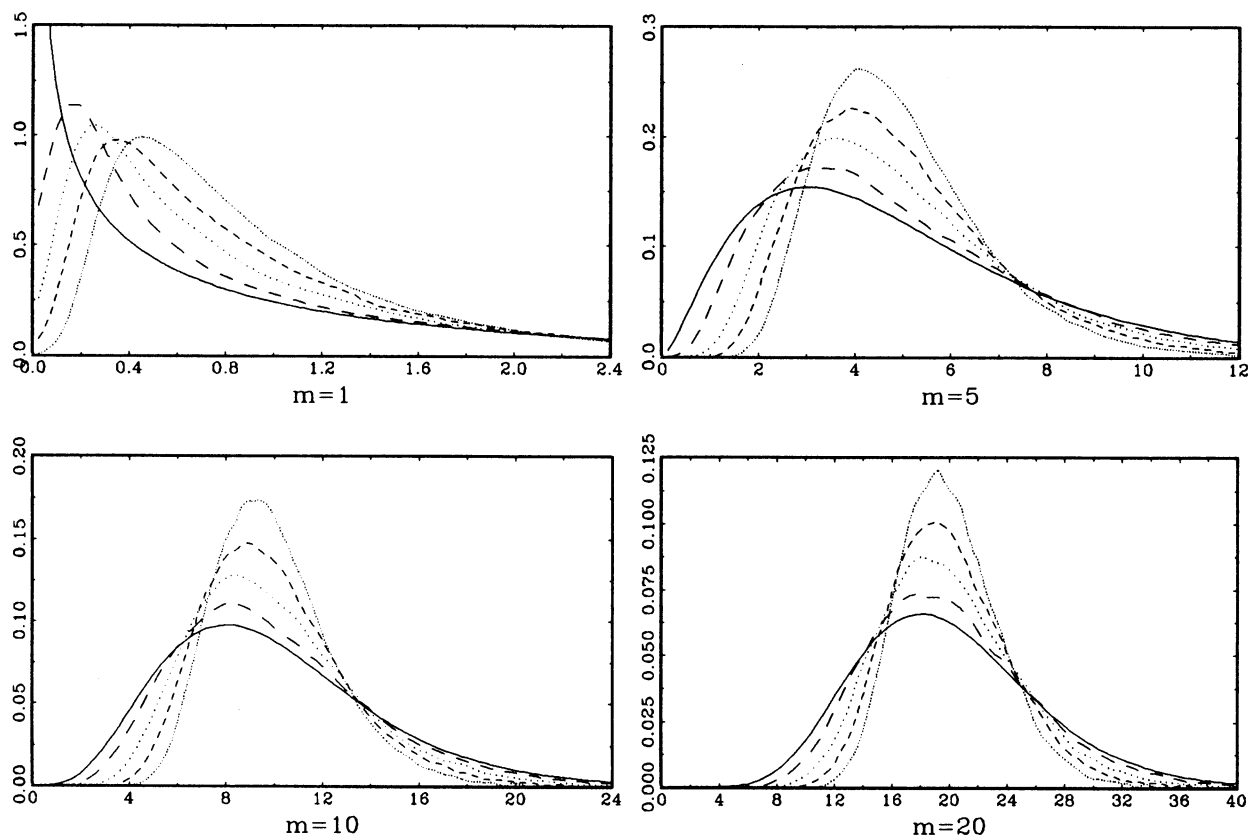


Figure 3. AveF Distribution.

take. After some experimentation, I settled on the following choice:

$$w(q) = \begin{cases} 1, & 0 \leq q \leq .1 \\ \left(\frac{.8-q}{.7}\right)^2, & .1 \leq q \leq .8 \\ 0, & .8 \leq q \leq 1.0. \end{cases} \quad (12)$$

The weight function $w(q)$ in (12) has the following features. It gives highest weight to p values in the region $[0, .1]$ and zero weight to those in the region $[.8, 1.0]$. It is continuous between these points, with a quadratic decay.

When $Q(q)$ is not analytic, I can replace the continuous region $[0, 1]$ by a discrete set $[q_1, \dots, q_N]$ to approximate the integral (11). I use the set $\{.001, .002, \dots, .999\}$ in the work that follows.

Table 1. Absolute Error in Fitted Distributions

p value	SupF distributions		ExpF distributions		AveF distributions	
	Median error	Maximum error	Median error	Maximum error	Median error	Maximum error
.00	.0001	.0006	.0001	.0013	.0001	.0009
.01	.0004	.0015	.0004	.0017	.0004	.0014
.02	.0005	.0019	.0005	.0021	.0005	.0018
.03	.0006	.0023	.0006	.0023	.0006	.0018
.04	.0006	.0025	.0006	.0021	.0006	.0018
.05	.0006	.0021	.0006	.0019	.0006	.0021
.06	.0006	.0021	.0006	.0019	.0005	.0021
.07	.0005	.0019	.0005	.0020	.0006	.0020
.08	.0005	.0019	.0006	.0024	.0005	.0019
.09	.0005	.0018	.0005	.0024	.0005	.0020
.10	.0005	.0018	.0005	.0019	.0005	.0018
.15	.0005	.0024	.0005	.0020	.0005	.0017
.20	.0005	.0020	.0006	.0021	.0005	.0024
.25	.0006	.0024	.0006	.0025	.0005	.0019
.30	.0005	.0020	.0006	.0027	.0006	.0023
.40	.0006	.0024	.0006	.0025	.0006	.0025
.50	.0006	.0030	.0006	.0025	.0006	.0023
.60	.0006	.0027	.0006	.0027	.0006	.0026
.70	.0008	.0024	.0008	.0032	.0008	.0028
.80	.0017	.0058	.0019	.0068	.0015	.0066

Table 2. SupF Distribution, $m \geq 1$

m	$\pi_0 = .01$			$\pi_0 = .05$			$\pi_0 = .15$			$\pi_0 = .25$			$\pi_0 = .35$		
	θ_0	θ_1	η	θ_0	θ_1	η	θ_0	θ_1	η	θ_0	θ_1	η	θ_0	θ_1	η
1	-1.79	1.17	4.5	-1.39	1.07	3.6	-.99	1.02	3.0	-.73	.98	2.5	-.50	.96	2.1
2	-3.06	1.18	6.1	-2.38	1.11	5.4	-1.65	1.06	4.7	-1.16	1.02	4.1	-.78	.97	3.5
3	-4.09	1.21	7.8	-3.31	1.10	6.5	-2.05	1.13	6.8	-1.61	1.03	5.5	-1.06	1.01	4.9
4	-5.33	1.21	8.9	-4.08	1.14	8.2	-2.52	1.11	8.0	-1.91	1.04	7.0	-1.45	.97	5.7
5	-6.39	1.18	9.4	-4.84	1.15	9.3	-3.46	1.07	8.3	-2.63	1.02	7.5	-1.82	1.00	7.0
6	-7.08	1.26	11.8	-5.37	1.19	11.2	-4.05	1.08	9.5	-2.94	1.05	9.0	-1.79	1.03	8.6
7	-8.49	1.17	11.1	-6.21	1.21	12.6	-4.42	1.10	11.0	-3.23	1.05	10.1	-2.21	1.01	9.3
8	-9.20	1.17	12.2	-7.24	1.13	11.9	-5.36	1.08	11.3	-3.65	1.06	11.4	-1.69	1.10	12.2
9	-10.22	1.14	12.3	-8.07	1.11	12.4	-5.43	1.10	13.1	-4.38	1.01	11.3	-2.83	1.00	11.1
10	-11.01	1.14	13.3	-8.84	1.11	13.2	-6.47	1.06	12.8	-4.97	1.01	12.0	-2.92	1.05	13.0
11	-11.90	1.11	13.4	-9.56	1.06	13.1	-6.79	1.04	13.5	-4.62	1.05	14.4	-3.26	1.01	13.4
12	-12.88	1.06	12.8	-10.35	1.09	14.5	-7.80	1.02	13.6	-5.32	1.05	15.1	-3.91	1.00	13.8
13	-13.88	1.09	14.1	-11.07	1.07	14.8	-7.93	1.07	15.9	-5.80	1.04	15.8	-4.14	1.00	14.9
14	-14.61	1.15	16.6	-11.52	1.11	16.8	-8.54	1.05	16.1	-5.90	1.05	17.2	-4.06	1.02	16.5
15	-15.49	1.04	14.1	-12.44	1.08	16.6	-9.05	1.05	17.2	-6.59	1.04	17.6	-3.10	1.08	20.0
16	-16.34	1.15	17.8	-12.27	1.20	21.4	-9.13	1.09	19.3	-7.00	1.04	18.5	-4.79	.99	17.6
17	-17.20	1.15	18.8	-13.73	1.15	20.1	-10.45	1.05	18.3	-7.23	1.05	19.8	-5.01	1.02	19.1
18	-18.10	1.17	20.0	-14.15	1.14	21.2	-10.63	1.05	19.5	-7.76	1.04	20.2	-5.11	1.02	20.2
19	-18.19	1.04	17.2	-14.94	.97	16.3	-12.14	.90	14.9	-9.84	.89	15.3	-7.09	.91	16.8
20	-18.99	1.02	17.0	-16.09	.99	17.0	-12.14	.97	18.3	-8.87	1.00	20.5	-5.94	1.00	21.3
25	-23.42	1.06	21.0	-19.06	1.06	23.4	-14.16	1.05	25.0	-10.65	1.03	25.7	-6.57	1.02	27.1
30	-27.30	1.03	22.5	-22.91	1.04	25.1	-17.06	1.03	27.8	-11.51	1.07	32.8	-6.79	1.05	34.1
35	-30.01	.92	20.0	-25.88	.97	25.2	-20.09	.98	28.4	-15.78	.97	30.2	-10.44	.98	33.1
40	-34.24	.97	24.8	-29.24	.98	28.0	-21.65	1.05	36.7	-14.18	1.07	42.7	-11.95	.94	35.0

Minimization of $d_r(\theta)$ yields the parameter value that best fits the approximation $p(x|\theta)$ to the true p -value function $p(x)$. Let the minimum value be denoted by θ^* :

$$\theta^* = \text{Argmin}_{\theta \in \Theta} d_r(\theta),$$

the loss-minimizing p -value function by $p^*(x) = p(x|\theta^*)$, and the approximate p values by $p_n^* = p(T_n|\theta^*)$.

Alternative choices for loss function (11) and weight function (12) may be made. For example, Hansen (1992) and Adda and Gonzalo (1996) set $r = 2$. This penalizes large errors less severely than my choice. My choice to set r high implies that I am concerned about large approximation errors and am not very willing to trade off a few large errors in return for many other small errors. I think this corresponds to my idea that I want a reported p value in any application to be accurate.

One can also view the fitting of $p(Q(q)|\theta)$ to q as a regression, a point made in particular by MacKinnon (1994). Because this regression has nonclassical statistical properties, MacKinnon (1994) suggested that reweighting be done to account for heteroscedastic errors. I do not believe this is appropriate. A correctly specified loss function, such as (11)–(12), incorporates all information necessary for loss-

minimizing curve fitting. I think it is most constructive to discuss the choice of loss function rather than the statistical qualities of the regression fit.

2.3 Approximating the Quantile Function

Minimization of the criterion (11) requires the computation of the true quantile function $Q(q)$, which is unknown. It may be numerically approximated using analytic techniques. DeLong (1981) provided expansions for the SupF distribution for $m \leq 4$. Anderson and Darling (1952) provided expansions for the AveF distribution for $m = 1$ and $\pi_0 = 0$. It is possible that these techniques could be generalized to handle my applications. It is not clear, however, that this is desirable. Such numerical approximations involve considerable analytic effort and in the end still produce approximations (such as truncated infinite sums).

Another approach is to use analytic methods to approximate $p(x)$ for large x (i.e., for small p values) as did Kim and Siegmund (1989). The downside is that this approach does not necessarily give good estimates for the entire support of the distribution.

I took the analytically simpler approach of Monte Carlo simulation. The cost is a relatively heavy use of computer resources. I approximated the distributions (2), (3), and (4)

Table 3. ExpF, $m = 1, 2, 3$

π_0	$m = 1$					$m = 2$				$m = 3$			
	θ_0	θ_1	θ_2	θ_3	η	θ_0	θ_1	θ_2	η	θ_0	θ_1	θ_2	η
.01	-.74	5.23	-1.16	.17	2.3	-1.34	3.32	-.14	3.1	-2.10	3.54	-.13	4.9
.05	-.61	4.66	-1.01	.16	2.1	-1.07	3.05	-.11	2.9	-1.66	3.29	-.11	4.6
.15	-.42	3.75	-.65	.10	1.7	-.77	2.66	-.07	2.6	-1.16	2.91	-.08	4.2
.25	-.30	3.33	-.54	.09	1.5	-.55	2.32	-.03	2.3	-.67	2.74	-.07	4.1
.35	-.19	2.81	-.41	.08	1.2	-.38	2.13	-.02	2.1	-.56	2.08	-.01	3.0

Table 4. *ExpF* Distribution, $m \geq 4$

m	$\pi_0 = .01$			$\pi_0 = .05$			$\pi_0 = .15$			$\pi_0 = .25$			$\pi_0 = .35$		
	θ_0	θ_1	η	θ_0	θ_1	η	θ_0	θ_1	η	θ_0	θ_1	η	θ_0	θ_1	η
4	-2.48	2.41	4.5	-2.05	2.34	4.5	-1.47	2.26	4.4	-1.09	2.12	4.1	-0.72	2.03	4.0
5	-3.19	2.34	5.2	-2.66	2.29	5.2	-1.95	2.20	5.2	-1.49	2.08	4.9	-0.89	2.05	5.0
6	-3.94	2.20	5.5	-3.34	2.17	5.6	-2.45	2.13	5.8	-1.77	2.09	5.9	-1.18	2.00	5.8
7	-4.67	2.23	6.3	-3.88	2.20	6.6	-2.73	2.19	7.1	-1.90	2.12	7.1	-1.26	2.05	7.0
8	-5.26	2.20	7.1	-4.37	2.19	7.5	-2.91	2.22	8.5	-1.86	2.23	9.0	-.80	2.20	9.3
9	-6.08	2.17	7.6	-5.07	2.20	8.3	-3.58	2.20	9.1	-2.58	2.12	9.0	-1.46	2.10	9.4
10	-6.74	2.17	8.4	-5.56	2.16	9.0	-4.03	2.11	9.4	-3.08	2.00	9.0	-1.78	2.03	9.8
11	-7.49	2.25	9.6	-6.21	2.23	10.1	-4.66	2.15	10.3	-3.13	2.15	11.1	-1.44	2.18	12.2
12	-8.19	2.10	9.2	-6.86	2.11	10.0	-5.20	2.07	10.5	-3.79	2.01	10.7	-2.24	2.00	11.4
13	-8.89	2.07	9.6	-7.39	2.12	10.9	-5.49	2.04	11.3	-4.16	2.00	11.5	-2.55	2.01	12.3
14	-9.65	2.16	11.0	-7.79	2.25	13.0	-5.56	2.17	13.5	-3.68	2.17	14.5	-2.15	2.11	14.6
15	-10.51	2.07	10.6	-8.97	2.13	12.0	-6.66	2.11	13.1	-5.27	2.00	12.7	-3.57	1.98	13.3
16	-11.26	2.00	10.5	-9.63	2.01	11.5	-6.92	2.11	14.2	-4.82	2.13	15.6	-2.48	2.11	16.7
17	-11.89	2.13	12.5	-10.07	2.11	13.4	-7.58	2.08	14.4	-5.34	2.09	15.8	-2.53	2.15	18.2
18	-12.57	1.98	11.5	-10.62	2.00	12.9	-8.15	1.94	13.5	-5.96	2.00	15.5	-4.05	1.96	15.9
19	-13.17	2.01	12.6	-11.26	1.98	13.2	-8.78	1.93	14.0	-6.51	1.99	16.0	-4.47	1.96	16.5
20	-13.84	2.03	13.4	-11.91	2.00	14.1	-8.96	2.02	16.1	-6.43	2.06	18.0	-4.17	2.01	18.6
25	-17.50	1.97	15.0	-15.05	1.92	15.8	-12.18	1.88	16.7	-9.27	1.91	18.9	-6.04	1.97	21.7
30	-20.92	1.90	16.4	-18.16	1.91	18.3	-14.19	1.93	21.0	-11.25	1.89	21.9	-8.47	1.85	22.6
35	-23.97	1.80	16.9	-20.86	1.87	20.2	-16.50	1.95	24.7	-12.00	2.01	28.8	-7.10	2.07	33.3
40	-27.29	1.84	19.9	-23.96	1.97	25.1	-18.64	1.93	27.8	-14.44	1.92	30.2	-5.07	2.20	43.8

using a grid on $[0, 1]$ with 1,000 evenly spaced points. This is equivalent to simulating (5) using a sample of size 1,000. I then constructed the empirical quantile function $\hat{Q}(q)$ from $R = 50,000$ independent replications. This should be sufficiently precise for my purposes. Indeed, let $\hat{p}(x) = \hat{Q}^{-1}(x)$ be the empirical p -value function. By the central limit theorem, $P(|\hat{p}(x) - p(x)| \geq .0044) \approx .05$. Thus at the 95% confidence level, the maximum simulation error is about .0044. Most simulation errors, of course, are much less than this amount.

To summarize, my p -value approximations involve two separate approximations. First, I estimate the true p values $p(x)$ by $\hat{p}(x)$ using simulation. Second, I use a parametric function $p(x|\theta)$ to approximate the estimated p values $\hat{p}(x)$. These two errors do not necessarily offset one another. To reduce the total error, I need to make both errors small, which is possible only by (a) increasing the number of simulation replications and (b) increasing the order v of the polynomial $\alpha_v(x|\theta)$.

2.4 Results

I fit p -value functions of the form (8) to the $\text{Sup}F(\pi_0)$, $\text{Exp}F(\pi_0)$, and $\text{Ave}F(\pi_0)$ distributions for $m = 1, 2, \dots, 40$ and $\pi_0 = .01, .03, .05, \dots, .49$. There are thus 3,000 distinct distributions. For each distribution, I selected the polynomial order v to get a good yet parsimonious fit. For the $\text{Sup}F$ distributions, I found that $v = 1$ was sufficient for

all m . For the $\text{Exp}F$ distributions, $v = 3$ was necessary for $m = 1, v = 2$ was needed for $m = 2$ and $m = 3$, and $v = 1$ was sufficient for $m \geq 4$. For the $\text{Ave}F$ distributions, $v = 3$ was used for $m = 1, v = 2$ for $m = 2$, and $v = 1$ for $m \geq 3$.

For any distribution, the absolute error from my parametric approximation is $d(x) = |p(x|\hat{\theta}) - \hat{p}(x)|$. Table 1 reports a summary of the errors for 20 p values of interest. The column "Median error" reports the median of the absolute errors across the 1,000 distributions for each test ($\text{Sup}F$, $\text{Exp}F$, and $\text{Ave}F$). The column "Maximum error" reports the maximum absolute error across all 1,000 distributions. It appears that the errors are quite small. For example, we see that for the 1% p value the distributions err at most by .0017, with a median error of only .0004. Interestingly, the numerical approximations are almost as precise for large p values. For example, at the 50% p value, the $\text{Sup}F$ distribution has a maximal error of .0030 and a median of .0006. The accuracy of these approximations is much better than necessary for empirical applications.

Despite the parsimony of the fitted models, I still have over 9,000 coefficients to report, which is too many to print in this article. The complete estimates are available in a GAUSS program. I report here the coefficients for $\pi_0 = .01, .05, .15, .25, .35$, and for $m = 1, 2, \dots, 20, 25, 30, 35$, and 40. Table 2 reports the coefficients for the $\text{Sup}F$ distributions. Tables 3 and 4 report those for the $\text{Exp}F$ distributions, and Tables 5 and 6 for the $\text{Ave}F$ distributions.

Table 5. *AveF* Distribution, $m = 1, 2$

π_0	$m = 1$					$m = 2$				
	θ_0	θ_1	θ_2	θ_2	η	θ_0	θ_1	θ_2	η	
.01	-1.02	5.39	-.95	.11	3.2	-1.78	3.12	-.10	4.0	
.05	-.74	4.95	-.84	.09	3.2	-1.42	2.80	-.08	3.8	
.15	-.47	3.63	-.51	.05	2.5	-.94	2.21	-.05	3.2	
.25	-.35	2.56	-.30	.03	1.8	-.62	1.70	-.02	2.7	
.35	-.22	1.79	-.16	.02	1.3	-.41	1.21	-.00	2.0	

Table 6. AveF Distribution, $m \geq 3$

<i>m</i>	$\pi_0 = .01$			$\pi_0 = .05$			$\pi_0 = .15$			$\pi_0 = .25$			$\pi_0 = .35$		
	θ_0	θ_1	η	θ_0	θ_1	η	θ_0	θ_1	η	θ_0	θ_1	η	θ_0	θ_1	η
3	-2.41	2.24	4.3	-2.03	2.06	4.2	-1.41	1.70	3.7	-1.01	1.40	3.2	-.59	1.22	3.1
4	-3.28	2.20	5.5	-2.76	2.02	5.3	-1.94	1.68	4.8	-1.29	1.45	4.5	-.72	1.26	4.3
5	-3.91	2.24	7.3	-3.27	2.07	7.1	-2.23	1.72	6.4	-1.59	1.44	5.6	-.89	1.24	5.3
6	-4.85	2.26	8.7	-4.09	2.07	8.4	-2.87	1.71	7.4	-1.86	1.48	7.0	-1.05	1.28	6.6
7	-5.30	2.21	10.2	-4.45	2.02	9.7	-3.24	1.63	8.2	-2.41	1.34	7.0	-1.50	1.17	6.7
8	-6.24	2.17	11.1	-5.27	1.99	10.7	-3.84	1.60	9.0	-2.95	1.32	7.6	-1.84	1.16	7.5
9	-7.39	2.07	11.3	-6.17	1.94	11.3	-4.34	1.62	10.2	-2.80	1.41	9.9	-1.77	1.21	9.1
10	-7.88	2.11	13.3	-6.65	1.94	12.8	-4.65	1.61	11.5	-3.20	1.38	10.6	-1.77	1.22	10.4
11	-8.45	2.17	15.5	-7.25	1.98	14.6	-4.74	1.70	14.0	-3.72	1.39	11.6	-2.37	1.21	10.9
12	-9.41	2.15	16.4	-7.94	1.98	15.9	-5.48	1.66	14.4	-3.45	1.45	14.0	-1.44	1.31	14.2
13	-10.43	2.16	17.7	-8.93	1.97	16.7	-6.38	1.63	14.8	-4.47	1.40	13.7	-2.69	1.22	13.2
14	-11.32	2.02	17.0	-9.89	1.82	15.6	-6.90	1.58	15.2	-4.35	1.43	15.6	-3.18	1.17	13.3
15	-12.01	2.17	20.4	-10.36	1.97	19.1	-7.19	1.67	17.9	-5.21	1.42	16.1	-2.98	1.26	15.9
16	-10.48	2.47	28.9	-8.66	2.25	27.3	-5.34	1.87	24.5	-3.24	1.57	21.8	-1.88	1.31	19.1
17	-12.92	2.22	24.9	-10.95	2.03	23.5	-7.44	1.71	21.5	-4.39	1.52	21.3	-1.75	1.34	21.0
18	-13.31	2.25	27.1	-10.94	2.09	26.6	-8.82	1.58	19.6	-5.92	1.39	19.1	-3.52	1.21	18.3
19	-13.02	2.36	31.9	-10.53	2.17	30.8	-8.30	1.68	23.6	-5.69	1.43	21.5	-4.37	1.17	17.9
20	-14.41	2.23	30.1	-12.65	1.97	26.8	-9.69	1.54	21.2	-6.95	1.33	19.6	-5.05	1.13	17.6
25	-19.33	2.10	33.2	-16.14	1.95	32.6	-11.79	1.60	28.1	-9.03	1.33	24.1	-5.88	1.17	23.3
30	-24.32	2.11	39.1	-19.98	2.01	40.4	-14.53	1.64	34.7	-10.24	1.39	31.5	-2.41	1.36	38.4
35	-26.94	2.19	49.8	-22.85	2.01	47.3	-16.38	1.66	41.6	-8.60	1.55	45.4	-4.38	1.33	42.0
40	-31.58	2.04	50.0	-27.55	1.83	45.8	-19.54	1.57	43.0	-12.15	1.42	44.5	-3.76	1.33	49.5

3. EMPIRICAL ILLUSTRATION

To illustrate the usefulness of these *p*-value approximations, I report two simple applications using autoregressive (AR) models. The first application is an AR(6) fit to the growth rate (log-difference) of U.S. monthly total personal income for the period 1946.1–1995.7, extracted from the Citibase file GMPY. The second is an AR(12) fit to the first-difference of the monthly three-month U.S. treasury-bill (T-bill) rate for the period 1947.1–1995.7, extracted from the Citibase file FYMG3. The results are reported in Table 7. I report the numerical value of the Sup LM, Exp LM, and Ave LM versions of the tests for constancy of all regression coefficients, setting $\pi_1 = .15$ and $\pi_2 = .85$ (so $\pi_0 = .15$). I also report the asymptotic 10%, 5%, and 1% critical values from Andrews (1993) and Andrews and Ploberger (1994). Finally, I report the approximate asymptotic *p* values.

To compute the *p* values, I use Formula (8) and the coefficients from Tables 2–6 for $\pi_0 = .15$. For example, take the Sup LM statistic for personal income. From Table 2, I find that, for $m = 7$ (the number of parameters in the regression) and $\pi_0 = .15, \theta_0 = -4.42, \theta_1 = 1.10$, and $\eta = 11.0$. Thus, I take the test value of 12.5, make the computation $-4.42 + 12.5 * 1.10 = 9.33$, and use the chi-

squared distribution with $\eta = 11.0$ df evaluated at 9.33 to obtain the *p* value of .59. Or consider the Ave LM statistic for the T-bill rate. From Table 6, I find that for $m = 14$ and $\pi_0 = .15$ that $\theta_0 = -6.38, \theta_1 = 1.63$ and $\eta = 14.8$. So my approximate *p* value for the test value of 18.4 is $1 - \chi^2(-6.38 + 1.63 * 18.4 | 14.8) = .07$.

A reading of the table shows that the *p* values yield much more information than simply the critical values. Take, for example, the SupLM test applied to the T-bill series. The test statistic of 25.0 appears to be quite “close” to the 10% critical value of 29.1, so on the basis of the critical values alone a researcher might conclude that the test is “close” to significant. But the asymptotic *p* value turns out to be only .27. Similarly, the ExpLM statistic is 10.2, which appears close to the 10% critical value of 11.1, yet has a *p* value of only .18. In summary, the *p* values are relatively simple to calculate, given the tables provided, yet they enable a researcher to come to more informed conclusions than would be possible simply on the basis of the asymptotic critical values.

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Table 7. Empirical Applications

	Personal income ($m = 7$)					T-bill rate ($m = 13$)				
	Test statistic	Critical value			Approximate <i>p</i> value	Test statistic	Critical value			Approximate <i>p</i> value
		10%	5%	1%			10%	5%	1%	
Sup LM	12.5	19.7	21.8	26.3	.59	25.0	29.1	31.8	37.0	.27
Exp LM	4.6	6.7	7.7	9.5	.41	10.2	11.1	12.3	14.6	.18
Ave LM	7.4	10.3	11.5	14.3	.37	18.4	17.2	18.8	22.5	.07

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