CONVERGENCE TO STOCHASTIC INTEGRALS FOR DEPENDENT HETEROGENEOUS PROCESSES

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This paper provides conditions to establish the weak convergence of stochastic integrals. The theorems are proved under the assumption that the innovations are strong mixing with uniformly bounded 2+ moments. Several applications of the results are given, relevant for the theories of estimation with $I(1)$ processes, $I(2)$ processes, processes with nonstationary variances, near-integrated processes, and continuous time approximations.

1. INTRODUCTION

A considerable research program has developed in econometrics concerning an asymptotic distribution theory for integrated and near-integrated processes. Frequently, the theory involves sequences of cadlag[1] processes $\{U_n(s), V_n(s)\}$ which converge weakly in the Skorohod topology to $\{U(s), V(s)\}$. It is often desirable to obtain the limiting distribution of the integral process $\int_0^s U_n \, dV_n$. In certain cases, $\int_0^s U_n \, dV_n \Rightarrow \int_0^s U^- \, dV$,[2] but it is known that this result is typically violated when V_n is not a martingale. See, for example, Phillips [16].

There is a growing literature studying the asymptotic distribution of stochastic integrals. Chan and Wei [4, Lemma 2.4] derive the distribution of $\int_0^1 U_n \, dV_n$ when U_n and V_n are martingale arrays with uniformly bounded conditional variances (which excludes many conditionally heteroskedastic processes, such as ARCH). Phillips [19] extends the results of [4] to allow the innovations to be general linear processes with independent identically distributed innovations. Strasser [22, Theorem 1.7] derives a functional limit theorem for $\int_0^s U_n \, dV_n$, for the case that V_n is a martingale array and U_n a Lipschitz function of a martingale array. In another paper, Phillips [18] attempts an alternative proof for mixing processes, but makes an error.[3] Jeganathan [9, Proposition 6] provides a proof for the case that V_n is a martingale array and (U, V) are continuous processes. The most general results

yet published for martingale arrays are contained in Kurtz and Protter [10]. These authors consider a variety of limit theorems when V_n is a semimartingale.

This paper provides an asymptotic theory for stochastic integrals which arise in econometric applications. The assumptions allow for weakly dependent heterogeneous data. Section 2 considers stochastic integrals with respect to a martingale process. Section 3 develops a martingale difference approximation for strong mixing sequences. Section 4 demonstrates the usefulness of the results for several examples of major interest. In all cases, the differences of V_n are assumed to be strong mixing. The first example assumes that the differences of U_n are strong mixing, which has applications in the theory of multivariate unit roots [4] and cointegration among $I(1)$ variables [8,12]. The second example assumes that U_n is the product of processes whose differences are strong mixing, which has applications in the theory of heteroskedastic cointegration [6] and nonstationary variances [7]. The third example assumes that the second differences of U_n are strong mixing, which has applications in the theory of cointegration among $I(2)$ variables [13]. The final example assumes that U_n is a vector process with a root local to unity, which has applications in the theory of near-integration [3,17,20] and continuous time approximations [14,15]. The appendix contains the proofs.

A word on notation. The symbol "$\equiv$" denotes equality in distribution, $\|B\|_r = \left(\sum_{ij} E|B_{ij}|^r \right)^{1/r}$ denotes the L^r-norm, "BM(Ω)" denotes vector Brownian motion with covariance matrix Ω, M^{km} denotes the real-valued, $k \times m$ matrices, $\mathfrak{D}$ denotes the space of cadlag functions, and "$\Rightarrow$" denotes weak convergence with respect to the Skorohod metric (as defined in [2]).

2. MARTINGALE DIFFERENCES

Consider random arrays $\{U_{nt}, Y_{nt} : 1 \le t \le n; n \ge 1\}$ where U_{nt} is a $k \times m$ matrix and Y_{nt} is an $m \times 1$ vector. We can transform these arrays into random elements on $[0,1]$ by defining

$$U_n(s) = U_{n[ns]}, \qquad Y_n(s) = Y_{n[ns]}, \qquad 0 \le s \le 1.$$

Also, define the differences $\epsilon_{nt} = Y_{nt} - Y_{n,t-1}$. We can then define the stochastic integral

$$\int_0^s U_n(r)\, dY_n(r) = \int_0^s U_n\, dY_n = \sum_{i=1}^{[ns]} U_{ni}\epsilon_{ni+1}.$$

To evaluate the limit distribution of $\int_0^s U_n\, dY_n$, we start by assuming that Y_{nt} is a martingale. This allows us to apply a result from Kurtz and Protter [10] to obtain the following theorem.

THEOREM 2.1. *For each* n, *let* $\{U_{nt}, Y_{nt}\}$ *be an* $\{\mathcal{F}_{nt}\}$-*adapted process with sample paths in* $\mathfrak{D}_{M^{km} \times \mathbb{R}^m}[0,1]$, *and let* Y_{nt} *be an* $\{\mathcal{F}_{nt}\}$-*martingale. If* $(U_n, Y_n) \Rightarrow (U, Y)$ *in* $\mathfrak{D}_{M^{km} \times \mathbb{R}^m}[0,1]$, *and* $\sup_n \sum_{i=1}^n E(\epsilon_{ni}^2) < \infty$, *then* Y *is a martingale with respect to a filtration to which* U *and* Y *are adapted, and*

$$\left(U_n, Y_n, \int U_n \, dY_n \right) \Rightarrow \left(U, Y, \int U^- \, dY \right) \text{ in } \mathfrak{D}_{M^{km} \times \mathbb{R}^m \times \mathbb{R}^k}[0,1]. \qquad \blacksquare$$

Theorem 2.1 is a special case of one of the several theorems provided in Kurtz and Protter [10]. Their results encompass a broad range of processes, including semimartingales with weaker moment conditions. For the applications considered in this paper, however, the level of generality provided in Theorem 2.1 is sufficient.

3. MIXING SEQUENCES

Theorem 2.1 gives general conditions for the convergence of stochastic integrals when the process ϵ_{ni} is a square integrable martingale difference. This is not sufficiently general for many applications in econometrics. Phillips [19] used a martingale difference approximation taken from Hall and Heyde [5] to derive analogous results for strictly stationary linear processes. We extend this analysis to cover strong mixing (α-mixing) sequences.

To facilitate the analysis, we assume that the array $\{V_{nt}\}$ is a normalized stochastic partial sum process:

$$V_{nt} = \frac{1}{\sqrt{n}} V_t, \qquad V_t = \sum_{i=1}^t v_i. \tag{1}$$

The stochastic integral of interest is

$$\int_0^s U_n \, dV_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{[ns]} U_{ni} v_{i+1}.$$

We assume that the sequence $\{v_t\}$ satisfies the following assumption.

Assumption 1. For some $p > \beta > 2$, $\{v_i\}$ is a zero mean, strong mixing sequence with mixing coefficients α_m of size $-p\beta/(p - \beta)$ and $\sup_{i \geq 1} \| v_i \|_p = C < \infty$. In addition, $(1/n) E(V_n V_n') \to \Omega < \infty$ as $n \to \infty$.

Set $\mathcal{F}_t = \sigma\{U_{ni}, v_i : i \leq t, n \geq 1\}$ to be the smallest sigma-field containing the past history of $\{U_{nt}, v_t\}$ for all n, and denote $E(X \mid \mathcal{F}_i)$ by $E_i X$. Define

$$\epsilon_i = \sum_{k=0}^\infty (E_i v_{i+k} - E_{i-1} v_{i+k}), \qquad z_i = \sum_{k=1}^\infty E_i v_{i+k}.$$

It is not hard to verify that

$$v_i = \epsilon_i + z_{i-1} - z_i, \qquad E_{i-1}\epsilon_i = 0. \tag{2}$$

This representation is useful because of the equality

$$\int_0^s U_n \, dV_n = \int_0^s U_n \, dY_n + \Lambda_n^*(s) \tag{3}$$

where $Y_n(s) = Y_{n[ns]}$, $Y_{nt} = Y_t/\sqrt{n}$, $Y_t = \sum_1^t \epsilon_i$, and

$$\Lambda_{nt}^* = \frac{1}{\sqrt{n}} \sum_{i=1}^t (U_{ni} - U_{ni-1})z_i' - \frac{1}{\sqrt{n}} U_{nt}z_{t+1}'.$$

Since $\{\epsilon_i, \mathcal{F}_i\}$ is a martingale difference sequence, $\int_0^s U_n \, dY_n$ can be analyzed via Theorem 2.1. This is stated in the following theorem.

THEOREM 3.1. *If Assumption 1 holds and* $(U_n, V_n) \Rightarrow (U, V)$ *in* $\mathfrak{D}_{M^{km} \times \mathbb{R}^m}[0,1]$ *then*

$$\int_0^s U_n \, dY_n \Rightarrow \int_0^s U^- \, dV,$$

with $V(s) = \mathrm{BM}(\Omega)$, *and* Ω *being defined in Assumption 1.*   ∎

Theorem 3.1 gives a convergence result for the martingale approximation $\int_0^s U_n \, dY_n$. To achieve a convergence result for the process $\int_0^s U_n \, dV_n$ itself, it is necessary to examine the process Λ_{nt}^*. We have not been able to establish a general result. Instead, in the next section we illustrate the results for several common applications. We will find the following two results useful in the sequel. Theorem 3.2 gives a mixingale-type bound for the sequence $\{v_i z_i' - E v_i z_i'\}$. Theorem 3.3 shows that L^1-mixingales are asymptotically uncorrelated with arrays which have continuous asymptotic sample paths.

THEOREM 3.2. *If Assumption 1 holds, then*

$$\|E_{i-m}(v_i z_i' - E v_i z_i')\|_{\beta/2} \le C^2 \psi_m$$

where $\psi_m = 12m\alpha_m^{2(1/\beta-1/p)} + 14\sum_{k=m}^\infty \alpha_k^{2(1/\beta-1/p)}$, *and* $\Psi = \sum_{m=1}^\infty \psi_m < \infty$.
    ∎

THEOREM 3.3. *Suppose* $U_n \Rightarrow U$ *in* $\mathfrak{D}_{M^{km}}[0,1]$ *and* $U(\cdot)$ *is almost surely continuous. For a random sequence* $\{e_i\}$ *and a sequence of nondecreasing sigma fields* $\{\mathcal{F}_i^e\}$ *to which* $\{e_i\}$ *is adapted, assume that* $\sup_i E|E(e_i|\mathcal{F}_{i-m}^e)| \to 0$ *as* $m \to \infty$. *Then*

$$\sup_{0 \le s \le 1} \left| \frac{1}{n} \sum_{i=1}^{[ns]} U_{ni} e_i \right| \underset{p}{\to} 0.$$
    ∎

Theorem 3.3 has the following interesting corollary.

COROLLARY. *For a random sequence $\{e_i\}$ and a sequence of non-decreasing sigma fields $\{\mathcal{F}_i^e\}$ to which $\{e_i\}$ is adapted, assume that* $\sup_i E|E(e_i|\mathcal{F}_{i-m}^e)| \to 0$ *as $m \to \infty$. Then*

$$\max_{t \le n} \left| \frac{1}{n} \sum_{i=1}^{t} e_i \right| \underset{p}{\to} 0.$$ ∎

4. APPLICATIONS

In the following applications, V_n is defined as in (1).

4.1. *I*(1) Processes

For our first application, we set $U_{nt} = V_{nt}$. Thus,

$$\int_0^s U_n \, dV_n' = \int_0^s V_n \, dV_n' = \frac{1}{n} \sum_{i=1}^{[ns]} V_i v_{i+1}'.$$

Convergence of matrices of this form is an important component of regression theory under cointegration (see [12] for an application involving $\int_0^1 V_n \, dV_n$, or [8] for an application using the stochastic integral process $\int_0^s V_n \, dV_n'$). The only existing valid result which allows for serial correlation is [19] which uses general linear processes with square integrable i.i.d. innovations. This is quite restrictive, excluding, for example, conditional heteroskedasticity. The following theorem allows for strong mixing processes, and provides the first proof of weak convergence of the stochastic integral process for serially correlated arrays. Define

$$\Lambda = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \sum_{j=i+1}^{\infty} E(v_i v_j').$$

THEOREM 4.1. *If Assumption 1 holds, then*

$$\int_0^s V_n \, dV_n' \Rightarrow \int_0^s B \, dB' + s\Lambda, \qquad as \ n \to \infty,$$

where $B \equiv BM(\Omega)$.

4.2. Products of *I*(1) Processes

For our second application we set $U_{nt} = V_{nt} \otimes V_{nt} = (1/n)V_t \otimes V_t$. Thus,

$$\int_0^s U_n \, dV_n' = \int_0^s (V_n \otimes V_n) \, dV_n' = \frac{1}{n^{3/2}} \sum_{i=1}^{[ns]} (V_t \otimes V_t)v_{t+1}'.$$

Stochastic integrals of this form have arisen in models with nonstationary variances [6,7].

THEOREM 4.2. *If Assumption* 1 *holds with* $\beta = 3$, *then*

$$\int_0^s (V_n \otimes V_n) \, dV_n' \Rightarrow \int_0^s (B \otimes B) \, dB' + \Lambda \otimes \int_0^s B + \int_0^s B \otimes \Lambda. \qquad \blacksquare$$

4.3. *I*(2) Processes

The third application sets $U_{nt} = (1/n)\sum_{i=1}^{t} V_{ni} = n^{-3/2} \bar{V}_t$, where $\bar{V}_t = \sum_{i=1}^{t} V_t$. Thus,

$$\int_0^s U_n \, dV_n' = n^{-2} \sum_{i=1}^{[ns]} \bar{V}_i v_{i+1}'.$$

$\bar{V}_i$ is known as an $I(2)$ process. Stochastic integrals of this form are studied, for example, in [13].

THEOREM 4.3. *If Assumption* 1 *holds, then*

$$\int_0^s U_n \, dV_n' \Rightarrow \int_0^s \bar{B} \, dB', \qquad as \quad n \to \infty, \qquad \blacksquare$$

where $\bar{B}(r) = \int_0^r B$.

4.4. Near-Integrated Processes

The final application is to near-integrated arrays, which have been studied by [3], [17], and [20], and relate to continuous time approximations [14,15]. Define for some matrix G the near-integrated array

$$X_{ni} = \exp(G/n)X_{ni-1} + v_i$$

and the limit diffusion

$$U_G(r) = \int_0^r \exp\{(r - \lambda)G\} \, dB(\lambda).$$

Set $U_{ni} = (1/\sqrt{n})X_{ni}$. Here

$$\int_0^s U_n \, dV_n' = \frac{1}{n} \sum_{i=1}^{[ns]} X_{ni} v_i'.$$

THEOREM 4.4. *If Assumption* 1 *holds, then*

(a) $U_{n[ns]} \Rightarrow U_G(s)$;
(b) $\int_0^s U_n \, dV_n' \Rightarrow \int_0^s U_G \, dB' + s\Lambda.$ $\qquad \blacksquare$

NOTES

1. A process is *cadlag* if it is right continuous with left limits.
2. $U^-(\cdot)$ is the left limit of $U(\cdot)$.
3. Equation (24) in the proof of Lemma 2.5 (b) in [18] is valid, but the subsequent critical step is invalid, for the processes w_t and L_G depend upon the matrix G.

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APPENDIX

Proof of Theorem 2.1. We simply need to verify the conditions of Theorem 4.6 of [10]. Since Y_n is a martingale, it is a semimartingale, and their condition (4.20) holds trivially with $\gamma_n^T(\eta) \equiv 0$. To show that their condition C2.2(i) holds, set $\delta = \infty$ and $\tau_n^a = 2a$. Thus, for all $a > 0$, $P\{\tau_n^a \le a\} = 0 \le 1/a$. In the notation of Kurtz–Protter, $M_n^\delta = Y_n^\delta = Y_n$, $A_n^\delta = 0$, and

$$\sup_n E\big([M_n^\delta]_{s\wedge\tau_n^a} + T_{s\wedge\tau_n^a}(A_n^\delta)\big)$$

$$= \sup_n E[Y_n]_{s\wedge 2a} = \sup_n E \sum_1^{[n(s\wedge 2a)]} \epsilon_{ni}^2 \le \sup_n E \sum_1^n \epsilon_{ni}^2 < \infty.$$

In this expression, $[Y_n]$ denotes the quadratic variation process of Y_n, and $T(A)$ denotes the total variation of A. ∎

Proof of Theorem 3.1. We first establish the moment condition for ϵ_{ni}. By Minkowski's inequality and McLeish's strong mixing inequality [11]

$$\|z_t\|_\beta = \left\| \sum_{k=1}^\infty E_{i-1} v_{i+k} \right\|_\beta \le \sum_{k=1}^\infty \|E_{i-1} v_{i+k}\|_\beta$$

$$\le \sum_{k=1}^\infty 6\alpha_k^{1/\beta - 1/p} \|v_{i+k}\|_p \le 6C \sum_{k=1}^\infty \alpha_k^{1/\beta - 1/p} < \infty, \tag{A.1}$$

uniformly in i. Set $\epsilon_{ni} = \epsilon_i/\sqrt{n}$. It follows that

$$\sum_{i=1}^n E\epsilon_{ni}^2 \le \sup_{i\le n} E\epsilon_i^2 = \big(\sup_{i\le n} \|v_i - z_{i-1} + z_i\|_2\big)^2$$

$$\le \big(\sup_{i\le n} \|v_i\|_\beta + 2 \sup_{i\le n} \|z_i\|_\beta\big)^2 < \infty. \tag{A.2}$$

Note that $V_n(s) \Rightarrow V(s) = \mathrm{BM}(\Omega)$ as shown by Wooldridge and White [23, Corollary 4.2]. It remains to show that $(U_n, Y_n) \Rightarrow (U, V)$. Since

$$(U_n, Y_n) = (U_n, V_n) + (0, (Y_n - V_n)),$$

the result follows from the continuous mapping theorem since $(U_n, V_n) \Rightarrow (U, V)$ and $(Y_n - V_n) \Rightarrow 0$. We now show the latter. For all $\eta > 0$,

$$P\big\{\sup_{i\le n} |z_i| > \sqrt{n}\eta\big\} \le \sum_{i=1}^n P\{|z_i| > \sqrt{n}\eta\}$$

$$\le \sum_{i=1}^n \frac{1}{\eta^\beta n^{\beta/2}} E|z_i|^\beta \le \left(\eta^{-1} 6C \sum_1^\infty \alpha_k^{1/\beta - 1/p}\right)^\beta n^{1-\beta/2} \to 0$$

since $\beta > 2$. Therefore

$$\sup_{t\le n} |Y_{nt} - V_{nt}| \le 2 \frac{1}{\sqrt{n}} \sup_{t\le n} |z_t| \xrightarrow{p} 0. \tag{A.3}$$

This establishes that $(Y_n - V_n) \Rightarrow 0$ in the uniform metric, which implies convergence in the Skorohod metric as well (since the zero function is continuous). The conditions of Theorem 2.1 are satisfied and we conclude that

$$\int_0^s U_n \, dY_n' \Rightarrow \int_0^s U^- \, dV'. \qquad \blacksquare$$

Proof of Theorem 3.2. By Minkowski's inequality

$$\|E_{i-m}[v_i z_i' - Ev_i z_i']\|_{\beta/2} = \left\| \sum_{k=1}^{\infty} (E_{i-m} v_i v_{i+k}' - Ev_i v_{i+k}') \right\|_{\beta/2}$$

$$\leq \sum_{k=1}^{m} \|E_{i-m} v_i v_{i+k}' - Ev_i v_{i+k}'\|_{\beta/2}$$

$$+ \sum_{k=m}^{\infty} (\|E_{i-m} v_i v_{i+k}'\|_{p/2} + \|Ev_i v_{i+k}'\|_{\beta/2}). \qquad \text{(A.4)}$$

By McLeish's strong mixing inequality [11], Minkowski's inequality, the Rao-Blackwell theorem, and Holder's inequality

$$\sum_{k=1}^{m} \|E_{i-m} v_i v_{i+k}' - Ev_i v_{i+k}'\|_{\beta/2}$$

$$\leq 6m\alpha_m^{2/\beta-2/p} \|v_i v_{i+k}' - Ev_i v_{i+k}'\|_{p/2}$$

$$\leq 6m\alpha_m^{2/\beta-2/p} (\|v_i v_{i+k}'\|_{p/2} + \|Ev_i v_{i+k}'\|_{p/2})$$

$$\leq 12m\alpha_m^{2/\beta-2/p} \|v_i\|_p \|v_{i+k}\|_p \leq 12C^2 m\alpha_m^{2/\beta-2/p}. \qquad \text{(A.5)}$$

In addition,

$$\|E_{i-m} v_i v_{i+k}'\|_{\beta/2} \leq \|v_i E_i v_{i+k}'\|_{\beta/2}$$

$$\leq \|v_i\|_p \|E_i v_{i+k}\|_{\beta p/(2p-\beta)} \leq 6C^2 \alpha_k^{2/\beta-2/p}. \qquad \text{(A.6)}$$

Finally, by an α-mixing inequality [5, Corollary A.2],

$$\|Ev_i v_{i+k}'\|_{\beta/2} \leq 8\alpha_k^{2/\beta-2/p} \|v_i v_{i+k}'\|_{p/2} \leq 8C^2 \alpha_k^{2/\beta-2/p}. \qquad \text{(A.7)}$$

(A.4), (A.5), (A.6), and (A.7) together establish the main result. Finally,

$$\sum_{m=1}^{\infty} \psi_m = \sum_{m=1}^{\infty} \left(12m\alpha_m^{2/\beta-2/p} + 14 \sum_{k=m}^{\infty} \alpha_k^{2/\beta-2/p} \right)$$

$$= 26 \sum_{m=1}^{\infty} m\alpha_m^{2(1/\beta-1/p)} < \infty,$$

since $\alpha_m^{1/\beta-1/p}$ is of size -1 by Assumption 1. $\qquad \blacksquare$

Proof of Theorem 3.3. For some $\delta \in (0,1)$, set $N = [1/\delta]$, $t_k = [kn/N] + 1$, and $t_k^* = t_{k+1} - 1$, $N_s^* = [(N-1)s]$, and $t_{sk}^* = \min(t_k^*, [ns])$. Then

$$\sup_{0 \le s \le 1} \left| \frac{1}{n} \sum_{i=1}^{[ns]} U_{ni} e_i \right| = \sup_{0 \le s \le 1} \left| \frac{1}{n} \sum_{k=0}^{N_s^*} \sum_{t=t_k}^{t_{sk}^*} U_{nt} e_t \right|$$

$$\le \sup_{0 \le s \le 1} \left| \frac{1}{n} \sum_{k=0}^{N_s^*} U_{nt_k} \sum_{t=t_k}^{t_{sk}^*} e_t \right| + \sup_{0 \le s \le 1} \left| \frac{1}{n} \sum_{k=0}^{N_s^*} \sum_{t=t_k}^{t_{sk}^*} (U_{nt} - U_{t_k}) e_t \right|$$

$$\le \frac{1}{n} \sum_{k=0}^{N-1} |U_{nt_k}| \left| \sum_{t=t_k}^{t_k^*} e_t \right| + \frac{1}{n} \sum_{k=0}^{N-1} \sum_{t=t_k}^{t_k^*} |U_{nt} - U_{t_k}| |e_t|$$

$$\le \sup_{0 \le s \le 1} |U_n(s)| \frac{1}{n} \sum_{k=0}^{N-1} \left| \sum_{t=t_k}^{t_k^*} e_t \right|$$

$$+ \sup_{|r-s| \le \delta} |U_n(r) - U_n(s)| \frac{1}{n} \sum_{i=1}^{n} |e_i|.$$

Since U_n converges in the Skorohod topology, $\sup_s |U_n(s)| = O_p(1)$, while $(1/n) \times \sum_1^n |e_i| = O_p(1)$ since e_i is uniformly integrable. Furthermore,

$$E \frac{1}{n} \sum_{k=0}^{N-1} \left| \sum_{t=t_k}^{t_k^*} e_t \right| \le \frac{N}{n} \sup_{0 \le k \le N-1} E \left| \sum_{t=t_k}^{t_k^*} e_t \right| \le \sup_{t \le n} E \left| \frac{1}{\delta n} \sum_{i=t}^{t+\delta n} e_i \right| \to 0$$

as $n \to \infty$, by Andrews [1, Theorem 1]. The proof is completed by noting that

$$\sup_{|r-s| \le \delta} |U_n(r) - U_n(s)| \xrightarrow[d]{} \sup_{|r-s| \le \delta} |U(r) - U(s)| \qquad \text{as} \quad n \to \infty,$$

$$\xrightarrow[p]{} 0 \qquad \qquad \text{as} \quad \delta \to 0,$$

since $U(\cdot)$ is continuous. ∎

Proof of Theorem 4.1. As discussed in the proof of Theorem 3.1,

$$U_n(s) = V_n(s) = \frac{1}{\sqrt{n}} \sum_1^{[ns]} v_i \Rightarrow B(s) \equiv \text{BM}(\Omega).$$

The martingale difference approximation (2) allows decomposition (3), which in this case is

$$\int_0^s V_n \, dV_n' = \int_0^s V_n \, dY_n' + \Lambda_{n[ns]}^*$$

with

$$\Lambda_{nt}^* = \frac{1}{n} \sum_{i=1}^{t} v_i z_i' - \frac{1}{n} V_t z_{t+1}'.$$

By Theorem 3.1, $\int_0^s V_n \, dY_n' \Rightarrow \int_0^s B \, dB'$. It remains to show that $\Lambda_{n[ns]}^* \Rightarrow s\Lambda$. Since the limit process is nonrandom, this is equivalent to uniform convergence in probability. It is therefore sufficient to consider the case where Λ_{nt}^* is scalar.

First, observe that

$$\sup_{t \le n} \frac{1}{n} |V_t z_{t+1}| \le \sup_{t \le n} \left| \frac{1}{\sqrt{n}} V_t \right| \frac{1}{\sqrt{n}} \sup_{t \le n} |z_t| \xrightarrow[p]{} 0$$

since $\sup_{t \le n} |V_t| = O_p(\sqrt{n})$ and $(1/\sqrt{n}) \sup_{t \le n} |z_t| \xrightarrow[p]{} 0$ as in (A3).

Second, by Theorem 3.2, the sequence $\{v_i z_i' - E v_i z_i'\}$ is an $L_{\beta/2}$-mixingale. Since $\beta > 2$, the sequence is therefore a uniformly integrable L_1-mixingale. Applying the Corollary to Theorem 3.3, we find

$$\sup_{t \leq n} \left| \frac{1}{n} \sum_{i=1}^{t} (v_i z_i - E v_i z_i) \right| \xrightarrow{p} 0.$$

Finally, observe that

$$E \frac{1}{n} \sum_{i=1}^{[ns]} v_i z_i' = \frac{1}{n} \sum_{i=1}^{[ns]} E \left(v_i \sum_{k=1}^{\infty} E_i v_{i+k}' \right) = \frac{1}{n} \sum_{i=1}^{[ns]} \sum_{k=1}^{\infty} E(v_i v_{i+k}') \xrightarrow[n \to \infty]{} s \Lambda. \qquad ■$$

Proof of Theorem 4.2. By the continuous mapping theorem, $U_n = V_n \otimes V_n \Rightarrow V \otimes V$. Employing the martingale difference approximation (2)-(3) and Theorem 3.1, we obtain

$$\int_0^s (V_n \otimes V_n) \, dY_n' \Rightarrow \int_0^s (V \otimes V) \, dV'.$$

To analyze the bias term Λ_{nt}^*, note that

$$\Lambda_{nt}^* = \frac{1}{n^{3/2}} \sum_{i=1}^{t} ((V_i \otimes V_i) - (V_{i-1} \otimes V_{i-1})) z_i' + o_p(1)$$

$$= \frac{1}{n^{3/2}} \sum_{i=1}^{t} ((V_{i-1} \otimes v_i) + (v_i \otimes V_{i-1}) + (v_i \otimes v_i)) z_i' + o_p(1)$$

$$= \frac{1}{n^{3/2}} \sum_{i=1}^{t} V_{i-1} \otimes (v_i z_i') + \frac{1}{n^{3/2}} \sum_{i=1}^{t} (v_i z_i') \otimes V_{i-1}$$

$$+ \frac{1}{n^{3/2}} \sum_{i=1}^{t} (v_i \otimes v_i) z_i' + o_p(1). \qquad \text{(A.8)}$$

First,

$$E \sup_{t \leq n} \left| \frac{1}{n^{3/2}} \sum_{i=1}^{t} (v_i \otimes v_i) z_i' \right| \leq \frac{1}{n^{3/2}} \sum_{i=1}^{n} \|v_i\|_3^2 \|z_i\|_3 \to 0 \qquad \text{(A.9)}$$

as $n \to \infty$ by (A.1). Second, set $e_i = e_i = v_i z_i' - \Lambda$, which by Theorem 3.2 satisfies the conditions of Theorem 3.3, yielding

$$\sup_{0 \leq s \leq 1} \left| \frac{1}{n} \sum_{i=1}^{[ns]} V_{ni-1} \otimes e_i \right| \xrightarrow{p} 0$$

and therefore

$$\frac{1}{n^{3/2}} \sum_{i=1}^{[ns]} V_{i-1} \otimes (v_i z_i') = \frac{1}{n^{3/2}} \sum_{i=1}^{[ns]} V_{i-1} \otimes \Lambda + \frac{1}{n} \sum_{i=1}^{[ns]} V_{ni-1} \otimes e_i \Rightarrow \int_0^s B \otimes \Lambda$$

$$\text{(A.10)}$$

by the continuous mapping theorem. Similarly,

$$\frac{1}{n^{3/2}} \sum_{i=1}^{[ns]} (v_i z_i') \otimes V_{i-1} \Rightarrow \Lambda \otimes \int_0^s B. \qquad \text{(A.11)}$$

(A.8)-(A.11) combine to yield the result. $\qquad ■$

Proof of Theorem 4.3. Since $V_n \Rightarrow V$, it follows that $U_n \Rightarrow \bar{V}$ by the continuous mapping theorem. Therefore by Theorem 3.1,

$$\int_0^s U_n \, dY_n' \Rightarrow \int_0^s \bar{V} \, dV'.$$

Additionally, $\Lambda_{n[ns]}^* \Rightarrow 0$ since

$$\left\| \sup_{0 \le s \le 1} \Lambda_n^*(s) \right\|_1 = \left\| \sup_{0 \le s \le 1} \left| n^{-2} \sum_{i=1}^{[ns]} V_i z_{i+1}' \right| \right\|_1 + o_p(1)$$

$$\le n^{-2} \sum_{i=1}^n \| V_i z_{i+1}' \|_1 + o_p(1) \le n^{-2} \sum_{i=1}^n \| V_i \|_2 \| z_{i+1}' \|_2 + o_p(1)$$

$$\to 0 \qquad \text{as } n \to \infty. \qquad\qquad \blacksquare$$

Proof of Theorem 4.4

(a) $U_{n[ns]} = \dfrac{1}{\sqrt{n}} \sum_{k=0}^{[ns]} \exp\left(-\left(\dfrac{[ns] - k}{n} \right) G \right) v_k' + o_p(1)$

$$= \exp\left(-\left(\dfrac{[ns]}{n} \right) G \right) \dfrac{1}{\sqrt{n}} \sum_{k=0}^{[ns]} \exp\left(\left(\dfrac{k}{n} \right) G \right) (\epsilon_k - z_{k-1} + z_k) + o_p(1).$$

Note that

$$\exp\left(-\left(\dfrac{[ns]}{n} \right) G \right) \dfrac{1}{\sqrt{n}} \sum_{k=0}^{[ns]} \exp\left(\left(\dfrac{k}{n} \right) G \right) \epsilon_k$$

$$\Rightarrow \exp(-sG) \int_0^s \exp(\lambda G) \, dB(\lambda)' = \int_0^s \exp(-(s - \lambda)G) \, dB(\lambda)' = U_G(s)$$

by Theorem 3.1. In additon, standard manipulations can show that

$$\dfrac{1}{\sqrt{n}} \sum_{k=0}^{[ns]} \exp\left(\left(\dfrac{k}{n} \right) G \right) (z_k - z_{k-1})' \Rightarrow 0$$

establishing the result.

(b) By Theorem 3.1, $\int_0^s U_n \, dY_n' \Rightarrow \int_0^s U_G \, dV$. In addition,

$$\Lambda_n^*(s) = \dfrac{1}{n} \sum_{i=1}^{[ns]} \Delta X_{ni} z_i' + o_p(1)$$

$$= \dfrac{1}{n} \sum_{i=1}^{[ns]} v_i z_i' + \left[\exp\left(-\dfrac{G}{n} \right) - I \right] \dfrac{1}{n} \sum_{i=1}^{[ns]} X_{ni-1} z_i' + o_p(1) \Rightarrow s\Lambda$$

as shown in the proof of Theorem 4.1. (Note that the $o_p(1)$ terms hold uniformly in $s \in [0,1]$.) $\qquad \blacksquare$