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Is assortative matching efficient?★

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Summary. This paper develops some general conditions under which complementarities between individual agents imply that assortative matching is efficient. Our analysis has four main findings. First, when agents are organized into equal-sized groups, just as in Becker (1973), the presence of within-group complementarities is sufficient for stratification to be efficient. Second, if group sizes vary, assortative matching may not be efficient even though complementarities are present, unless particular functional form assumptions are imposed. Third, the connection between assortative matching, complementarities and efficiency reemerges if one considers sequences of replications of the economy in which individual coalitions are uniformly bounded in size. Fourth, the presence of feedbacks from the composition of group memberships has important effects on efficient allocations and breaks any simple link between assortative matching and efficiency. Together, these results suggest that the characterization of the cross-section evolution of an efficiently sorted economy is likely to be highly complex.

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1 Introduction

Recent work in economic theory, ranging from models of economic development to income inequality has begun to focus on the question of stratification, defined as the

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tendency of agents with similar characteristics to interact with one another in isolation of others. Two natural examples within a given economy are the assignment of workers to firms (Kremer, 1993; Kremer and Maskin, 1996) and the assignment of families to neighborhoods (Bénabou, 1993, 1996a,b; Durlauf, 1996a,b). When the productivity of each worker in an organization depends positively on the productivity of coworkers, incentives will exist for relatively high skilled workers to form firms that exclude their lower skilled counterparts. Similarly, when the quality and or quantity of a public good, such as education, within a neighborhood is an increasing function of the income distribution of the neighborhood due to tax base, role model influences or other effects, incentives exist for wealthier or better educated families to isolate themselves from poorer or less educated ones. Underlying these different environments is the common assumption that interactions between agents exhibit positive spillover effects, usually in the form of complementarities. In turn, each of these models can be thought of as a solution to a sorting problem, whose details are embedded in the economic environment of interest.

While the conditions under which stratification will emerge as an equilibrium allocation have been studied in a number of contexts, there has been less attention given to the efficiency of such allocations. Any relationship between equilibrium and efficiency in these models is not self-evident, of course, due to the many spillover effects and associated incomplete markets that typically are built into the economies under study. Neighborhood models, for example, typically exhibit role model and peer group effects between students attending a common school that are not adjudicated through market mechanisms. The major exception to this lack of attention is Becker's seminal work (1973) on the marriage problem. Becker considers the allocation of men and women into marriages in which the "productivity" of each marriage is assumed to depend on the ability levels of each of the partners. Becker then provides conditions under which the efficient allocation of partners results in "assortative matching," i.e. the most able male is matched with the most able female, etc. Specifically, the efficiency of assortative matching is shown to depend on the presence of positive cross-partial derivatives between the abilities of the partners in the output of a marriage. The generalization of this assumption - strategic complementarities between individuals - is typically assumed in describing interactions between agents in the more recent literature on interactions and stratification; as a result, Becker's proof is often used as evidence that in the presence of strategic complementarities, efficiency will induce stratification, suggesting that the stratification found in equilibrium models will at least qualitatively generalize when markets are complete.

The purpose of this paper is to determine whether the presence of complementarities between individuals is sufficient to determine whether stratification of agents by an attribute such as ability is efficient. We do this by employing a particular formalization of the notion of complementarity, increasing differences, which allows one to work with fairly general payoff structures. Basic properties of functions exhibiting increasing differences lead to four general conclusions. First, when agents are organized into equal-sized groups, increasing differences is a sufficient condition for stratification. This shows how the assortative matching results for the marriage problem may be generalized. Second, when group sizes vary, stratification may not be efficient even in the presence of increasing differences. Third, we show that the connection between stratification and efficiency will reemerge when one considers t-fold replications of the economy. Fourth, when there are incentive effects associated with the allocation of agents across coalitions, then stratification by ability may be inefficient even in a large economy limit.

These results in turn have two implications for the existing literatures on inequality. First, from the theory side, these results clarify the importance of market imperfections or specific functional forms, as opposed to complementarities *per se*, in generating stratified equilibria. In this sense, stratification is not a primitive feature of complementarities-driven economies. From the empirical side, these results indicate that stratification is not, as asserted by Herrnstein and Murray (1994), a logical consequence of the breakdown of barriers to mobility in society, and therefore cannot be treated as a self-evident explanation for persistent income inequality. Further, by clarifying the relationship between individual characteristics and group assignments, the analysis is important for determining the identifiability of group spillover effects in different economic and social contexts, as is clear from the work of Manski (1993) and Brock and Durlauf (2000, 2001).

Section 2 of the paper specifies the basic model under study. Section 3 develops the relationship between stratification and efficiency for economies exhibiting strategic complementarities. Equal and variable coalition-sizes are considered. Section 4 analyzes economies in which some additional restrictions are placed either on the coalition-specific production functions or on the cross-section distribution of abilities. Section 5 introduces incentive effects into the analysis of efficient allocations of agents. Section 6 discusses some examples that illustrate the general claims of the paper. Section 7 provides a summary and conclusions.

2 Model specification

Consider an economy consisting of N agents denoted by n. Each agent is associated with a nonnegative ability level a_n, which may be interpreted as any scalar attribute that relates to productivity. The collection of the ability levels for the agents, $\{a_1, \ldots, a_N\}$, is denoted by $\mathcal{A}$.

Agents are organized into coalitions. Coalition k, of size I can be represented as the vector $\underset{\sim}{a}_k = a_{n_1}, \ldots, a_{n_I}$ when it is comprised of agents n_1 to n_I. There exists a sequence of production functions indexed by the number of agents that interact within a coalition. Total output of the coalition equals

$$\Phi_I \left(\underset{\sim}{a}_k \right) . \tag{1}$$

Abilities are measured so that $\Phi_I(\underset{\sim}{0}) = 0 \forall I$. Any permutation of the ability levels within a coalition is assumed to leave output unchanged. Throughout, we will be interested in allocations of individuals across coalitions that maximize aggregate output.

We will employ two definitions in the subsequent discussion. The first definition formalizes the notion of strategic complementarities for functions that are not required to be continuous or differentiable, thereby generalizing the positive cross-partial derivative assumption of Becker to arbitrary coalition-specific payoff functions.

Definition 1. *Increasing differences*
An I-size coalition-specific production function exhibits increasing differences if for any pair of J-length ability vectors $\underset{\sim}{b}$ and $\underset{\sim}{b}'$ and $I-J$ length vector $\underset{\sim}{c}$,

$$\Phi_I\left(\underset{\sim}{b},\underset{\sim}{c}\right)-\Phi_I\left(\underset{\sim}{b}',\underset{\sim}{c}\right) \text{ is strictly increasing in } \underset{\sim}{c} \text{ if } \underset{\sim}{b} > \underset{\sim}{b}'.$$

When $\Phi_I(\cdot)$is twice-differentiable, strict increasing differences is equivalent to the condition that all cross-partial derivatives of this function are positive, and thus corresponds exactly to the notion of strategic complementarities studied by Cooper and John (1988) and many others.

Topkis (1978) shows that if a function (with domain and range defined on the reals) exhibits increasing differences, it will also exhibit the property of supermodularity. A given size-specific coalition production function is strictly supermodular if for any nonscalar vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ such that $\underset{\sim}{a} \neq \underset{\sim}{b}$,

$$\Phi_I\left(\underset{\sim}{a}\vee\underset{\sim}{b}\right)+\Phi_I\left(a\wedge\underset{\sim}{b}\right)>\Phi_I\left(\underset{\sim}{a}\right)+\Phi_I\left(\underset{\sim}{b}\right), \tag{2}$$

where for any two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, $\underset{\sim}{a}\vee\underset{\sim}{b}=(\max(a_1,b_1),\ldots,\max(a_I,b_I))$ and $\underset{\sim}{a}\wedge\underset{\sim}{b}=(\min(a_1,b_1),\ldots,\min(a_I,b_I))$.

This equivalence between increasing differences and supermodularity will prove to be very useful below.[1]

The second definition formalizes what we mean by a stratified allocation.

Definition 2. *Stratified allocations*
An allocation of agents is said to be stratified if

A. Agents are allocated to at least 2 distinct coalitions.

B. For any pair of coalitions $\underset{\sim}{a}$ and $\underset{\sim}{b}$ either $\min_n a_n \in \{\underset{\sim}{a}\} \geq \max_n b_n \in \{\underset{\sim}{b}\}$or $\max_n a_n \in \{\underset{\sim}{a}\} \leq \min_n b_n \in \{\underset{\sim}{b}\}$.

3 Efficient allocations

i) Fixed coalition size

We first consider the problem of the efficient allocation of agents across coalitions when the size of each coalition is fixed at some I. Assume that N/I equals some

[1] Milgrom and Roberts (1990) provide a comprehensive introduction to increasing differences and supermodularity as well as a survey of its use in studying a wide variety of economic environments; see Milgrom and Shannon (1994) for many additional results. Cooper and John (1988) provide an excellent overview of the macroeconomic implications of strategic complementarities.

integer K. An efficient allocation of agents across coalitions is one that maximizes

$$\sum_{k=1}^{K} \Phi_I\left(\underset{\sim}{a}_k\right), \tag{3}$$

subject to the conditions that all agents are allocated, i.e.

$$\bigcup_{k=1}^{K}\left\{\underset{\sim}{a}_k\right\} = \mathcal{A}. \tag{4}$$

Our first result indicates how strict increasing differences induces stratification in this case. The result is a slight generalization of the Becker condition for assortative mating in marriages, extending the assumptions of that paper to include cases where 1) more than two agents are matched, 2) agents within a coalition are of a common type, (as opposed to distinguishable between men and women), and 3) the coalition payoff functions are not differentiable.

Proposition 1. *Efficiency of stratification under increasing differences and fixed coalition size.*

Suppose that all coalitions must be of the same size I and that the associated payoff function exhibits strict increasing differences. Then stratification is necessary for output maximization.

Proof. Since N is finite, there always exists at least one optimal coalition configuration. Within such an output maximizing allocation, let $\underset{\sim}{b}$ and $\underset{\sim}{c}$ denote allocations to any particular pair of coalitions; associated with these vectors let $\underset{\sim}{b}^{(I)}$denote the reordering of $\underset{\sim}{b}$ in ascending order of ability levels and $\underset{\sim}{c}_{(I)}$ denote the reordering of $\underset{\sim}{c}$ in descending order of abilities. In order for $\underset{\sim}{b}$ and $\underset{\sim}{c}$ to be part of an optimal allocation, it must be the case that $\Phi_I(\underset{\sim}{b}) + \Phi_I(\underset{\sim}{c})$ maximizes output among all possible coalitions of agents in$\{\underset{\sim}{b}\} \cup \{\underset{\sim}{c}\}$. This implies, since the ordering of agents within a coalition has no effect on output, that

$$\begin{aligned}&\Phi_I\left(\underset{\sim}{b}^{(I)} \vee \underset{\sim}{c}_{(I)}\right) + \Phi_I\left(\underset{\sim}{b}^{(I)} \wedge \underset{\sim}{c}_{(I)}\right) \leq \Phi_I\left(\underset{\sim}{b}^{(I)}\right)\\ &+\Phi_I\left(\underset{\sim}{c}_{(I)}\right) = \Phi_I\left(\underset{\sim}{b}\right) + \Phi_I\left(\underset{\sim}{c}\right).\end{aligned} \tag{5}$$

This latter condition can hold given (2), only if either $\underset{\sim}{b}^{(I)} \vee \underset{\sim}{c}_{(I)} = \underset{\sim}{b}^{(I)}$ or $\underset{\sim}{b}^{(I)} \vee \underset{\sim}{c}_{(I)} = \underset{\sim}{c}_{(I)}$, which implies that across the coalitions, the agents are stratified. □

ii) Variable coalition size

Proposition 1 indicates how stratification will be efficient for models of such organizations as marriages or athletic teams in which social norms have established a fixed set of coalition sizes. On the other hand, when one considers organizations such as firms or neighborhoods, the assumption of fixed coalition sizes is no longer sensible. When the coalition size is a choice variable, then the property of increasing differences no longer implies that multiple stratified coalitions are efficient. The efficiency of multiple stratified coalitions depends critically on whether large coalitions with low ability levels among some agents are less efficient than groups of small coalitions. Proposition 2 formalizes this by providing a condition under which stratification is never efficient - namely that an additional agent can never reduce the payoff of a coalition.

Proposition 2. *Condition for efficiency of integration under increasing differences and variable coalition size.*

Suppose that all production functions exhibit increasing differences and that coalitions may vary in size. If

$$\Phi_{I+J}\left(\underset{\sim}{a}, \underset{\sim}{0}\right) \geq \Phi_I\left(\underset{\sim}{a}\right) \forall I, \tag{6}$$

then output is maximized by a single coalition of all agents, regardless of the distribution of abilities.

Proof. If output is maximized under multiple coalitions, then

$$\Phi_{I+J}\left(\underset{\sim}{b}, \underset{\sim}{c}\right) < \Phi_I\left(\underset{\sim}{b}\right) + \Phi_J\left(\underset{\sim}{c}\right), \tag{7}$$

for any coalitions $\underset{\sim}{b}$ and $\underset{\sim}{c}$ which are part of the optimal allocation of agents. Given (2) this implies

$$\begin{aligned}\Phi_{I+J}\left(\underset{\sim}{b}, \underset{\sim}{c}\right) &+ \Phi_{I+J}\left(\underset{\sim}{0}, \underset{\sim}{0}\right)\\ &< \Phi_{I+J}\left(\underset{\sim}{b}, \underset{\sim}{0}\right) + \Phi_{I+J}\left(\underset{\sim}{0}, \underset{\sim}{c}\right). \end{aligned} \tag{8}$$

But $(\underset{\sim}{b}, \underset{\sim}{c}) = (\underset{\sim}{b}, \underset{\sim}{0}) \vee (\underset{\sim}{0}, \underset{\sim}{c})$ and $(\underset{\sim}{0}, \underset{\sim}{0}) = (\underset{\sim}{b}, \underset{\sim}{0}) \wedge (\underset{\sim}{0}, \underset{\sim}{c})$, which means that (8) contradicts (2). □

We now consider the case where

$$\Phi_{I+J}\left(\underset{\sim}{a}, \underset{\sim}{0}\right) < \Phi_I\left(\underset{\sim}{a}\right), \tag{9}$$

which, by Proposition 2, is necessary for multiple coalitions to be efficient under some conditions. The existence of a link between increasing differences and stratification can now be considered. In fact, the existence of multiple coalition sizes means that no such link exists. To see this, suppose that at an efficient allocation, agents are allocated into two coalitions of sizes $N-K$ and K respectively, where $N-K>K$. Suppose that the agents in these coalitions are ordered in terms of increasing ability and are divided into three non-overlapping ability groups, $\underset{\sim}{a}_{\text{low}}$, $\underset{\sim}{a}_{\text{mid}}$ and $\underset{\sim}{a}_{\text{high}}$ such that low and high groups each have K members. Segregation will be efficient if and only if either $\Phi_{N-K}(\underset{\sim}{a}_{\text{low}}, \underset{\sim}{a}_{\text{mid}}) + \Phi_K(\underset{\sim}{a}_{\text{high}})$ or $\Phi_{N-K}(\underset{\sim}{a}_{\text{mid}}, \underset{\sim}{a}_{\text{high}}) + \Phi_K(\underset{\sim}{a}_{\text{low}})$ maximizes output relative to all possible configurations of agents. Consider the latter case; symmetric reasoning applies to the former. Let $\gamma_K(\cdot) = \Phi_{N-K}(\underset{\sim}{a}_{\text{mid}}, \cdot)$. For segregation to be efficient, such an allocation must be necessary to maximize the sum of two distinct K-size increasing differences functions, $\gamma_K(\cdot) + \Phi_K(\cdot)$. Such a requirement is not implied by, and is quite different from, the supermodularity condition that characterizes the individual functions. Following this logic, under variable coalition size, the assumption of increasing differences no longer implies any necessary link between stratification and efficiency.

Proposition 3. *Possible efficiency of integration under increasing differences and variable coalition size.*

There exist sets of size-specific payoff functions such that output is maximized by integrated coalitions even though each payoff function exhibits increasing differences.

Proof. We verify the proposition by example. Suppose that there are three agents with ability levels $a_1 = 1, a_2 = 1.5$ and $a_3 = 2$ respectively. The three size-specific payoff functions are:

$$\Phi_1(a_i) = 0.001a_i^2 + 1.1\max(a_i - 1, 0), \tag{10}$$

$$\Phi_2(a_i, a_j) = 1.5(a_i \cdot a_j), \tag{11}$$

and

$$\Phi_3(a_i, a_j, a_k) = .1(a_i \cdot a_j \cdot a_k). \tag{12}$$

These functions exhibit increasing differences yet the output-maximizing configuration of agents would place those with abilities a_1 and a_3 in one coalition, leaving the agent with ability a_2 isolated. □

While the example in the proposition's proof is ungainly, it illustrates some general ideas of interest. The high ability of agent 3 is productive in conjunction with another agent, requiring at least one multiple agent coalition. The isolation of the least able agent will leave him totally unproductive. In turn, his integration with agent 3 represents the most productive allocation. This example indicates how the integration of very skilled and unskilled workers in an organization may be more efficient than the integration of very skilled and moderately skilled workers even when each coalition exhibits complementarities.

The example in Proposition 3 possesses an additional feature of interest. Suppose that the ability of agent 1 is increased from 1 to 1.4. In this case, the efficient allocation of agents places 2 and 3 together, and leaves 1 in isolation. Therefore, an inequality-decreasing change in the cross-section distribution of abilities leads to an increase in the degree of stratification. To see why this can hold more generally, suppose that at some initial distribution of abilities, it is efficient to stratify into coalitions $\underset{\sim}{b}$ and $\underset{\sim}{c}$, *i.e.* $\Phi_I(\underset{\sim}{b}) + \Phi_J(\underset{\sim}{c}) > \Phi_{I+J}(\underset{\sim}{b}, \underset{\sim}{c})$. Assuming that $\underset{\sim}{b}$ comprises the higher ability agents, a monotonic decrease in ability from $\underset{\sim}{c}$ to $\underset{\sim}{c}'$, would necessarily preserve this stratification, relative to an integration of both coalitions only if

$$\Phi_{I+J}\left(\underset{\sim}{b}, \underset{\sim}{c}\right) - \Phi_{I+J}\left(\underset{\sim}{b}, \underset{\sim}{c}'\right) + \Phi_J\left(\underset{\sim}{c}\right) - \Phi_J\left(\underset{\sim}{c}'\right) > 0. \tag{13}$$

This condition, in turn, is equivalent to the condition of increasing differences for the composite payoff function $\Phi_I(\underset{\sim}{b}) + \Phi_J(\underset{\sim}{c}) - \Phi_{I+J}(\underset{\sim}{b}, \underset{\sim}{c})$, which is not an implication of the fact that each function exhibits increasing differences individually.[2] These considerations are summarized in Proposition 4.

Proposition 4. *Lack of relationship between degree of cross-section inequality and stratification*

Consider an economy in which all production functions exhibit increasing differences and in which all agent allocations are efficient.

A. For an initial distribution of abilities such that in the efficient allocation no agents with abilities above $\bar{a}$ form coalitions with agents with abilities below $\underline{a}$, this stratification will not necessarily be preserved if all abilities above $\bar{a}$ are increased whereas all abilities below $\underline{a}$ are decreased.

B. A rightward shift in the distribution of abilities has no necessary implication for the degree of efficient stratification.

Propositions 3 and 4 illustrate that stratification cannot be treated as a generic property of economies exhibiting complementarities and some minimum degree of cross-section inequality without additional restrictions on the payoff functions under study.[3]

4 Stratification in restricted economic environments

A link between stratification and efficiency can be re-established with additional restrictions on the economic environment under study. One possibility concerns restrictions on the coalition-specific production functions. Alternatively, one can restrict the distribution of abilities across agents.

[2] We thank Paul Milgrom for discussion of this point.

[3] This result is related to Becker's (1974) analysis of assortative mating in polygamous societies. Becker argues that efficiency in the marriage market may be achieved equally well by matching a relatively able male with either one able or two less able females. Our analysis shows how this type of result may be strictly efficiency-enhancing in a number of alternative contexts; for example, we do not require any necessary link between the efficiency of integration and coalition size.

i) Restrictions on coordination costs

One approach to delimiting the class of production functions so as to restore a link between stratification and efficient allocations is to parameterize the costs of coordination in larger coalitions. Proposition 5 indicates how a restriction on coordination costs, namely the representation of these costs as additive and a function only of coalition size, leads to the efficiency of stratification. The proposition additionally shows that this sort of restriction on costs produces a relationship between the relative ability distributions of any two coalitions and their relative sizes.

Proposition 5. *Efficiency of stratification under increasing differences and variable coalition size when increased coalition size induces fixed costs.*

If

$$\Phi_{I+J}\left(\underset{\sim}{a},0\right)=\Phi_I\left(\underset{\sim}{a}\right)-C\left(I,J\right), \tag{14}$$

where $C\left(I,J\right)$ is positive, then

A. Any output maximizing configuration is stratified.

B. When coalitions are ordered by ability, if one coalition contains abilities which are greater than another, then that coalition will be at least as large as the other.

Proof. At an equilibrium it must be the case that any pair of coalitions $\underset{\sim}{b}$ and $\underset{\sim}{c}$ must maximize $\Phi_I\left(\underset{\sim}{b}\right)+\Phi_J\left(\underset{\sim}{c}\right)$, relative to any reallocation of agents across coalitions which preserves their size. Without loss of generality, take $I>J$. Using eq. (14) this sum can be rewritten as

$$\Phi_{I+J}\left(\underset{\sim}{b},\underset{\sim}{0}_J\right)+\Phi_{I+J}\left(\underset{\sim}{c},\underset{\sim}{0}_I\right)+C\left(I,J\right)+C\left(J,I\right), \tag{15}$$

where the subscripts on the 0 vectors denote length. It is clear that eq. (15) can only be maximized if $\left(\underset{\sim}{b},\underset{\sim}{0}_J\right)=\left(\underset{\sim}{b},\underset{\sim}{0}_J\right)\vee\left(\underset{\sim}{c},\underset{\sim}{0}_I\right)$ and $\left(\underset{\sim}{c},\underset{\sim}{0}_I\right)=\left(\underset{\sim}{b},\underset{\sim}{0}_J\right)\wedge\left(\underset{\sim}{c},\underset{\sim}{0}_I\right)$, which can only occur when the elements of $\underset{\sim}{b}$ are at least as large as those of $\underset{\sim}{c}$, i.e. the coalitions are stratified. This proves A.

To prove B, observe that if $I<J$, then the argument still goes through, only in this case $\underset{\sim}{c}>\underset{\sim}{b}$ as the larger coalition must have more nonzero elements, given the argument in A. □

ii) Multiplicative interactions

A second type of production function which always produces stratification in efficient allocations is the so-called O-ring production function, introduced by Kremer

(1993). Kremer's production function is a special case of coalition-specific payoff functions of the form

$$\Phi_I\left(\underset{\sim}{a}\right) = \Gamma\left(I\right) \underset{n}{\Pi}\, \varphi\left(a_n\right). \tag{16}$$

In this expression, $\Gamma\left(\cdot\right)$ is a scaling factor which reflects economies (or diseconomies) of scale for coalitions of different sizes. The individual ability component $\varphi\left(\cdot\right)$ is assumed to be nonnegative and increasing.[4] As Proposition 6 states, such production functions will always produce stratification for efficient allocations.

Proposition 6. *Relationship between O-ring production functions and efficiency of stratification*

If Eq. (1) characterizes the set of coalition-specific production functions, then any pair of distinct coalitions will be stratified.

Proof. Since any pair of coalitions of equal size must be stratified by Proposition 1, we only need to consider any pair of coalitions with sizes $I > J$ and associated allocations $\underset{\sim}{a}$ and $\underset{\sim}{b}$. Rewrite $\underset{\sim}{a}$ as $\left(\underset{\sim}{c}, \underset{\sim}{d}\right)$, where $\underset{\sim}{c}$ contains the smallest J elements of $\underset{\sim}{a}$. Total production from the two coalitions is

$$\frac{\Gamma\left(I\right)}{\Gamma\left(J\right)} \overset{I-J}{\underset{n=1}{\Pi}} \varphi\left(d_n\right) \Phi_J\left(\underset{\sim}{c}\right) + \Phi_J\left(\underset{\sim}{b}\right) = K\Phi_J\left(\underset{\sim}{c}\right) + \Phi_J\left(\underset{\sim}{b}\right) \tag{17}$$

for some nonnegative K. If $K < 1$, maximization requires that $\underset{\sim}{c} = \underset{\sim}{c} \wedge \underset{\sim}{b}$. Further, since these coalitions are output maximizing, then they must be invariant under any alternative partition of $\underset{\sim}{a}$, as alternative partitions will lower the value of K. Similarly, if $K = 1$, then we can arbitrarily assign $\underset{\sim}{c} = \underset{\sim}{c} \wedge \underset{\sim}{b}$ and repeat the same argument. When $K > 1$, it is clear that this expression can only be maximized if $\underset{\sim}{c} = \underset{\sim}{c} \vee \underset{\sim}{b}$. Since the elements of $\underset{\sim}{d}$ exceed those of $\underset{\sim}{c}$, they must exceed those of $\underset{\sim}{b}$, and the coalitions are again stratified. □

iii) Limiting behavior of replicated economies

While the above results illustrate how, in a finite economy, integrated economies may be efficient even in the presence of complementarities, these results can disappear when one considers replications of the economy. In particular, one can consider t-fold replications of the agents of an economy comprised of a finite number of agents with ability set $\mathcal{A}$, i.e. increasing collections of agents with associated abilities $\mathcal{A}_t$ such that

$$\mathcal{A}_t = \mathcal{A}_{t-1} \cup \mathcal{A}\,. \tag{18}$$

[4] Kremer (1993) studies a particular parameterization of this function, $\Phi_I(\underset{\sim}{a}) = \Gamma\left(I\right) \underset{n}{\Pi}\, a_n$ where $\Gamma\left(\cdot\right)$ is increasing and a_n is bounded between 0 and 1.

In such sequences, the number of agents with identical abilities becomes unbounded for each element of the support of abilities, while the number of agent types will remain unchanged. This turns out to have a critical effect on the limiting behavior of the economy, as it affects the capacity for completely homogeneous coalitions to emerge.

Proposition 7. *Asymptotic stratification with fixed upper coalition bound on coalition size.*

Suppose that the maximum coalition size never exceeds some finite I. Then as t becomes large in a sequence of t-fold replications of a finite collection of agents, the fraction of agents located in completely homogeneous coalitions will approach 1.

Proof. At an optimal allocation of agents across coalitions for the t'th replication, suppose that $C_{i,t}$, agents are located in coalitions of size i, so that $\sum_{i=1}^{I} C_{i,t} = tN$, the number of agents in the economy at that replication. For agents in coalitions of size i, let $K_{i,t}$ be the largest integer such that $C_{i,t} - i \cdot K_i$ is non-negative. For the $i \cdot K_{i,t}$ agents in coalitions of size i to be optimally assigned, Proposition 1 implies those agents must be in homogeneous coalitions, since such coalitions are feasible within the i-size class. Therefore, at most $\sum_{i=1}^{I} (C_{i,t} - i \cdot K_{i,t}) < I^2$ agents can lie in nonhomogeneous coalitions, given the upper bound on the number of types. Therefore, the fraction of agents in homogeneous coalitions can be no smaller than $1 - I^2 / tN$, which converges to 1 as tN becomes large. □

Observe that this result does not depend on the assumption that the maximum coalition size is exogenously limited; the proposition will still hold if, along any sequence of efficient allocations corresponding to increasing the number of replications of the original economy, the size of the largest coalition is uniformly bounded.

The applicability of Proposition 7 will depend, of course on the environment under study. In the case of firms, for example, it seems plausible that difficulties in the coordination of activities can impose an upper bound on the size of individual coalitions that is small relative to the pool of available workers.[5] On the other hand, for the problem of determining the number and size of school districts, it is clearly plausible that returns to scale and/or politically imposed restrictions on the number of districts will mean that the size of districts is large relative to the size of the population of families which is to be allocated.

5 Interactions between sorting allocations and productive inputs

The discussion thus far has focused on the efficiency properties of sorting when the determinants of output, namely ability, are fixed. When effort is integrated into production, the relationship between sorting and efficiency becomes much more complicated.

In order to generalize the earlier discussion, we focus on the case in which all coalitions must be of size I. Individuals supply effort e_i as well as ability. For ease

[5] See Becker and Murphy (1992) for an analysis of this issue.

of exposition, we take the effort level to be binary so that $e_i \in \{\underline{e}, \bar{e}\}$. Each member of a coalition receives a payoff of the form

$$\phi_{I,i}\left(a_i, \underset{\sim}{a}_{-i}, e_i, \underset{\sim}{e}_{-i}\right) \tag{19}$$

Total coalition output may be written

$$\Phi_I\left(\underset{\sim}{a}, \underset{\sim}{e}\right) = \sum_{i=1}^{I} \phi_{I,i}\left(a_i, \underset{\sim}{a}_{-i}, e_i, \underset{\sim}{e}_{-i}\right)^6 \tag{20}$$

Milgrom and Shannon (1994) verify that the coalition output functions exhibit increasing differences so long as the individual payoff functions do so, given the additive form of (20). Notice that by allowing the individual payoff functions to differ within a coalition, agents with different abilities can receive different levels of compensation.

For any allocation of agents across coalitions, Milgrom and Roberts verify the following. Given an ability vector $\underset{\sim}{a}$ within a coalition, there will exist at least one vector $\underset{\sim}{e}$ such that each e_i solves

$$\max_{e_i \in \{\underline{e}, \bar{e}\}} \phi_{I,i}\left(a_i, \underset{\sim}{a}_{-i}, e_i, \underset{\sim}{e}_{-i}\right) \tag{21}$$

We now consider the allocation of individuals across coalitions. Observe that conditional on common effort levels across all agents, output maximization will require that all coalitions are stratified by ability. Further, if the effort level of an agent with given ability is always at least as high as the effort level of an agent with lesser ability, then stratification will also be efficient. However, when effort is endogenous, stratification may be inefficient due to its effects on the set of efforts across agents.

In particular, suppose $I = 2$ and that $N/2$ agents have high ability $\bar{a}$ and $N/2$ agents have low ability $\underline{a}$. Suppose as well that the following inequalities hold:

$$\phi_{2,i}\left(\bar{a}, \underline{a}, \bar{e}, \underline{e}\right) > \phi_{2,i}\left(\bar{a}, \underline{a}, \underline{e}, \underline{e}\right) \tag{22}$$

$$\phi_{2,i}\left(\underline{a}, \underline{a}, \bar{e}, \underline{e}\right) < \phi_{2,i}\left(\underline{a}, \underline{a}, \underline{e}, \underline{e}\right) \tag{23}$$

In this case, it is clear stratification will always be output maximizing if

$$\left|\Phi_2\left(\bar{a}, \bar{a}, \bar{e}, \bar{e}\right) - \Phi_2\left(\bar{a}, \underline{a}, \bar{e}, \bar{e}\right)\right| > \left|\Phi_2\left(\bar{a}, \underline{a}, \bar{e}, \bar{e}\right) - \Phi_2\left(\underline{a}, \underline{a}, \underline{e}, \underline{e}\right)\right| \tag{24}$$

Increasing differences imply only that

$$\left|\Phi_2\left(\bar{a}, \bar{a}, \bar{e}, \bar{e}\right) - \Phi_2\left(\bar{a}, \underline{a}, \bar{e}, \bar{e}\right)\right| > \left|\Phi_2\left(\bar{a}, \underline{a}, \bar{e}, \bar{e}\right) - \Phi_2\left(\underline{a}, \underline{a}, \bar{e}, \bar{e}\right)\right| \tag{25}$$

[6] It might seem more natural to assume that each individual chooses an effort level which maximizes a utility function whose arguments include effort as well as compensation from the coalition. In fact, our current formulation can be rewritten this way without any qualitative change of results. The current formulation, by folding any utility aspects of effort into the coalition output function, avoids any ambiguity in what is meant by "efficient."

which makes clear the effect of incentives on efficiency. Proposition 8 verifies that these incentive effects can be powerful enough to render integrated coalitions efficient, even for a fixed-coalition size economy.

Proposition 8. *Efficiency of integration under increasing differences and constant coalition size in the presence of endogenous effort.*

There exist fixed-size coalition and individual payoff functions such that

A. Total payoffs are maximized by integrated coalitions even though the payoff functions exhibit increasing differences jointly in effort and ability.

B. The efficiency of integration holds for arbitrary replications of the economy.

Proof. We prove by example. Suppose that $\underline{a} = 1$, $\bar{a} = 6$ and $e_i \in \{1, 2\}$. Let the individual payoff functions within each coalition obey

$$\phi_{2,i}\left(a_i, \underset{\sim}{a}_{-i}, e_i, \underset{\sim}{e}_{-i}\right) = a_i \cdot a_j \cdot e_i \cdot e_j - 5e_i + 100e_j^2 \tag{26}$$

It is easy to verify that high ability agents will always choose $\bar{e}$ whereas low ability agents will choose $\underline{e}$ when matched with one another versus $\bar{e}$ when paired with high ability agents. A coalition of two high ability agents will produce a payoff of 534 for each, a coalition of two low ability agents will produce a payoff of 0 for each, and a mixed coalition will produce a payoff of 414 for each, which implies that integration will maximize total payoffs, which proves A. Part B is immediate since replications of the agents, keeping the distribution of abilities constant, do not change the set of feasible coalition types in the economy. □

One interesting feature of the efficient allocation of agents across coalitions in this example is that once the effort levels are set, the allocation is *ex post* inefficient; as in fact must always be true given Proposition 1. Durlauf (1996c) argues that this feature makes it difficult to assess the efficiency of programs such as affirmative action which may have desirable effects on unobservable variables such as effort by altering the way agents are sorted.

Finally, observe that if the coalition payoff function is replaced by $(a_i - 5)\, e_i + 100e_ie_j$ and the effort support is {0,1}, then while any coalition with a high ability agent will be associated with high effort by both members, a coalition with two low ability members will exhibit multiple Nash equilibria in effort as {0,0} and {1,1} are both self-reinforcing choices. This suggests that integrated allocations can help overcome coordination problems. This basic idea has application far beyond this simple example. For example, Brock and Durlauf (2001) show how the presence of multiple equilibria due to coordination failure in binary choice environments depends on the interplay of a relatively strong interdependence of choices on "social utility" effects (the large coefficient on $100e_ie_j$in this case) with a relatively weak dependence of private utility on choices (the negative coefficient in $(a_i - 5)\, e_i$ for low ability agents). Alterations of the cross-section characteristics of agents within groupings, each of which obeys that model, can eliminate the presence of multiple equilibria. Durlauf (1995c) shows that this type of argument suggests that certain classes of affirmative action policies may be efficiency enhancing.

6 Examples

i) Endogenous growth

One variant of endogenous growth models of the type pioneered by Romer (1986) and Lucas (1988) may be thought of as positing individual production functions of the form[7]

$$\phi\left(a_n, F_{N_k}, \mu\left(N_k\right)\right), \tag{27}$$

whereN_k denotes the interaction neighborhood of agent n, F_{N_k} denotes the empirical distribution function of agent abilities in the neighborhood and $\mu\left(N_k\right)$ denotes the population of the neighborhood. These models typically assume a large population of individuals so that no single agent's actions affect the characteristics of the whole population. The function $\phi\left(\cdot,\cdot,\cdot\right)$ is usually taken to exhibit increasing differences with respect to the ability levels of other agents (in the sense that a rightward shift in F_{N_k} increases the marginal product of a_n), and to exhibit increasing returns in all arguments whereas $\partial^2\phi\left(\cdot,\cdot,\cdot\right)/\partial a_n^2 < 0$, so that the function is concave in its first argument. The aggregate production function Φ is simply the sum of the individual production functions so that

$$\Phi\left(F_{N_1},\ldots,F_{N_K},\mu\left(N_1\right),\ldots,\mu\left(N_k\right)\right) = \sum_n \phi\left(a_n, F_{N_k}, \mu\left(N_k\right)\right). \tag{28}$$

As before, conditional on any distribution of firms across neighborhoods, the aggregate production function inherits the concavity and increasing differences properties of the individual production functions.

One possible form for the individual production function (27) is

$$\phi\left(a_n, \mu\left(N_k\right)^{\rho}\cdot\int_{N_k} a\cdot dF_{N_k}\left(a\right)\right), \tag{29}$$

so that the population and ability density terms interact multiplicatively.

Different choices of the function ρ allow one to distinguish the extent to which spillovers depend on total versus average ability levels as well as the output-maximizing configuration of firms across neighborhoods for a given distribution of abilities if one generalizes these models to allow agents to choose with whom they interact.[8] In particular, two extreme cases exist. When $\rho = 1$, individual productivity depends only on the magnitude of the ability aggregate. In this case, the requirement of Proposition 2 holds, so that if the interaction range is endogenous, then all agents will choose to interact together. This configuration corresponds to

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7 To be precise, this exercise should be thought of as describing sorting behavior in a model with capital externalities in which firm-specific capital stocks are given.

8 The appropriateness of endogenizing the spillover environments in such models will of course depend upon the spillover in question. For example, it certainly seems reasonable *a priori* that firms will organize and locate themselves to account for human capital spillovers both internally and externally, as occurs, for example in the Silicon Valley. Similarly, the voluntary allocation of families into neighborhoods and associated house price and rental barriers is dependent on the feedback from neighborhood characteristics into offspring outcomes.

the case studied by Romer and Lucas, in which all agents spillover symmetrically onto one another. Of course, the same fully integrated outcome occurs when $\rho = 0$.

On the other hand, if $\rho = -1$, then individual productivity depends on the ability mean and Proposition 2's requirements are violated. In this case, the economy will completely stratify. This is intuitively obvious and can be verified as follows. Observe that for any ability distribution where a fraction λ of all agents have ability $\bar{a}$ and a fraction $1-\lambda$ have ability $\underline{a}$, concavity in the first argument of the individual production function implies (supposing other arguments of the function)

$$\begin{gathered}\phi\left(\lambda\bar{a}+(1-\lambda)\underline{a},\lambda\bar{a}+(1-\lambda)\underline{a}\right)>\\ \lambda\phi\left(\bar{a},\lambda\bar{a}+(1-\lambda)\,\underline{a}\right)+(1-\lambda)\,\phi\left(\underline{a},\lambda\bar{a}+(1-\lambda)\,\underline{a}\right).\end{gathered}\tag{30}$$

Further note that by social increasing returns

$$\lambda\phi\left(\bar{a},\bar{a}\right)+(1-\lambda)\,\phi\left(\underline{a},\bar{a}\right)>\phi\left(\lambda\bar{a}+(1-\lambda)\,\underline{a},\lambda\bar{a}+(1-\lambda)\,\underline{a}\right).\tag{31}$$

Together these inequalities imply

$$\begin{aligned}\lambda\phi\left(\bar{a},\bar{a}\right)+(1-\lambda)\,\phi\left(\underline{a},\underline{a}\right)&>\lambda\phi\left(\bar{a},\lambda\bar{a}+(1-\lambda)\,\underline{a}\right)\\ &+(1-\lambda)\,\phi\left(\underline{a},\lambda\bar{a}+(1-\lambda)\,\underline{a}\right)\end{aligned}\tag{32}$$

which means that integration is never efficient for any mixture of types. Repeated use of this argument can be used to show that the economy will always break up into isolated individuals when the spillover effects are based on mean ability. By extension, since the efficiency of the economy is unaffected whenever individuals of equal ability inhabit the same coalition or different coalitions, the economy will completely stratify whenever the number of feasible coalitions equals the number of different types.[9] For $-1 < \rho < 0$, the model can exhibit stratification or integration, depending on the distribution of abilities.

ii) Classroom size

To further illustrate the connection, or lack thereof, between the efficiency of stratification and complementarity, we consider the question of allocating students across classrooms when teachers are of differing qualities. The issue of class size, and the manner in which it influences individual learning has been a contentious one in the education literature. To model this problem, one can employ the same basic framework as in the endogenous growth case. However, the introduction of matching between students and teachers introduces some important differences.

We assume that the production function that determines a student's human capital h_n is, instead of (29)

$$\Psi\left(a_n,\mu\left(N_k\right)^{\rho}\cdot\int_{N_k}a\cdot dF_{N_k}\left(a\right),H_n\right)\tag{33}$$

[9] This follows from the fact that the average product of two identical agents in separate size-1 coalitions will equal their average product when they combine since the spillover associated with the mean ability level does not change.

where H_n denotes the human capital of the teacher assigned to a student with ability a_n.

Given a vector of student abilities, $\underset{\sim}{a}$ and a vector of teacher human capital levels $\underset{\sim}{H}$, the question under consideration is the matching of teachers and students so as to maximize total human capital among students. (Classrooms are implicitly defined by sets of students assigned to a common teacher.) This problem turns out to be complicated because of the possibilities that classroom sizes differ. For ease of exposition, we consider a special case of (33),

$$a_n^{\alpha} \cdot \mu \left(N_k\right)^{\rho} \cdot H_n^{\gamma} \left(\int_{N_k} a \cdot dF_{N_k}\left(a\right)\right)^{\beta} \tag{34}$$

Here, γ is the elasticity of a student's human capital with respect to his teacher's human capital. Observe that if $\alpha, \gamma > 0$, then student ability a_n and H_n are complementary in determining the student's human capital. When $\gamma = 0$, we are back to the endogenous growth case.

As before, ρ determines whether classmates are complementary inputs in human capital production. In particular, regardless of the value of γ, if $\rho > 0$, then the conditions of Proposition 2 go through, and integration is efficient. When $\rho < 0$, efficient allocations are again difficult to characterize. Hence, we use some numerical examples to illustrate the main workings of the model.

Imagine that there are three students with ability levels, 1, 1.5 and 2.0 and two teachers with human capital levels, 0.5 and 0.6. Further, assume that $\alpha = \beta = 0.5$. Finally, assume that teachers can go unassigned. Table 1 depicts the various efficient allocations for alternative values of ρ and γ.

Table 1 illustrates some basic ideas. First, consider allocations for different values of ρ when γ is fixed. Higher values of ρ imply larger negative effects of group size on individual achievement. This in turns renders stratification relatively efficient. Further, there is substitutability between group size and teacher human capital when ρ is negative. This implies that efficient allocations assign low ability students to high human capital teachers. When γ is small, the effect of substitutability between group size and individual achievement dominates the effects of complementarity between individual ability and teacher human capital. Again, lower ability students are assigned to better teachers. As γ increases, the latter effect dominates and more able students are assigned to better teachers.

What happens as ρ varies for a fixed γ? Unsurprisingly, higher values of ρ induce stratification. Further, notice that when $-1.6 < \rho < -1.2$, the efficient allocation assigns weaker students to stronger teachers. As in the case of varying γ, there are two forces at work. When ρ is sufficiently negative, the direct negative effect of group size on individual achievement is so strong that efficiency requires matching more able students and teachers. When γ and ρ are both high, stratification of students across classrooms and assignment of more able students to better teachers is required for efficiency.

Taken as a whole, these results suggest that the determination of efficient assignments when groups sizes are endogenous is potentially quite complex, and that the link between the efficiency of stratification and the degree of complementarity

Table 1. Effect of stratification on efficiency

$\rho\backslash\gamma$	0.1	0.2	1.5
-1	$\{a_1 \Leftrightarrow H_1\}, \{(a_2,a_3) \Leftrightarrow H_2\}$	$\{(a_1,a_2,a_3) \Leftrightarrow H_2\}$	$\{(a_1,a_2,a_3) \Leftrightarrow H_2\}$
-1.2	$\{(a_1,a_2) \Leftrightarrow H_2\}, \{a_3 \Leftrightarrow H_1\}$	$\{(a_1,a_2) \Leftrightarrow H_2\}, \{a_3 \Leftrightarrow H_1\}$	$\{a_1 \Leftrightarrow H_1\}, \{(a_2,a_3) \Leftrightarrow H_2\}$
-1.4	$\{(a_1,a_2) \Leftrightarrow H_2\}, \{a_3 \Leftrightarrow H_1\}$	$\{(a_1,a_2) \Leftrightarrow H_2\}, \{a_3 \Leftrightarrow H_1\}$	$\{a_1 \Leftrightarrow H_1\}, \{(a_2,a_3) \Leftrightarrow H_2\}$
-1.6	$\{(a_1,a_2) \Leftrightarrow H_2\}, \{a_3 \Leftrightarrow H_1\}$	$\{(a_1,a_2) \Leftrightarrow H_2\}, \{a_3 \Leftrightarrow H_1\}$	$\{a_1 \Leftrightarrow H_1\}, \{(a_2,a_3) \Leftrightarrow H_2\}$
-1.8	$\{(a_1,a_2) \Leftrightarrow H_1\}, \{a_3 \Leftrightarrow H_2\}$	$\{(a_1,a_2) \Leftrightarrow H_1\}, \{a_3 \Leftrightarrow H_2\}$	$\{(a_1,a_2) \Leftrightarrow H_1\}, \{a_3 \Leftrightarrow H_2\}$

The rows depict varying values for ρ, while the columns stand for different values of γ, the elasticity of a student's human capital with respect to his teacher's human capital. Each cell indicates the efficient allocation of students to teachers. For instance, $\{a_1 \Leftrightarrow H_1\}, \{(a_2,a_3) \Leftrightarrow H_2\}$ means that the student with the lowest ability a_1 is matched to the teacher with the lowest human capital H_1, while the other two students are assigned to the better teacher H_2.

will depend on a range of factors. And of course, this analysis does not consider the question of efficiency in dynamic contexts, where additional considerations arise. For example, Seshadri (2000) shows how the assignment of students to teachers has subtle implications for both intergenerational mobility and growth.

7 Conclusions

This paper has examined conditions under which the efficient allocation of agents produced stratification by ability. The assortative mating solution proposed by Becker (1973) was shown to generalize in two senses. First, assortative mating will hold for all economies comprised of identical sized coalitions. Second, replications of the population will lead to asymptotic stratification when the number of agents per coalition is bounded. On the other hand, when agents are free to choose coalition size, strategic complementarities were shown to be compatible with the efficiency of integrated equilibria. Similar results were obtained when the configuration of individuals across coalitions influenced the degree of effort made by each individual. The implication of these results is that the link between stratification and efficiency, which in the marriage problem holds whenever strategic complementarities exist between spouses, does not generalize to a wide range of sorting problems of interest. Therefore the wide range of stratified equilibria which have emerged as a robust implication of the new theoretical literature on inequality carry no presumption of efficiency. By extension, stratification as an equilibrium phenomenon must therefore depend on the presence of market incompleteness, except in relatively special cases. For example, it suggests that various assertions in Herrnstein and Murray (1994) on the role of economic forces in the rise of cognitive stratification are incorrect. Finally, neither the evidence of increasing stratification of firms by skill (Kremer and Maskin, 1995) nor of neighborhoods by income (Jargowsky, 1997) is self-evidently explained by efficiency considerations. In fact, the implication of models such as those of Bénabou (1993) and Durlauf (1995a,b), is that such increased stratification may be explained by the interaction of changes in the cross-section distribution of individual characteristics with the presence of either direct externalities or restrictions on the compensation rules available to coalitions.

One important extension of the current analysis is suggested by the possibility that the contemporaneous allocation of agents by coalition affects the distribution of abilities next period, as would occur in the neighborhood-based human capital models or in models in which worker ability is influenced by learning-by-doing in an environment conditioned by coworkers. In this context, there will exist dynamic efficiency considerations beyond those which we have explored. Bénabou (1995) provides an interesting analysis of dynamic efficiency of this type in a comparison of completely stratified versus completely integrated economies; a useful complement to that analysis would consider the dynamics of efficient stratification.

Finally, our analysis suggests the importance of developing a metric for identifying features of heterogeneous economies which are robust with respect to changes in the distribution of cross-section characteristics as well as with respect to functional form specifications. One fundamental difference between models in economic science and models in the natural sciences is the relative lack of guidance provided by economic theory on the specifics of individual agent behavior. Yet it is precisely these details which will determine the cross-section allocation of a heterogeneous population. Research in complex systems has already identified, in many contexts, equivalence classes of dynamic processes with similar limiting behavior. A similar research program represents an important ingredient in the development of a complex systems approach to endogenous groupings of individuals.

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