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GARCH(1, 1) processes are near epoch dependent

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This paper obtains conditions under which GARCH(1, 1) processes satisfy near epoch dependence without imposing strict stationarity. This substantially generalizes the conditions under which weak and strong laws of large numbers, central limit theorems and invariance principles hold for GARCH processes.

1. Introduction

Since the introduction of ARCH and GARCH by Engle (1982) and Bollerslev (1986), many applied economists have used ARCH to model conditional variances. See Bollerslev et al (1990) for a review. Despite the explosion of empirical applications, theoretical knowledge of the behavior of ARCH and GARCH processes is quite limited. All existing asymptotic proofs rely heavily upon the assumption that the driving innovations are independent and identically distributed (iid) so that the resulting process is strictly stationary. Asymptotic results for heterogeneous (non-stationary) non-linear processes typically require mixing or near epoch dependent (NED) conditions. In fact, many applications appeal to theorems which have been demonstrated only under a weak dependence condition such as mixing. It is not known, however, under what conditions GARCH processes are mixing. A proof has never been attempted, and it is not even intuitively obvious that the result need be true. It is known, for example, that simple autoregressive processes with iid innovations need not be mixing; see, for example, the counter-example given by Andrews (1984). Proofs that linear processes are mixing are tedious and difficult. See, for example, Gorodetski (1977) and Withers (1981).

Many theorems which can be demonstrated for mixing processes can also be shown under the assumption of near epoch dependence. Also called 'functions of mixing processes', near epoch dependence was introduced by Ibragimov (1962), and used by Billingsley (1968), McLeish (1975), Gallant and White (1988), Wooldridge and White (1988), and Andrews (1988), among others. Near epoch dependence is a simpler condition than mixing to verify, although somewhat harder to use in applications.

Section 2 shows that certain GARCH(1, 1) processes are near epoch dependent, allowing for a moderate degree of heterogeneity, and not requiring strict stationarity. Section 3 applies this result to demonstrate weak and strong laws of large numbers, a central limit theorem, and an invariance principle for GARCH processes. The appendix contains the proof of the theorem.

2. Main theorem

Consider some sequence of martingale differences $\{x_t\}$. Define the natural filtration $\mathscr{G}_t = \sigma(\ldots, x_{t-1}, x_t)$ and the conditional variance $\sigma_t^2 = \mathrm{E}(x_t^2 \mid \mathscr{G}_{t-1})$. Assume that the conditional variance follows a GARCH process:

Assumption 1. $\sigma_t^2 < \infty$ a.s. and $\sigma_t^2 = \omega + \beta\sigma_{t-1}^2 + \alpha x_{t-1}^2$.

Define the renormalized variable $e_t = x_t/\sigma_t$, and the following filtrations on e_t: $\mathscr{F}_i^j = \sigma(e_i, \ldots, e_j)$, and $\mathscr{F}_i = \sigma(\ldots, e_i)$. By construction, $\{e_t, \mathscr{F}_t\}$ is a martingale difference sequence with $\mathrm{E}(e_t^2 \mid \mathscr{F}_{t-1}) = 1$ (a.s.).

It is conventional in the theoretical ARCH literature to assume that the variable $\{e_t\}$ is iid, and thus σ_t^2 is strictly stationary. We relax this unusually strict assumption by allowing $\{e_t\}$ to be strong mixing. The α-mixing coefficients for $\{e_t\}$ are:

$$\alpha_m = \sup_j \sup_{\{F \in \mathscr{F}_{-\infty}^j, G \in \mathscr{F}_{j+m}^\infty\}} |P(G \cap F) - P(G)P(F)|.$$

Rewriting σ_t^2 as a function of $\{e_t\}$ by repeated back-substitution, we find

$$\sigma_t^2 = \omega + \omega \sum_{k=1}^{\infty} \prod_{i=1}^{k} (\beta + \alpha e_{t-i}^2) = \omega + \omega \sum_{k=1}^{\infty} Z_t(k), \tag{1}$$

where $Z_t(k) = \prod_{i=1}^k (\beta + \alpha e_{t-i}^2) = \prod_{i=1}^k u_{t-i}$, and $u_t = (\beta + \alpha e_t^2)$. We see that σ_t^2, and hence x_t, depends upon the infinite history of the innovations $\{e_t\}$. This is true for ARCH(1) processes ($\beta = 0$) as well as for general GARCH processes. It seems reasonable, however, that if we consider the m-dependent variable $\sigma_{mt}^2 = \omega + \omega\sum_{k=1}^m Z_t(k)$ for sufficiently large m, that $|\sigma_{mt}^2 - \sigma_t^2|$ can be made arbitrarily small. This is essentially the idea behind near epoch dependence. The following definition was introduced by Andrews (1988).

Definition 1. $\{Y_t\}$ is L^r-near epoch dependence (L^r-NED) with respect to $\{e_t\}$ if $\{e_t\}$ is α-mixing and there exist non-negative constants $\{d_t : t \geq 1\}$ and $\{v_m : m \geq 0\}$ such that $v_m \downarrow 0$ as $m \uparrow \infty$ and $\| Y_t - \mathrm{E}(Y_t \mid \mathscr{F}_{t-m}^{t+m}) \|_r \leq d_t v_m$.

We will require the following conditional moment bound

$$\left(\mathrm{E}\left[\left(\beta + \alpha e_t^2\right)^r \mid \mathscr{F}_{t-1}\right]\right)^{1/5} \leq c < 1 \text{ a.s. for all } t, \tag{2}$$

which reduces to $\sup_{t \geq 1} \| \beta + \alpha e_t^2 \|_r < 1$ when $\{e_t\}$ is iid.

Theorem 1. If (2) *holds then* $\{\sigma_t^2, x_t\}$ *is* $L^r - NED$ *with respect to* $\{e_t\}$. *In addition,* $d_t = 2\omega c/(1-c)$ *for all t and* $v_m = c^m$.

Note that the decay rate given for the NED numbers is exponential.

3. Applications

This section illustrates the usefulness of the Theorem by proving a weak law of large numbers (WLLN), strong law of large numbers (SLLN), central limit theorem (CLT) and invariance principle (IP) for GARCH(1, 1) processes. Set $S_n = \Sigma_{t=1}^{n} x_t$.

Our first result applies Theorem 1 of Andrews (1988) and the fact that L^q-bounded ($q > 1$) L^1-NED sequences are L^1-mixingales. Note that setting $r = 1$, (2) holds if $\beta + \alpha < 1$.

$$\limsup n^{-1} \sum_{t=1}^{n} \| x_t \|_q < \infty. \tag{3}$$

Corollary 1 (WLLN). If $\beta + \alpha < 1$, $\alpha_m \downarrow 0$, *and* (3) *holds for some* $q > 1$, *then* $n^{-1} S_n \to_p 0$.

Our next result applies Theorem 2 of Hansen (1990).

Corollary 2 (SLLN). If (2) *holds for some* $r > 1$, (3) *holds for some* $q > r$, *and* $\Sigma_{m=1}^{\infty} \alpha_m^{1/2} < \infty$, *then* $n^{-1} S_n \to_{\text{a.s.}} 0$.

How restrictive is (2) with $r > 1$? Note that

$$\left(\mathrm{E}\left[\left(\beta + \alpha e_t^2\right)^5 \mid \mathscr{F}_{t-1}\right]\right)^{1/r} \leq \beta + \alpha\left(\mathrm{E}\left(e_t^{2r} \mid \mathscr{F}_{t-1}\right)\right)^{1/5}. \tag{4}$$

If $\{e_t\}$ is iid, (4) equals $\beta + \alpha \| e_t \|_{2r}^2$. Since the L^r-norm is continuous in r, for any $\delta > 0$ we can find some r close to but greater than unity such that

$$\| e_t \|_{2r}^2 < \| e_t \|_1^2 + \delta = 1 + \delta,$$

which implies that $\{e_t\}$ iid and $\beta + \alpha < 1$ is sufficient for (2). If $\{e_t\}$ is not iid, then by (4), a sufficient condition for (2) is the existence of some $r > 1$ such that

$$\left(\mathrm{E}\left(e_t^{2r} \mid \mathscr{F}_{t-1}\right)\right)^{1/5} < 1 + \delta \text{ a.s. for all } t,$$

and $\beta + \alpha(1 + \delta) < 1$.

We now turn to the central limit theorem and invariance principle. The following two results are direct consequences of Corollary 3.1 of Wooldridge and White (1988). Here we need (2) to hold for $r = 2$. This is equivalent to

$$\beta^2 + 2\alpha\beta + \alpha^2 \kappa_t < 1 \text{ a.s. for all } t, \tag{5}$$

where $\kappa_t = \mathrm{E}(e_t^4 \mid \mathscr{F}_{t-1})$ is the conditional kurtosis. If $\{e_t\}$ is iid, then κ_t is simply the unconditional kurtosis. In many applications, the distribution of e_t is assumed to be N(0, 1), in which case $\kappa = 3$.

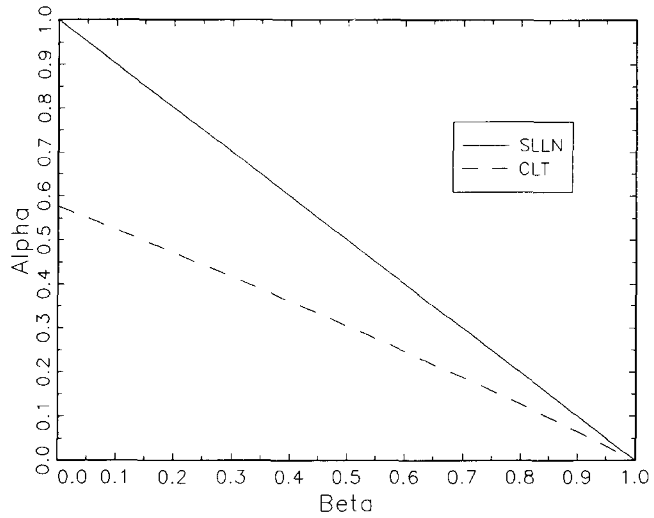

Fig. 1. Trade-offs under normality.

The region for (α, β) which satisfies (5) in this case is plotted in fig. 1.

$$n^{-1} \sum_{i=1}^{n} \mathrm{E}x_t^2 \rightarrow \sigma^2 > 0, \tag{6}$$

Corollary 3. If (5) *and* (6) *hold, for some* $q > 2$, $\sup_{t \geq 1} \| x_t \|_q < \infty$, *and* $\{\alpha_m\}$ *are of size* $-q/(q-2)$, *then* $n^{-1/2} S_n \rightarrow_d \mathrm{N}(0, \sigma^2)$.

Now define the process

$$W_n(s) = \frac{1}{\sqrt{n}\,\sigma} \sum_{i=1}^{[ns]} x_i,$$

where $[\cdot]$ denotes 'integer part'. Let '$\Rightarrow$' denote weak convergence in the sense of Billingsley (1968) and W(s) denote a continuous time process distributed as a standard Brownian motion.

Corollary 4. Under the conditions of Corollary 3, $W_n(s) \Rightarrow W(s)$.

Corollaries 1 through 4 show that the assumption of conditional heteroskedasticity of the GARCH form need not preclude the application of standard asymptotic theory, if appropriate regularity conditions are satisfied. The conditions, however, are quite strict, and may preclude estimated GARCH processes, since in applications the estimates frequently fall near the region of integration ($\alpha + \beta = 1$). This poses questions which cannot be addressed in the current study, but will remain of interest to empirical users of GARCH models.

Appendix

Proof of Theorem 1. We first prove that $\{\sigma_t^2\}$ is NED. Note that $Z_t(k)$ is measurable with respect to $\mathscr{F}_{t-m}^{t+m}$ for $k \le m$. For $k > m$, $Z_t(k) = Z_t(m)(\prod_{i=m+1}^{k} u_i)$. Thus by Minkowski's inequality

$$\left\| \sigma_t^2 - \mathrm{E}(\sigma_t^2 \mid \mathscr{F}_{t-m}^{t+m}) \right\|_r = \omega \left\| \sum_{k=1}^{\infty} \left(Z_t(k) - \mathrm{E}[Z_t(k) \mid \mathscr{F}_{t-m}^{t+m}] \right) \right\|_r,$$

$$\le \omega \sum_{k=1}^{\infty} \left\| Z_t(k) - \mathrm{E}[Z_t(k) \mid \mathscr{F}_{t-m}^{t+m}] \right\|_r,$$

$$= \omega \sum_{k=m+1}^{\infty} \left\| Z_t(m) \prod_{i=m+1}^{k} u_i - Z_t(m) \mathrm{E}\left[\prod_{i=m+1}^{k} u_i \mid \mathscr{F}_{t-m}^{t+m} \right] \right\|_r,$$

$$= \omega \sum_{k=m+1}^{\infty} \left(\mathrm{E}\left(Z_t(m)^r R_{mk}^r \right) \right)^{1/r}, \tag{A.1}$$

where

$$R_{mk} = \prod_{i=m+1}^{k} u_i - \mathrm{E}\left[\prod_{i=m+1}^{k} u_i \mid \mathscr{F}_{t-m}^{t+m} \right].$$

By repeated conditional expectations and (2), we find that

$$\mathrm{E}\left(Z_t(m)^r R_{mk}^r \right) = \mathrm{E}\left(\prod_{i=1}^{m} u_{t-i}^r R_{mk}^r \right) = \mathrm{E}\left(\mathrm{E}\left(u_{t-1}^r \mid \mathscr{F}_{-\infty}^{t-2} \right) \prod_{i=2}^{m} u_{t-i}^r R_{mk}^r \right),$$

$$\le c^r \mathrm{E}\left(\prod_{i=2}^{m} u_{t-i}^r R_{mk}^r \right) = c^r \mathrm{E}\left(\mathrm{E}\left(u_{t-2}^r \mid \mathscr{F}_{-\infty}^{t-3} \right) \prod_{i=3}^{m} u_{t-i}^r R_{mk}^r \right),$$

$$\le c^{2r} \mathrm{E}\left(\prod_{i=3}^{m} u_{t-i}^r R_{mk}^r \right) \le c^{rm} \mathrm{E}\left(R_{mk}^r \right). \tag{A.2}$$

By Minkowski's inequality and Blackwell's theorem

$$\| R_{mk} \|_r \le \left\| \prod_{i=m+1}^{k} u_i \right\|_r + \left\| \mathrm{E}\left[\prod_{i=m+1}^{k} u_i \mid \mathscr{F}_{t-m}^{t+m} \right] \right\|_r,$$

$$\le 2 \left\| \prod_{i=m+1}^{k} u_i \right\|_r = 2 \left(\mathrm{E} \prod_{i=m+1}^{k} u_i^r \right)^{1/r} \le 2 c^{(k-m-1)}, \tag{A.3}$$

where the final inequality uses the same argument as in (A.2). Putting (A.1), (A.2) and (A.3) together we find

$$\left\| \sigma_t^2 - \mathrm{E}\left(\sigma_t^2 \mid \mathscr{F}_{t-m}^{t+m}\right) \right\|_r \leq \omega \sum_{k=m+1}^{\infty} \left(c^{rm} \mathrm{E}\left(R_{mk}^r\right)\right)^{1/r},$$

$$\leq 2\omega \sum_{k=m+1}^{\infty} c^m c^{(k-m-1)} = d v_m,$$

where $d = 2\omega c/(1-c)$ and $v_m = c^m$. Thus $\{\sigma_t^2\}$ is L^r-NED with respect to $\{e_t\}$.

Noting that $g(x) = x^{1/2}$ satisfies the Lipschitz condition, $\sigma_t = (\sigma_t^2)^{1/2}$ is L^{2r}-NED with respect to $\{e_t\}$ by Theorem 4.2 of Gallant and White (1988). Finally, $x_t = \sigma_t e_t$ is L^r-NED with respect to $\{e_t\}$ by Corollary 4.3.b of Gallant and White (1988). □

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