Evolutionary Game Theory

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1. Introduction

Population games are used to model strategic interactions in which

(i) the number of agents is large,
(ii) each agent is small,
(iii) agents are anonymous, with each agent's payoffs depending on his own strategy and the distribution of others' strategies.

Typically it is also assumed that

(iv) the number of populations is finite,
(v) agents in each population are identical.

Prototypical example: traffic networks.

Many examples in economics, computer science, biology, sociology, and other fields.

The traditional approach to prediction in games is equilibrium analysis.

It is based on strong assumptions about what players know:

- that players fully understand the game they are playing,
- that they are able to correctly anticipate how others will act.

These assumptions are especially demanding in games with many agents.

An alternative approach: a dynamic, disequilibrium analysis.

One assumes that agents occasionally receive opportunities to switch strategies.

An object called a revision protocol specifies when and how they do so.

The definition of the protocol reflects

> what information is available to agents when they make decisions, and
>
> how this information is used.

A population game, a population size, and a revision protocol generate a Markov process on the set of population states—that is, of distributions over pure strategies.

We will study this process over a fixed time horizon in the large population limit.

A suitable law of large numbers leads to a deterministic limit: the mean dynamic.

The analysis uses tools from the theory of dynamical systems.

Alternatively, stochastic stability analysis focuses on infinite horizon behavior as either a noise parameter or the population size approaches its limit.

This analysis uses tools from the theory of stochastic processes.

Outline of course:

1. Introduction
2. Population games
3. Revision protocols and mean dynamics
4. Deterministic evolutionary dynamics
5. Families of evolutionary dynamics
6. Potential games
7. Evolutionarily stable states and contractive games
8. Iterative solution concepts, supermodular games, and equilibrium selection
9. Nonconvergence of evolutionary dynamics
10. Connections and further developments

2. Population games

2.1 Definitions

We consider games played by a single unit-mass population.

$S = \{1, \ldots, n\}$	strategies
$X = \{x \in \mathbb{R}^n_+ \colon \sum_{i \in S} x_i = 1\}$	population states/mixed strategies
$F_i \colon X \to \mathbb{R}$	payoffs to strategy i (continuous)
$F \colon X \to \mathbb{R}^n$	payoffs to all strategies

x^* is a Nash equilibrium if

(1) $\quad x_i^* > 0$ implies that $F_i(x^*) \geq F_j(x^*)$ for all $j \in S$.

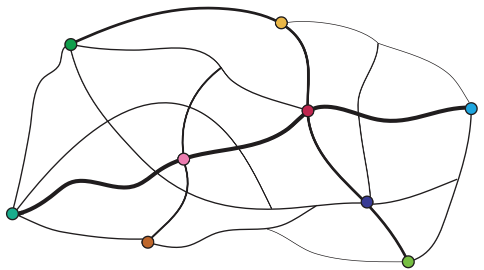
Theorem. *Every population game admits at least one Nash equilibrium.*

2.2 Examples

ex. 1. Matching in (symmetric two-player) normal form games

$A \in \mathbb{R}^{n \times n}$	payoff matrix
$A_{ij} = e_i' A e_j$	payoff for playing $i \in S$ against $j \in S$
$F_i(x) = e_i' A x = (Ax)_i$	total payoff for playing i against $x \in X$
$F(x) = Ax$	payoffs to all strategies

ex. 2. Congestion games (Beckmann, McGuire, and Winsten (1956))

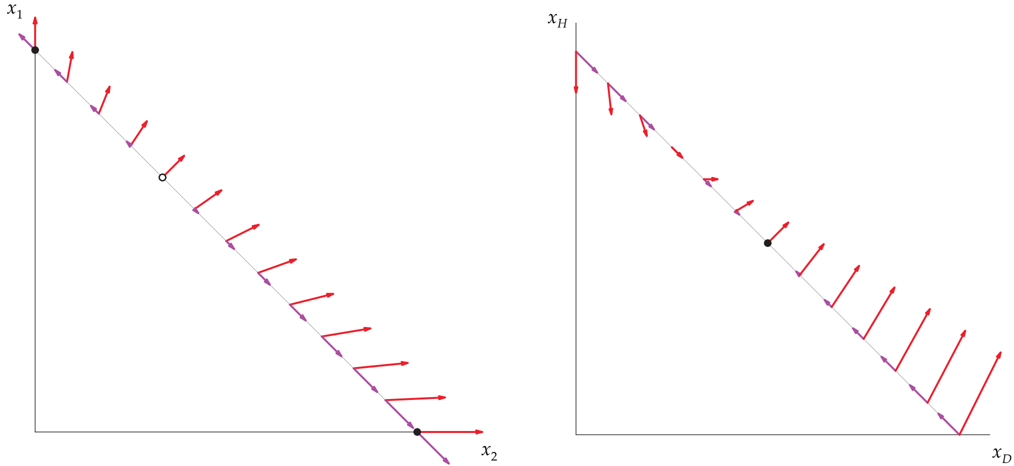


Home and Work are connected by paths $i \in S$ consisting of links $\ell \in \mathcal{L}$.

The payoff to choosing path i is

$-$(the delay on path i) = $-$(the sum of the delays on links in path i)

$F_i(x) = -\sum_{\ell \in \mathcal{L}_i} c_\ell(u_\ell(x))$	payoff to path i
x_i	mass of players choosing path i
$u_\ell(x) = \sum_{i:\, \ell \in \mathcal{L}_i} x_i$	utilization of link ℓ
$c_\ell(u_\ell)$	(increasing) cost of delay on link ℓ



(i) 12 Coordination (ii) Hawk-Dove

$$F^{C2}(x) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ 2x_2 \end{pmatrix}; \qquad F^{HD}(x) = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_H \\ x_D \end{pmatrix} = \begin{pmatrix} 2x_D - x_H \\ x_D \end{pmatrix}.$$

The simplex is $X = \{x \in \mathbb{R}^n_+ \colon \sum_{i\in S} x_i = 1\}$.

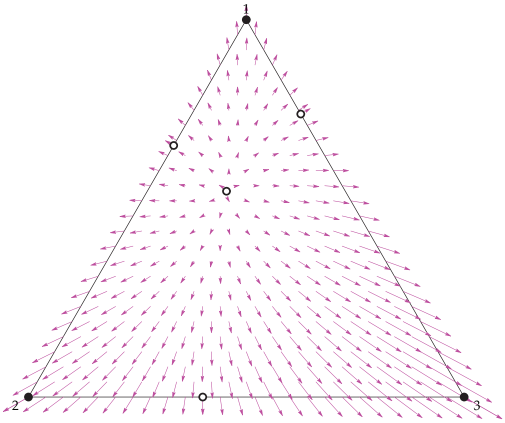
Its tangent space is $TX = \mathbb{R}^n_0 \equiv \{z \in \mathbb{R}^n \colon \sum_{i\in S} z_i = 0\}$.

The matrix $\Phi = I - \frac{1}{n}\mathbf{1}\mathbf{1}' \in \mathbb{R}^{n\times n}$ is the orthogonal projection of $\mathbb{R}^n$ onto TX.

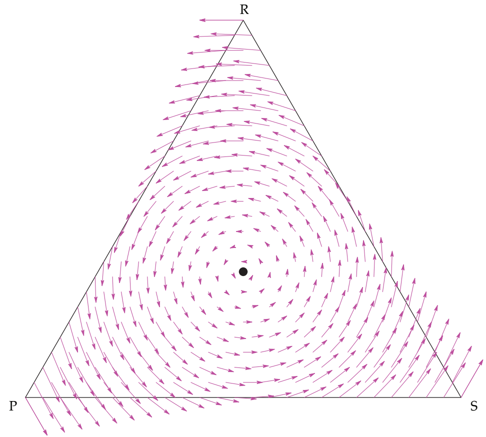
If $\pi \in \mathbb{R}^n$ then

$$\Phi\pi = \pi - \mathbf{1}\frac{1}{n}\sum_{i\in S}\pi_i \equiv \pi - \mathbf{1}\bar{\pi}$$

Φ eliminates information about average payoffs, while preserving information about payoff differences, and hence about the incentives that revising agents face.



(i) 123 Coordination



(ii) standard Rock-Paper-Scissors

$$F^{C3}(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ 2x_2 \\ 3x_3 \end{pmatrix}; \; F^{RPS}(x) = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_R \\ x_P \\ x_S \end{pmatrix} = \begin{pmatrix} x_S - x_P \\ x_R - x_S \\ x_P - x_R \end{pmatrix}.$$

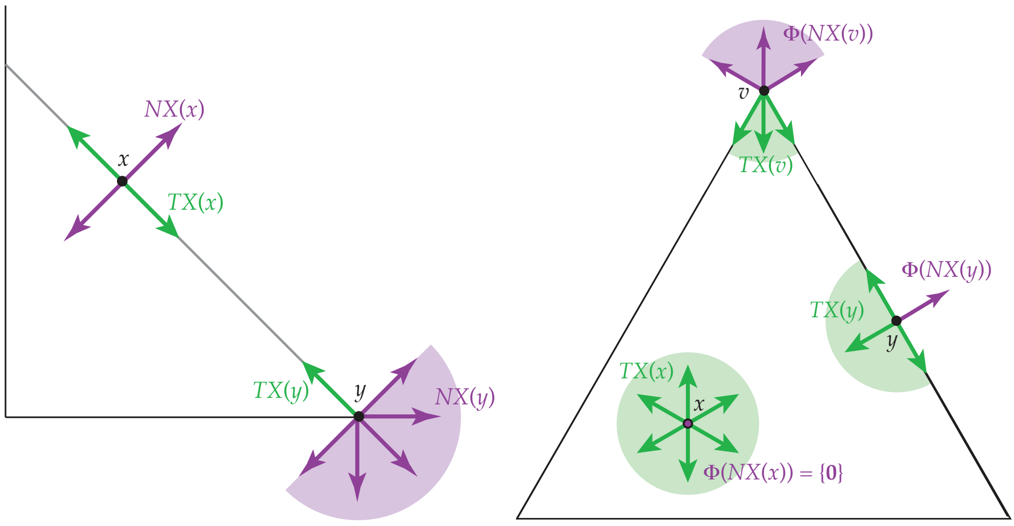
The tangent cone of X at x is be the set of directions of motion from x that do not cause the state to leave the simplex X:

$$TX(x) = \{z \in \mathbb{R}^n : z = \alpha\,(y - x) \text{ for some } y \in X \text{ and some } \alpha \geq 0\}\,.$$

The normal cone of X at x is the polar of the tangent cone of X at x.

It contains directions that form an obtuse or right angle with every vector in $TX(x)$:

$$NX(x) = (TX(x))^\circ = \{y \in \mathbb{R}^n : y'z \leq 0 \text{ for all } z \in TX(x)\}\,.$$



(i) Two-strategy games

(ii) Three-strategy games

Theorem. *Let F be a population game. Then $x \in NE(F)$ if and only if $F(x) \in NX(x)$.*

3. Revision Protocols and Mean Dynamics

Evolutionary dynamics for population games are designed to capture two basic assumptions.

Inertia: agents only occasionally consider switching strategies.

Myopia: agents base their decisions on the information they have about the current strategic environment.

3.1 Revision Protocols

ρ	revision protocol
$\rho^F : X \to \mathbb{R}^{n\times n}_+$	the revision protocol for game F
$\rho^F_{ij}(x)$	conditional switch rate

A population game F, a revision protocol ρ, and a finite population size N together define a stochastic evolutionary process on the discrete grid $\mathcal{X}^N = X \cap \frac{1}{N}\mathbb{Z}^n$.

Each agent has a rate R stochastic alarm clock.

The clock's ring signals the arrival of a revision opportunity for the clock's owner.

If he is playing $i \in S$, he switches to strategy $j \neq i$ with probability $\frac{1}{R}\rho^F_{ij}(x)$;
he continues to play strategy i with probability $1 - \frac{1}{R}\sum_{j\neq i}\rho^F_{ij}(x)$.

In general $\rho^F_{ii}(x)$ has no meaning.

It does if $\sum_{j\in S}\rho^F_{ij}(x) = 1$.

Then we set $R = 1$ and call $\rho^F_{ij}(x)$ a conditional switch probability.

3.2 Information Requirements for Revision Protocols

Reactive protocols only depend on the game by way of the current payoff:

$$\rho^F(x) = \rho(F(x), x).$$

Most protocols in the literature are of this form.

The remaining protocols are called prospective.

They require agents to know enough about the payoff functions to engage in counterfactual reasoning.

Protocols can also be distinguished by the amount and types of data about the current strategic environment that they require.

A third distinction separates continuous and discontinuous protocols.

3.3 The Stochastic Evolutionary Process and Mean Dynamics

The game F, the protocol ρ, and a finite population size N
define a Markov process $\{X_t^N\}_{t\geq 0}$
on the finite state space $\mathcal{X}^N = X \cap \frac{1}{N}\mathbb{Z}^n$.

Since each of the N agents receives revision opportunities at rate R, revision opportunities arrive in the population as a whole at rate NR.

Let τ_k denote the arrival time of the kth revision opportunity.

Then the transition law of the process $\{X_t^N\}$ is

$$\mathbb{P}\left(X^N_{\tau_{k+1}} = y \,\middle|\, X^N_{\tau_k} = x\right) = \begin{cases} \dfrac{x_i \rho_{ij}^F(x)}{R} & \text{if } y = x + \frac{1}{N}(e_j - e_i),\ j \neq i, \\ 1 - \displaystyle\sum_{i\in S}\sum_{j\neq i} \frac{x_i \rho_{ij}^F(x)}{R} & \text{if } y = x, \\ 0 & \text{otherwise.} \end{cases}$$

The mean dynamic

We consider the behavior of $\{X_t^N\}$ over $[0, T]$ as N grows large.

Over the next dt time units, starting from state x:

the expected number of opportunities arriving during is $NR\,dt$.

the expected number received by current strategy i players is $Nx_i\,R\,dt$.

the expected number of these that lead to switches to strategy j is $Nx_i\,\rho_{ij}^F(x)\,dt$.

$\therefore$ the expected change in the proportion of agents using strategy i is

$$\left(\sum_{j\in S} x_j\rho_{ji}^F(x) - x_i\sum_{j\in S}\rho_{ij}^F(x)\right)dt.$$

The mean dynamic induced by F and ρ is thus

$$\dot{x} = V^F(x), \quad \text{where } V_i^F(x) = \sum_{j\in S} x_j\rho_{ji}^F(x) - x_i\sum_{j\in S}\rho_{ij}^F(x). \tag{3}$$

Example (Pairwise proportional imitation and the replicator dynamic).

$$\rho_{ij}^{F}(x) = \rho_{ij}(F(x), x) = x_j[F_j(x) - F_i(x)]_+. \tag{4}$$

An agent who receives a revision opportunity chooses an opponent at random.

This opponent is a strategy j player with probability x_j.

The agent imitates the opponent only if the opponent's payoff is higher than his own, doing so with probability proportional to the payoff difference.

$$
\begin{aligned}
\dot{x}_i &= \sum_{j \in S} x_j x_i [F_i(x) - F_j(x)]_+ - x_i \sum_{j \in S} x_j [F_j(x) - F_i(x)]_+ \\
&= x_i \sum_{j \in S} x_j (F_i(x) - F_j(x)) \\
&= x_i \left(F_i(x) - \sum_{j \in S} x_j F_j(x) \right).
\end{aligned}
$$

This is the replicator dynamic of Taylor and Jonker (1978), the best known dynamic in evolutionary game theory.

After 3.3: Ordinary differential equations

A continuous vector field $V\colon \mathbb{R}^n \to \mathbb{R}^n$ defines an ordinary differential equation on $\mathbb{R}^n$:

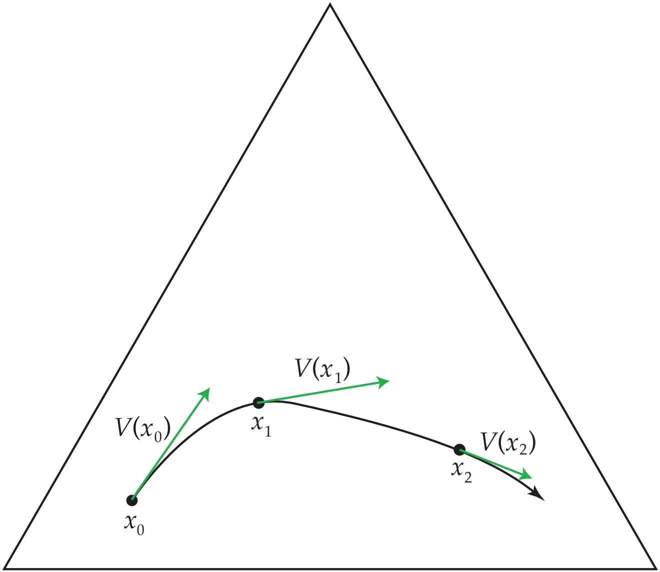
$$\tfrac{\mathrm{d}}{\mathrm{d}t} x_t = V(x_t).$$

Often we write $\dot{x}_t$ for $\frac{\mathrm{d}}{\mathrm{d}t} x_t$; we also express the previous equation as

(D) $\qquad \dot{x} = V(x).$

When the current state is x_t, the current velocity of state is $V(x_t)$. (Show picture.)

The trajectory $\{x_t\}_{t \in I}$ is a solution to (D) if $\dot{x}_t = V(x_t)$ at all times t in the interval I.



$V(x_0)$
$V(x_1)$
$V(x_2)$
x_0
x_1
x_2

Example: $\dot{x} = ax$

We call $f\colon O \to \mathbb{R}^n$ Lipschitz continuous if there exists a scalar K such that

$$\left|f(x) - f(y)\right| \leq K\left|x - y\right| \text{ for all } x, y \in O.$$

Theorem (The Picard-Lindelöf Theorem)**.** *Let $V\colon O \to \mathbb{R}^n$ be Lipschitz continuous. Then for each $\xi \in O$, there exists a scalar $T > 0$ and a unique trajectory $x\colon (-T, T) \to O$ with $x_0 = \xi$ such that $\{x_t\}$ is a solution to* (D).

Theorem ((Forward) invariance on compact convex sets)**.**

Let $C \subset \mathbb{R}^n$ be a compact convex set, and let $V \colon C \to \mathbb{R}^n$ be Lipschitz continuous.

(*i*) *Suppose that $V(\hat{x}) \in TC(\hat{x})$ for all $\hat{x} \in C$. Then for each $\xi \in C$, there exists a unique $x \colon [0, \infty) \to C$ with $x_0 = \xi$ that solves* (D)*.*

(*ii*) *Suppose that $V(\hat{x}) \in TC(\hat{x}) \cap (-TC(\hat{x}))$ for all $\hat{x} \in C$. Then for each $\xi \in C$, there exists a unique $x \colon (-\infty, \infty) \to C$ with $x_0 = \xi$ that solves* (D)*.*

The semiflow $\phi\colon [0,\infty)\times C \to C$ generated by (D) is defined by $\phi_t(\xi) = x_t$, where $\{x_t\}_{t\geq 0}$ is the solution to (D) with initial condition $x_0 = \xi$.

If we fix $\xi \in C$ and vary t, then $\{\phi_t(\xi)\}_{t\in[0,\infty)}$ is the solution orbit of (D) through initial condition ξ.

Note that ϕ satisfies the group property $\phi_t(\phi_s(\xi)) = \phi_{s+t}(\xi)$.

If we instead fix t and vary ξ, then $\{\phi_t(\xi)\}_{\xi\in C'}$ describes the positions at time t of solutions to (D) with initial conditions in $C' \subseteq C$.

Theorem (Continuity of solutions in initial conditions)**.**

Suppose that $V\colon C \to \mathbb{R}^n$ is Lipschitz continuous with Lipschitz constant K.

Let ϕ be the semiflow of (D)*, and fix $t \in [0,\infty)$.*

Then $\phi_t(\cdot)$ is Lipschitz continuous with Lipschitz constant $\mathrm{e}^{K|t|}$:
for all $\xi, \chi \in C$, we have that $\left|\phi_t(\xi) - \phi_t(\chi)\right| \leq |\xi - \chi|\,\mathrm{e}^{K|t|}$.

3.4 Finite Horizon Deterministic Approximation

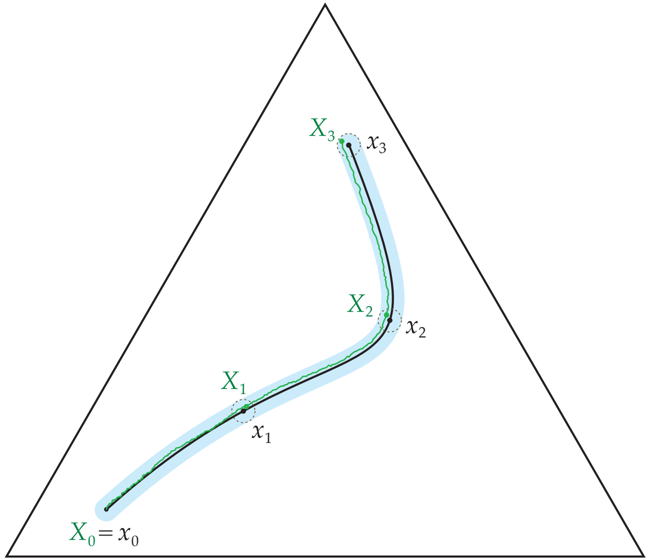
Theorem. *Suppose that the mean dynamic V^F is Lipschitz continuous.*

Let the initial conditions $X_0^N = x_0^N$ converge to state $x_0 \in X$.

Let $\{x_t\}_{t\geq 0}$ be the solution to the mean dynamic (3) *starting from x_0.*

Then for all $T < \infty$ and $\varepsilon > 0$,

$$\lim_{N\to\infty} \mathbb{P}\left(\sup_{t\in[0,T]} \left| X_t^N - x_t \right| < \varepsilon \right) = 1. \tag{5}$$



X_3
x_3
X_2
x_2
X_1
x_1
$X_0 = x_0$

4. Deterministic Evolutionary Dynamics

4.1 Definition

Let $\mathcal{F}$ be a set of population games $F\colon X \to \mathbb{R}^n$ (with fixed number of strategies n).

Let $\mathcal{D}$ be the set of Lipschitz continuous ODEs $\dot{x} = V(x)$ on the simplex X, where $V\colon X \to \mathbb{R}^n$ satisfies $V(x) \in TX(x)$ for all $x \in X$.

A map that assigns each game $F \in \mathcal{F}$ a differential equation in $\mathcal{D}$ is called a deterministic evolutionary dynamic.

Every well-behaved revision protocol implicitly defines a deterministic evolutionary dynamic.

4.2 Incentives and Aggregate Behavior

We introduce conditions that relate the evolution of aggregate behavior under the dynamics to the incentives in the underlying game.

The two most important conditions are:

(PC) *Positive correlation* $V^F(x) \neq \mathbf{0} \;\Rightarrow\; V^F(x)'F(x) > 0.$

(NS) *Nash stationarity* $V^F(x) = \mathbf{0} \;\Leftrightarrow\; x \in NE(F).$

(PC) *Positive correlation* $V^F(x) \neq \mathbf{0} \;\Rightarrow\; V^F(x)'F(x) > 0.$

Game-theoretic interpretation:

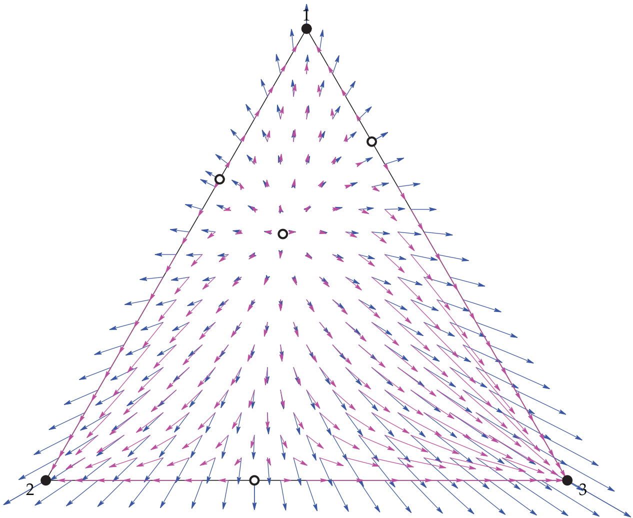
Requires a positive correlation between growth rates and payoffs under the uniform probability distribution on strategies:

$$\mathbb{E}(V^F(x)) = \sum_{k\in S} \tfrac{1}{n} V_k^F(x) = 0, \text{ so that}$$

$$\mathrm{Cov}(V^F(x), F(x)) = \mathbb{E}(V^F(x)\, F(x)) - \mathbb{E}(V^F(x))\, \mathbb{E}(F(x)) = \tfrac{1}{n} V^F(x)' F(x).$$

Geometric interpretation:

If the growth rate vector $V^F(x)$ is nonzero,
it forms an acute angle with the payoff vector $F(x)$.



(NS) *Nash stationarity* $V^F(x) = \mathbf{0} \;\Leftrightarrow\; x \in NE(F).$

Interpretation.

For the ($\Leftarrow$) direction, we have

Proposition. *If V^F satisfies* (PC), *then $x \in NE(F)$ implies that $V^F(x) = \mathbf{0}$.*

5. Families of Evolutionary Dynamics

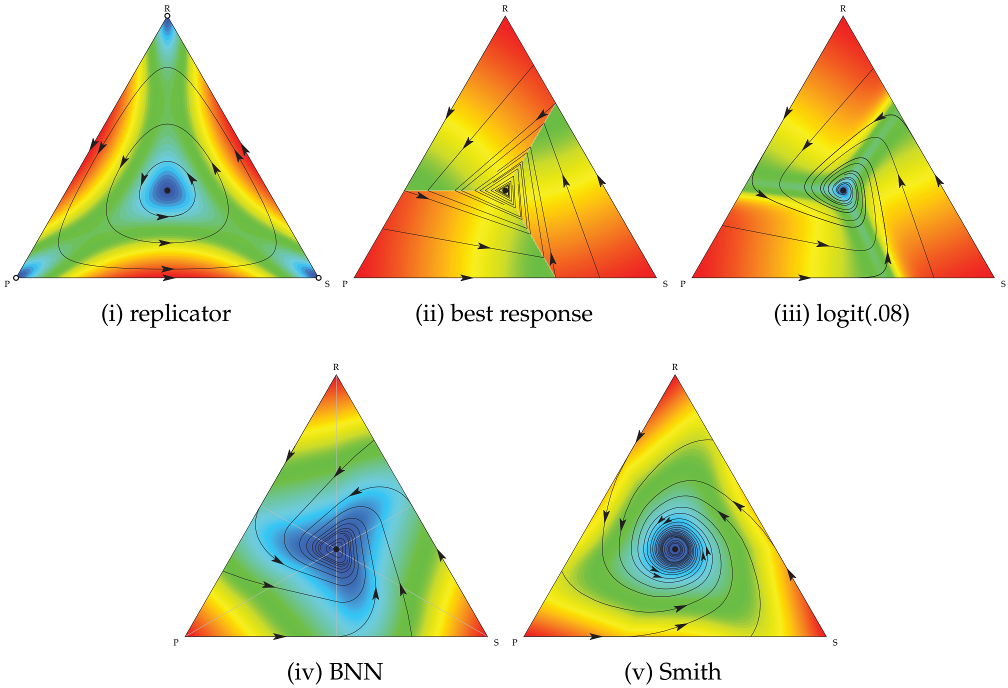
The five basic protocols and dynamics:

All of the protocols are reactive: $\rho_{ij}^{F}(x) = \rho_{ij}(F(x), x)$.

$\bar{F}(x) = \sum_{i \in S} x_i F_i(x)$	population average payoff
$\hat{F}_i(x) = F_i(x) - \bar{F}(x)$	excess payoff to strategy i
$M(\pi) = \underset{y \in X}{\operatorname{argmax}}\ y'\pi$	(mixed) maximizer correspondence

Revision protocol	Mean dynamic	Name
$\rho_{ij} = x_j[\pi_j - \pi_i]_+$	$\dot{x}_i = x_i \hat{F}_i(x)$	replicator
$\rho_{i\bullet} = M(\pi)$	$\dot{x} \in M(F(x)) - x$	best response
$\rho_{ij} = \dfrac{\exp(\eta^{-1}\pi_j)}{\sum_{k\in S}\exp(\eta^{-1}\pi_k)}$	$\dot{x}_i = \dfrac{\exp(\eta^{-1}F_i(x))}{\sum_{k\in S}\exp(\eta^{-1}F_k(x))} - x_i$	logit
$\rho_{ij} = [\pi_j - \sum_{k\in S} x_k\pi_k]_+$	$\dot{x}_i = [\hat{F}_i(x)]_+ - x_i \sum\limits_{j\in S}[\hat{F}_j(x)]_+$	BNN
$\rho_{ij} = [\pi_j - \pi_i]_+$	$\dot{x}_i = \sum\limits_{j\in S} x_j[F_i(x) - F_j(x)]_+ - x_i \sum\limits_{j\in S}[F_j(x) - F_i(x)]_+$	Smith

Table 1: Five basic deterministic dynamics.



The basic dynamics in standard RPS. Red is fastest, blue is slowest.

5.1 Imitative Dynamics

5.1.1 Definition

The Lipschitz continuous revision protocol ρ is an imitative protocol if

(6a) $\rho_{ij}(\pi, x) = x_j r_{ij}(F(x), x),$ where

(6b) $\pi_j \geq \pi_i \iff [r_{kj}(\pi, x) - r_{jk}(\pi, x) \geq r_{ki}(\pi, x) - r_{ik}(\pi, x)$ for all $i, j, k \in S]$.

The values of r_{ij} are called conditional imitation rates.

Condition (6b) is called net monotonicity of conditional imitation rates.

The dynamics generated by protocol satisfying (6) are called imitative dynamics:

(7) $$\dot{x}_i = x_i \sum_{j \in S} x_j \Big(r_{ji}(F(x), x) - r_{ij}(F(x), x)\Big).$$

Formula	Restriction	Interpretation
$\rho_{ij}(\pi, x) = x_j\, \phi(\pi_j - \pi_i)$	$\mathrm{sgn}(\phi(d)) = \mathrm{sgn}([d]_+)$	imitation via pairwise comparisons
$\rho_{ij}(\pi, x) = a(\pi_i)\, x_j$	a decreasing	imitation driven by dissatisfaction
$\rho_{ij}(\pi, x) = x_j\, c(\pi_j)$	c increasing	imitation of success
$\rho_{ij}(\pi, x) = \dfrac{x_j\, w(\pi_j)}{\sum_{k\in S} x_k\, w(\pi_k)}$	w increasing	imitation of success with repeated sampling

Table 2: Some specifications of imitative revision protocols.

5.1.2 Examples

We now consider some important instances of these protocols and the dynamics they induce.

Example (The replicator dynamic).

$$\dot{x}_i = x_i \hat{F}_i(x). \tag{8}$$

From imitation via pairwise comparisons: $\rho_{ij}(\pi, x) = x_j[\pi_j - \pi_i]_+$.

From imitation driven by dissatisfaction: $\rho_{ij}(\pi, x) = (K - \pi_i)x_j$

From imitation of success: $\rho_{ij}(\pi, x) = x_j(\pi_j - K)$.

Example (The Maynard Smith replicator dynamic).

Assume imitation of success with repeated sampling: $\rho_{ij}(\pi, x) = \dfrac{x_j\, w(\pi_j)}{\sum_{k \in S} x_k\, w(\pi_k)}$.

Suppose payoffs are positive, and let $w(\pi_j) = \pi_j$.

$$\text{(9)} \qquad \dot{x}_i = \frac{x_i\, \hat{F}_i(x)}{\bar{F}(x)}.$$

Example (The imitative logit dynamic).

Assume imitation of success with repeated sampling: $\rho_{ij}(\pi, x) = \dfrac{x_j\, w(\pi_j)}{\sum_{k\in S} x_k\, w(\pi_k)}$.

Let $w(\pi_j) = \exp(\eta^{-1}\pi_j)$ with noise level $\eta > 0$.

$$\dot{x}_i = \frac{x_i \exp(\eta^{-1} F_i(x))}{\sum_{k\in S} x_k \exp(\eta^{-1} F_k(x))} - x_i.$$

Before 5.1.3: Extinction and invariance under imitative dynamics

(7) $$\dot{x}_i = x_i \sum_{j \in S} x_j \Big(r_{ji}(F(x), x) - r_{ij}(F(x), x) \Big).$$

(⇒) imitative dynamics satisfy extinction:

(†) If $x_i = 0$, then $V_i(x) = 0$.

(⇒) $V(x) \in TX(x) \cap (-TX(x))$

Proposition (Forward and backward invariance)**.**

Every imitative dynamic admits a unique solution trajectory in $\mathcal{T}_{(-\infty,\infty)} = \{x\colon (-\infty,\infty) \to X \mid x \textit{ is continuous}\}$ *from every initial condition in* X*.*

Theorem (Support invariance)**.**

If $\{x_t\}$ *is a solution trajectory of an imitative dynamic, then the sign of component* $(x_t)_i$ *is independent of* $t \in (-\infty,\infty)$*.*

5.1.3 Basic properties

(10) $\dot{x}_i = V_i(x) = x_i G_i(x),$ where $G_i(x) = \sum_{k \in S} x_k \left(r_{ki}(F(x), x) - r_{ik}(F(x), x)\right).$

If strategy $i \in S$ is in use, then $G_i(x) = V_i(x)/x_i$ is the percentage growth rate of the number of agents using i.

Condition (6b) implies monotonicity of percentage growth rates

(11) $G_i(x) \geq G_j(x)$ if and only if $F_i(x) \geq F_j(x)$,

This property, a strong restriction on percentage growth rates, implies positive correlation (PC), a weak restriction on absolute growth rates.

(PC) $V^F(x) \neq \mathbf{0} \;\Rightarrow\; V^F(x)' F(x) > 0.$

With 5.1.3: Monotone percentage growth rates and positive correlation

Theorem. *All imitative dynamics satisfy positive correlation* (PC).

With 5.1.3: Rest points and restricted equilibria

Recall the definition of Nash equilibrium:

$$NE(F) = \{x \in X \colon x_i > 0 \Rightarrow F_i(x) = \max_{j \in S} F_j(x)\}.$$

Define the set of restricted equilibria of F by

$$RE(F) = \{x \in X \colon x_i > 0 \Rightarrow F_i(x) = \max_{j \in S : x_j > 0} F_j(x)\}.$$

In words, x is a restricted equilibrium of F if it is a Nash equilibrium of a restricted version of F in which only strategies in the support of x can be played.

Theorem. *If $\dot{x} = V^F(x)$ is an imitative dynamic, then $RP(V^F) = RE(F)$.*

Non-Nash rest points of imitative dynamics
are not natural predictions of play:
they cannot be locally stable,
nor can they be approached by any interior solution trajectory.

Even so, continuous dynamics move slowly near rest points,
so escape from the vicinity of non-Nash rest points is necessarily slow.

5.1.4 Inflow-outflow symmetry

Compare the general equation for mean dynamics with that for imitative dynamics:

$$\text{(3)} \qquad \dot{x}_i = \sum_{j \in S} x_j \rho^F_{ji}(x) - x_i \sum_{j \in S} \rho^F_{ij}(x),$$

$$\text{(7)} \qquad \dot{x}_i = x_i \sum_{j \in S} x_j \left(r_{ji}(F(x), x) - r_{ij}(F(x), x) \right).$$

The latter exhibits inflow-outflow symmetry:
the rates of switches from j to i and from i to j
are both proportional to both x_i and x_j.

This is the source of various special properties of imitative dynamics

Another dynamic satisfying inflow-ouflow symmetry is the projection dynamic

It is defined on the interior of the simplex by

$$\dot{x} = \Phi F(x). \tag{12}$$

It is defined in general by

$$\dot{x} = \mathrm{Proj}_{TX(x)}(F(x)),$$

where $\mathrm{Proj}_{TX(x)}$ represents the closest point projection onto $TX(x)$.

5.2 The Best Response Dynamic and Related Dynamics

5.2.1 Target protocols and target dynamics

Under a *target protocol*, an agent's conditional switch rates do not depend on his current strategy.

Protocols with this feature have identical rows:
$\rho^F_{ij}(x) = \rho^F_{\hat{\imath}j}(x)$ for all $x \in X$ and $i, \hat{\imath}, j \in S$.

In this case, we write $\tau^F \equiv \rho^F_{i\cdot}$.

Target protocols generate mean dynamics of the form

$$\text{(13)} \qquad \dot{x}_i = \tau^F_i(x) - x_i \sum_{j \in S} \tau^F_j(x),$$

which we call target dynamics.

$$(13) \qquad \dot{x}_i = \tau_i^F(x) - x_i \sum_{j \in S} \tau_j^F(x),$$

If $\lambda^F(x) = \sum_{j \in S} \tau_j^F(x) \neq 0$, define $\sigma^F(x) \in X$ by $\sigma_j^F(x) = \dfrac{\tau_j^F(x)}{\lambda^F(x)}$.

Then (13) becomes

$$(14) \qquad \dot{x} = \lambda^F(x)(\sigma^F(x) - x).$$

Thus the state moves from its current position x
toward the target state $\sigma^F(x)$ at rate $\lambda^F(x)$.

If $\sum_{j \in S} \tau_j^F(x) \equiv 1$, then the τ_j^F are conditional switch probabilities.
In this case, we write σ^F in place of τ^F, and (14) becomes

$$(15) \qquad \dot{x} = \sigma^F(x) - x.$$

5.2.2 The best response dynamic

The best response protocol:

$$\text{(16a)} \quad \sigma(\pi) = M(\pi), \quad \text{where}$$

$$\text{(16b)} \quad M(\pi) = \operatorname*{argmax}_{y \in X} y'\pi,$$

is the (mixed) maximizer correspondence.

Substituting into (15) yields the best response dynamic

$$\text{(17)} \qquad \dot{x} \in M(F(x)) - x.$$

Equivalently,

$$\dot{x} \in B^F(x) - x,$$

where $B^F = M \circ F$ is the (mixed) best response correspondence for F.

Since M is set-valued and discontinuous, this dynamic is a differential inclusion.

Thus, the basic results on existence and uniqueness of solutions and on deterministic approximation do not apply.

Fortunately, versions of both of these results are available for the current setting.

M is a convex-valued and upper hemicontinuous correspondence.

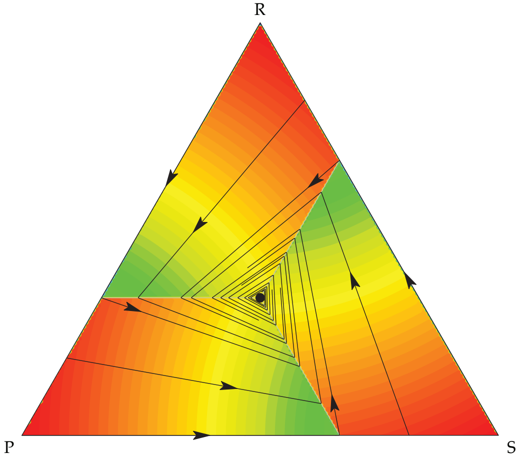
($\Rightarrow$) from every initial condition, there exists a Carathéodory solution:
a Lipschitz continuous trajectory $\{x_t\}_{t\geq 0}$ that satisfies $\dot{x}_t \in V(x_t)$ for almost all $t \geq 0$.

Solutions are generally not unique.

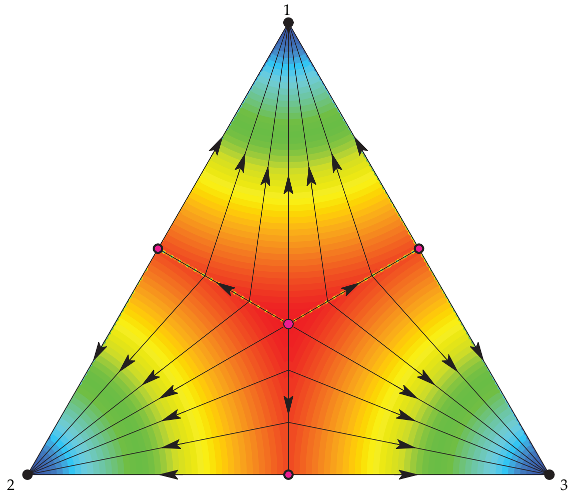
But in regions where the best response is unique, solutions take a simple form:

$$B^F(x) = \{i\} \quad \Rightarrow \quad \dot{x} = e_i - x \quad \Rightarrow \quad x_t = (1 - \mathrm{e}^{-t})\, e_i + \mathrm{e}^{-t}\, x_0.$$

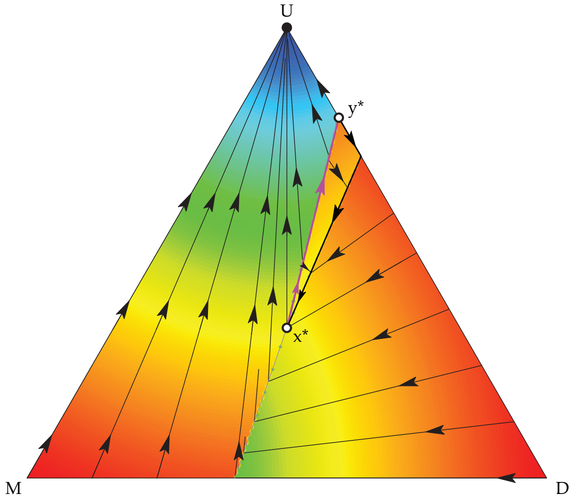
With 5.2.2: Construction of solutions of the best response dynamic



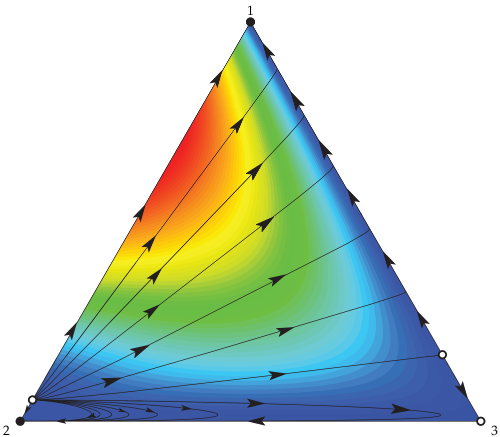
$$\text{Standard RPS, } A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}.$$



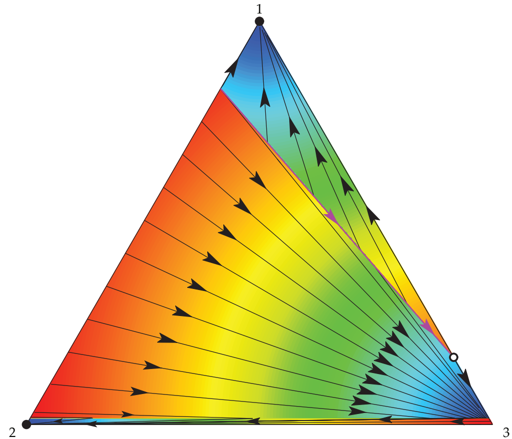
Pure Coordination, $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.



$$\text{Zeeman's game, } A = \begin{pmatrix} 0 & 6 & -4 \\ -3 & 0 & 5 \\ -1 & 3 & 0 \end{pmatrix}.$$



(i) replicator



(ii) best response

The Golman-Page game, $A = \begin{pmatrix} 1 & -k & -\frac{1}{k} \\ 2-k^3 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix}$.

$$(17) \qquad \dot{x} \in M(F(x)) - x.$$

The best response dynamic satisfies versions of (PC) and (NS) suitable for differential inclusions:

If $y \in M(F(x))$ is any best response to x, we have

$$(y - x)'F(x) = \max_{j \in S} F_j(x) - \bar{F}(x) = \max_{j \in S} \hat{F}_j(x) \geq 0,$$

with equality only when $x \in NE(F)$ (as we shall see).

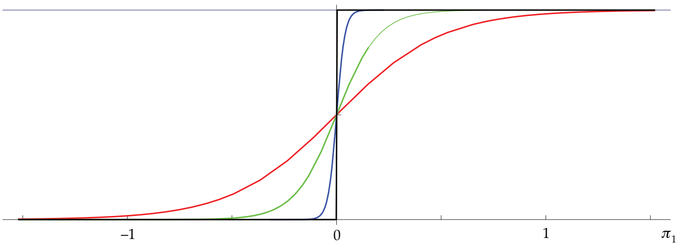
Clearly, $\mathbf{0} \in M(F(x)) - x \;\Leftrightarrow\; x \in NE(F)$.

5.2.3 Perturbed best response dynamics

Perturbed best response protocols are target protocols of the form

$$\sigma(\pi) = \tilde{M}(\pi), \tag{18}$$

the perturbed best response function $\tilde{M}\colon \mathbb{R}^n \to \operatorname{int}(X)$ is a smooth approximation of M.



Such protocols (under restrictions to come) generate perturbed best response dynamics:

$$\dot{x} = \tilde{M}(F(x)) - x. \tag{23}$$

$\tilde{M}$ is defined most conveniently in terms of a deterministic perturbation $v\colon \operatorname{int}(X) \to \mathbb{R}$ of the payoff to each mixed strategy:

$$\text{(19a)} \quad \tilde{M}^{v}(\pi) = \underset{y \in \operatorname{int}(X)}{\operatorname{argmax}} \left(y'\pi - v(y)\right), \quad \text{where}$$

$$\text{(19b)} \quad z'\nabla^2 v(y) z > 0 \text{ for all } z \in TX \text{ and } y \in \operatorname{int}(X), \text{ and}$$

$$\text{(19c)} \quad \lim_{k\to\infty} y_k \in \operatorname{bd}(X) \;\Rightarrow\; \lim_{k\to\infty} \left|\nabla v(y_k)\right| = \infty.$$

It is more natural to define $\tilde{M}$ using stochastic perturbations of the payoff of each pure strategy:

$$\text{(21)} \quad \tilde{M}^{\varepsilon}_{i}(\pi) = \mathbb{P}\left(i = \underset{j \in S}{\operatorname{argmax}}\, \pi_j + \varepsilon_j\right),$$

where ε is a random vector that admits a positive density function that is smooth enough that $\tilde{M}$ is continuously differentiable (e.g., if the ε_i are independent with bounded densities).

With 5.2.3: More on perturbed best response dynamics

Perturbed optimization: a representation theorem

(19) $$\tilde{M}^{v}(\pi) = \operatorname*{argmax}_{y \in \operatorname{int}(X)} (y'\pi - v(y))$$

(21) $$\tilde{M}_i^{\varepsilon}(\pi) = \mathbb{P}\left(i = \operatorname*{argmax}_{j \in S} \pi_j + \varepsilon_j \right)$$

In both cases, $\tilde{M}(\pi) = \tilde{M}(\Phi\pi)$ for all $\pi \in \mathbb{R}^n$,
so we can focus on the restriction $\bar{M} \colon \mathbb{R}_0^n \to \operatorname{int}(X)$. $\quad$ ($\mathbb{R}_0^n = TX = \{z \in \mathbb{R}^n \colon z'1 = 0\}$.)

Theorem. *Let $\tilde{M}$ be a perturbed maximizer function defined in terms of an admissible stochastic perturbation ε via equation* (21).

Then $\tilde{M}$ satisfies equation (19) *for some admissible deterministic perturbation v.*

In fact, $\bar{M} = \tilde{M}|_{\mathbb{R}_0^n}$ and ∇v are invertible, and $\bar{M} = (\nabla v)^{-1}$.

Logit choice and the logit dynamic

The logit choice function with noise level η:

$$(22) \qquad \tilde{M}_i(\pi) = \frac{\exp(\eta^{-1}\pi_i)}{\sum_{j\in S} \exp(\eta^{-1}\pi_j)}.$$

The logit dynamic with noise level η:

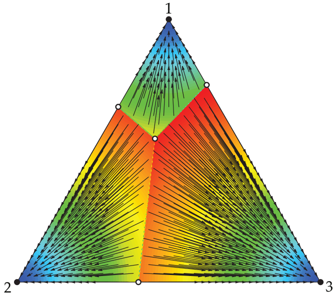
$$(\mathrm{L}) \qquad \dot{x}_i = \frac{\exp(\eta^{-1}F_i(x))}{\sum_{j\in S} \exp(\eta^{-1}F_j(x))} - x_i \,.$$

Rest points are called logit equilibria (or quantal response equilibria).

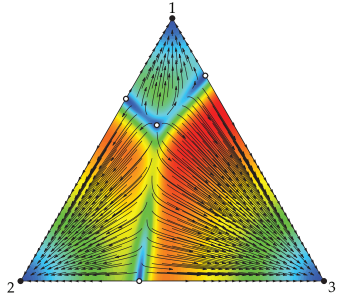
Stochastic derivation: ε_i are i.i.d. double exponential:
$\mathbb{P}(\varepsilon_i \leq c) = \exp(-\exp(-\eta^{-1}c - \gamma))$.

Deterministic derivation: v is negated entropy: $v(y) = \eta \sum_{j\in S} y_j \log y_j$.

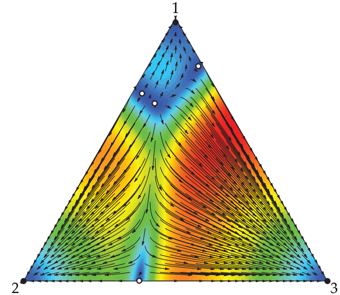
Example: 123 Coordination



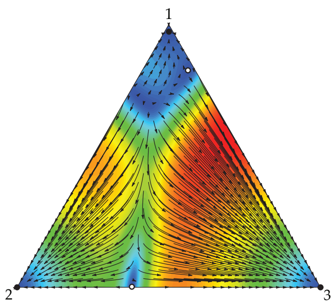
(i) $\eta = .001$



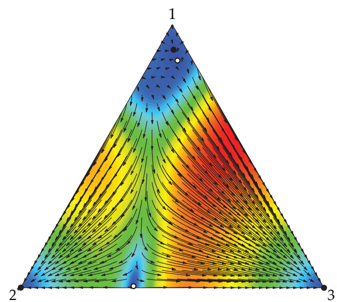
(ii) $\eta = .1$



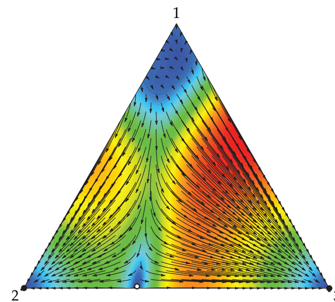
(iii) $\eta = .2$



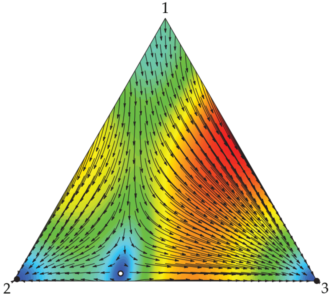
(iv) $\eta = .22$



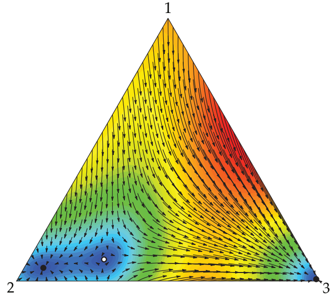
(v) $\eta = .27$



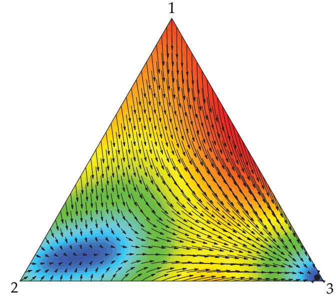
(vi) $\eta = .28$



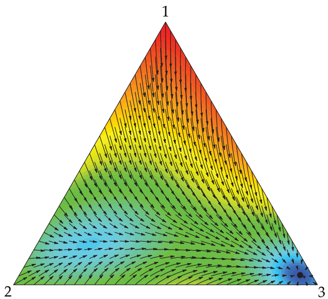
(vii) $\eta = .4$



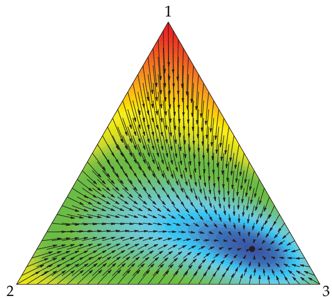
(viii) $\eta = .6$



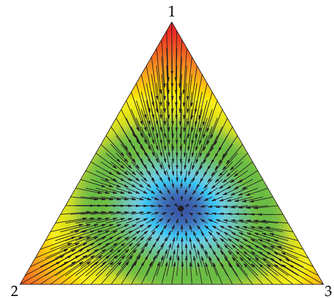
(ix) $\eta = .68$



(x) $\eta = .85$



(xi) $\eta = 1.2$



(xii) $\eta = 3$

Perturbed incentive properties via virtual payoffs

It is enough to consider deterministic perturbations.

Define the set of perturbed equilibria of the pair (F, v) by

$$PE(F, v) = \{x \in X \colon x = \tilde{M}(F(x))\}.$$

Observation. *All perturbed best response dynamics satisfy* perturbed stationarity*:*

$$V(x) = \mathbf{0} \text{ if and only if } x \in PE(F, v).$$

Define the virtual payoffs $\tilde{F} : \mathrm{int}(X) \to \mathbb{R}^n$ for the pair (F, v) by

$$\tilde{F}(x) = F(x) - \nabla v(x).$$

Intuitively, strategies that very few agents use have high virtual payoffs.

Theorem. *Let $x \in X$ be a social state. Then $x \in PE(F, v)$ if and only if $\Phi\tilde{F}(x) = \mathbf{0}$.*

Define virtual positive correlation:

(24) $\quad V(x) \neq \mathbf{0}$ implies that $V(x)'\tilde{F}(x) > 0$.

Theorem. *All perturbed best response dynamics satisfy virtual positive correlation* (24).

5.3 Excess Payoff and Pairwise Comparison Dynamics

Can Nash equilibrium can be interpreted as stationary behavior among agents who employ simple myopic rules? (Compare Nash (1950)!)

Imitative dynamics fail (NS).

The best response dynamic is discontinuous.

Perturbed best response dynamics only approximately satisfy (NS).

Can we do better?

5.3.1 Excess payoff dynamics

Recall the excess payoff function $\hat{F}\colon X \to \mathbb{R}^n$:

$$\hat{F}_i(x) = F_i(x) - \bar{F}(x).$$

Clearly, $\hat{F}(x)$ cannot be in $\mathrm{int}(\mathbb{R}^n_-)$.

Proposition. *$x \in NE(F)$ if and only if $\hat{F}(x) \in \mathrm{bd}(\mathbb{R}^n_-)$.*

Explanation:

Let $\mathbb{R}^n_* = \mathbb{R}^n \setminus \mathrm{int}(\mathbb{R}^n_-)$ denote the set of vectors in $\mathbb{R}^n$ with at least one nonnegative component. Then $\mathrm{bd}(\mathbb{R}^n_*) = \mathrm{bd}(\mathbb{R}^n_-)$.

The proposition says that the Nash equilibria are the states at which no strategy receives an above-average payoff.

With 5.3.1: Characterization of Nash equilibrium via excess payoffs

A target protocol τ an excess payoff protocol if it is Lipschitz continuous and satisfies

(25a) $\tau_j(\pi, x) = \tau_j(\hat{\pi}),$ where $\hat{\pi}_i = \pi_i - x'\pi,$ and

(25b) $\hat{\pi} \in \operatorname{int}(\mathbb{R}^n_*) \Rightarrow \tau(\hat{\pi})'\hat{\pi} > 0.$

Condition (25b), acuteness, requires that away from Nash equilibrium, strategies with higher growth rates tend to be those with higher excess payoffs.

These protocols generate the excess payoff dynamics:

(26) $$\dot{x}_i = \tau_i(\hat{F}(x)) - x_i \sum_{j \in S} \tau_j(\hat{F}(x)).$$

Example (The BNN dynamic).

$$\tau_i(\hat{\pi}) = [\hat{\pi}_i]_+, \tag{27}$$

(⇒) the Brown-von Neumann-Nash (BNN) dynamic:

$$\dot{x}_i = [\hat{F}_i(x)]_+ - x_i \sum_{j \in S} [\hat{F}_j(x)]_+. \tag{28}$$

It is not difficult to verify that the BNN dynamic satisfies (PC) and (NS).

In fact, one can show that all excess payoff dynamics satisfies these properties.

Of course, excess payoff dynamics require agents to know excess payoffs!

They thus do not provide a satisfactory foundation for Nash equilibrium prediction.

5.3.2 Pairwise comparison dynamics

Pairwise comparison protocols are Lipschitz continuous protocols $\rho\colon \mathbb{R}^n \times X \to \mathbb{R}_+^{n\times n}$ that satisfy sign preservation:

$$\text{(29)} \qquad \operatorname{sgn}(\rho_{ij}(\pi, x)) = \operatorname{sgn}([\pi_j - \pi_i]_+) \text{ for all } i, j \in S.$$

The resulting evolutionary dynamics (3) are called pairwise comparison dynamics.

Example (The Smith dynamic).

$$\text{(30)} \qquad \rho_{ij}(\pi, x) = [\pi_j - \pi_i]_+,$$

Inserting this into (3) yields the *Smith dynamic*:

$$\text{(31)} \qquad \dot{x}_i = \sum_{j\in S} x_j[F_i(x) - F_j(x)]_+ - x_i \sum_{j\in S} [F_j(x) - F_i(x)]_+.$$

Compare (30) to pairwise proportional imitation: $\rho_{ij}(\pi, x) = x_j[\pi_j - \pi_i]_+$.

With 5.3.2: Analysis of pairwise comparison dynamics

Theorem. *Every pairwise comparison dynamic satisfies* (PC) *and* (NS).

The theorem follows from three lemmas.

Lemma. $x \in NE(F) \Leftrightarrow$ *For all* $i \in S$, $x_i = 0$ *or* $\sum_{j \in S} [F_j(x) - F_i(x)]_+ = 0.$

Proof. Both statements say that each strategy in use at x is optimal. ■

Lemma. $V(x) = \mathbf{0} \Leftrightarrow$ *For all* $i \in S$, $x_i = 0$ *or* $\sum_{j \in S} \rho_{ij}(F(x)) = 0.$

Proof. ($\Leftarrow$) is immediate.

Lemma.

(*i*) $V(x)'F(x) \geq 0.$

(*ii*) $V(x)'F(x) = 0 \Leftrightarrow$ *For all* $i \in S, x_i = 0$ *or* $\sum_{j \in S} \rho_{ij}(F(x))[F_j(x) - F_i(x)]_+ = 0.$

With 5.3.2: Multiple revision protocols and hybrid dynamics

How compatible is imitation with Nash stationarity?

Suppose an agent uses revision protocols ρ^V and ρ^W at intensities a and b.

Then his behavior is described by the new revision protocol $\rho^C = a\rho^V + b\rho^W$.

Since mean dynamics are linear in conditional switch rates,
the mean dynamic for the combined protocol is $C^F = aV^F + bW^F$.

Theorem. *Suppose that V^F satisfies* (PC), *that W^F satisfies* (PC) *and* (NS), *and that $a, b > 0$. Then $C^F = aV^F + bW^F$ also satisfies* (PC) *and* (NS).

Family	Example	Continuity	Data Req.	(PC)	(NS)
imitation	replicator	yes	weak	yes	no
optimization	best response	no	moderate	yes	yes
perturbed optimization	logit	yes	moderate	approx.	approx.
excess payoff	BNN	yes	strong	yes	yes
pairwise comparison	Smith	yes	weak	yes	yes

Table 3: Families of revision protocols and evolutionary dynamics, and their properties.

Remark: Imitation vs. (nonimitative) pairwise comparison.

6. Potential Games

In potential games, all information about incentives can be captured by a scalar-valued function on the set of population states.

Dynamics satisfying positive correlation (PC) ascend this function and converge to its maximizers, which are Nash equilibria of the game.

6.1 Population Games and Full Population Games

Population games are models of multilateral externalities.

What is the effect of adding new agents playing strategy j
on the payoffs of agents currently choosing strategy i?

In principle, it is $\dfrac{\partial F_i}{\partial x_j}$.

But since payoffs are only defined on the simplex,
this partial derivative does not exist.

We therefore consider *full population games*,
in which payoffs are defined on the positive orthant $\mathbb{R}^n_+$.

These describe the payoffs that would arise were the population size to change.

6.2 Definition, Characterization, and Interpretation

Let $F : \mathbb{R}^n_+ \to \mathbb{R}^n$ be a (full) population game.

We call F a potential game if there exists a C^1 function $f : \mathbb{R}^n_+ \to \mathbb{R}$, called a potential function, satisfying

$$\nabla f(x) = F(x) \text{ for all } x \in \mathbb{R}^n_+, \text{ or equivalently}$$
$$\frac{\partial f}{\partial x_i}(x) = F_i(x) \text{ for all } i \in S \text{ and } x \in \mathbb{R}^n_+.$$

If the payoff function F is C^1, then F is a potential game if and only if it satisfies full externality symmetry:

$$DF(x) \text{ is symmetric for all } x \in \mathbb{R}^n_+, \text{ or equivalently}$$
$$\frac{\partial F_i}{\partial x_j}(x) = \frac{\partial F_j}{\partial x_i}(x) \text{ for all } i, j \in S \text{ and } x \in \mathbb{R}^n_+.$$

Intuition:

Suppose that some members of the population switch from strategy i to strategy j, so that the state moves in direction $z = e_j - e_i$.

If these switches improve the payoffs of those who switch, then

$$\frac{\partial f}{\partial z}(x) = \nabla f(x)'z = F(x)'z = F_j(x) - F_i(x) > 0.$$

Thus profitable strategy revisions increase potential.

For more general sorts of adjustment, we have the following simple lemma:

Lemma. *Let F be a potential game with potential function f, and suppose the dynamic V^F satisfies positive correlation* (PC). *Then along any solution trajectory $\{x_t\}$, we have $\frac{\mathrm{d}}{\mathrm{d}t} f(x_t) > 0$ whenever $\dot{x}_t \neq \mathbf{0}$.*

Proof: $\dfrac{\mathrm{d}}{\mathrm{d}t} f(x_t) = \nabla f(x_t)'\dot{x}_t = F(x_t)'V^F(x_t) \geq 0$, with equality only if $V^F(x_t) = \mathbf{0}$.

6.3 Examples

Example (Matching in games with common interests).

$F(x) = Ax$, $A \in R^{n \times n}$ symmetric.

Example (Congestion games).

$$F_i(x) = -\sum_{\ell \in \mathcal{L}_i} c_\ell \left(u_\ell(x) \right).$$

$$F_i(x) = -\sum_{\ell \in \mathcal{L}_i} c_\ell \left(u_\ell(x)\right).$$

$$(\Rightarrow) \quad \frac{\partial F_i}{\partial x_j}(x) = -\sum_{\ell \in \mathcal{L}_i \cap \mathcal{L}_j} c'_\ell(u_\ell(x)) = \frac{\partial F_j}{\partial x_i}(x).$$

In fact, we have

$$f(x) = -\sum_{\ell \in \mathcal{L}} \int_0^{u_\ell(x)} c_\ell(z)\,\mathrm{d}z. \tag{32}$$

In general f differs from the average payoff function,

$$\bar{F}(x) = -\sum_{\ell \in \mathcal{L}} u_\ell(x) c_\ell(u_\ell(x)).$$

But if $c_\ell(u) = a_\ell u^\eta$ for some $\eta \geq 0$, then $f(x) = \frac{1}{\eta+1}\bar{F}(x)$.

With 6.3: Efficiency in homogeneous full potential games

Definition. *We call a full potential game F* homogeneous of degree k *if each of its payoff functions $F_i : \mathbb{R}^n \to \mathbb{R}$ is a homogeneous function of degree k (that is, if $F_i(tx) = t^k F_i(x)$ for all $x \in \mathbb{R}^n$ and $t > 0$), where $k \neq -1$.*

Example: Matching in normal form games with common interests.

Example: Isoelastic congestion games.

Theorem. *The full potential game F is homogeneous of degree $k \neq -1$ if and only if the normalized aggregate payoff function $\frac{1}{k+1}\bar{F}(x)$ is a full potential function for F and is homogeneous of degree $k + 1 \neq 0$.*

Example (Games generated by variable pricing schemes).

Given population game F with average payoff function $\bar{F}$, define a new game $\tilde{F}$ by

$$\tilde{F}_i(x) = F_i(x) + \sum_{j \in S} x_j \frac{\partial F_j}{\partial x_i}(x).$$

Then

$$\frac{\partial \bar{F}}{\partial x_i}(x) = \frac{\partial}{\partial x_i} \sum_{j \in S} x_j F_j(x) = F_i(x) + \sum_{j \in S} x_j \frac{\partial F_j}{\partial x_i}(x) = \tilde{F}_i(x).$$

6.4 Characterization of equilibrium

Consider the problem of maximizing potential over the set of population states:

$$\max f(x) \quad \text{subject to} \quad \sum_{j \in S} x_j = 1 \;\text{ and }\; x_i \geq 0 \;\text{ for all } i \in S.$$

The Lagrangian for this maximization problem is

$$L(x, \mu, \lambda) = f(x) + \mu \left(1 - \sum_{i \in S} x_i \right) + \sum_{i \in S} \lambda_i x_i \, ,$$

so the Kuhn-Tucker first-order necessary conditions for maximization are

$$\text{(33a)} \quad \frac{\partial f}{\partial x_i}(x) = \mu - \lambda_i \quad \text{for all } i \in S,$$

$$\text{(33b)} \quad \lambda_i x_i = 0, \qquad \text{for all } i \in S, \text{ and}$$

$$\text{(33c)} \quad \lambda_i \geq 0 \qquad \text{for all } i \in S.$$

$$\frac{\partial f}{\partial x_i}(x) = \mu - \lambda_i \quad \text{for all } i \in S, \tag{33a}$$

$$\lambda_i x_i = 0, \quad \text{for all } i \in S, \text{ and} \tag{33b}$$

$$\lambda_i \geq 0 \quad \text{for all } i \in S. \tag{33c}$$

Theorem. *Let F be a potential game with potential function f.*

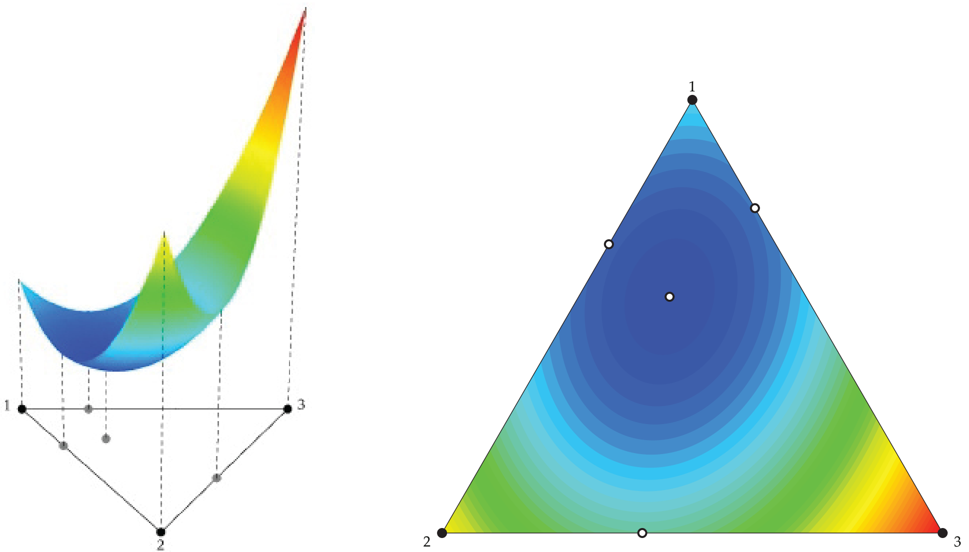
Then x is a Nash equilibrium of F if and only if (x, μ, λ) satisfies (33a)–(33c) *for some $\lambda \in \mathbb{R}^n$ and $\mu \in \mathbb{R}$.*

This result allows a simple proof of existence of Nash equilibrium.

Example (Nash equilibria in a potential game).

$$F^{C3}(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ 2x_2 \\ 3x_3 \end{pmatrix}$$

$$(\Rightarrow) \quad f^{C3}(x) = \frac{1}{2}((x_1)^2 + 2(x_2)^2 + 3(x_3)^2).$$

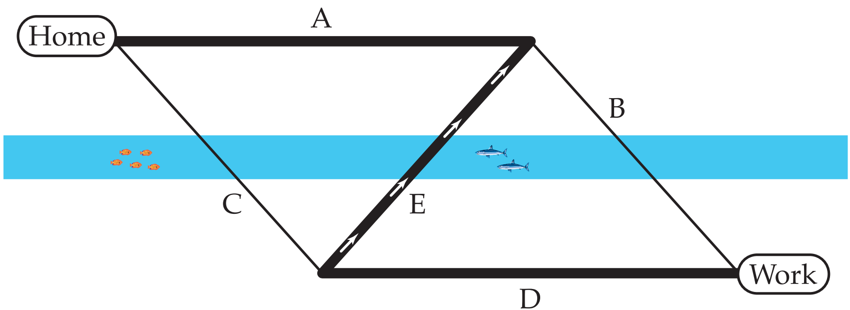


(i) (ii)

After 6.4: Inefficiency and inefficiency bounds in congestion games

Inefficiency and inefficiency bounds ("price of anarchy") in congestion games

Example (Braess's paradox).

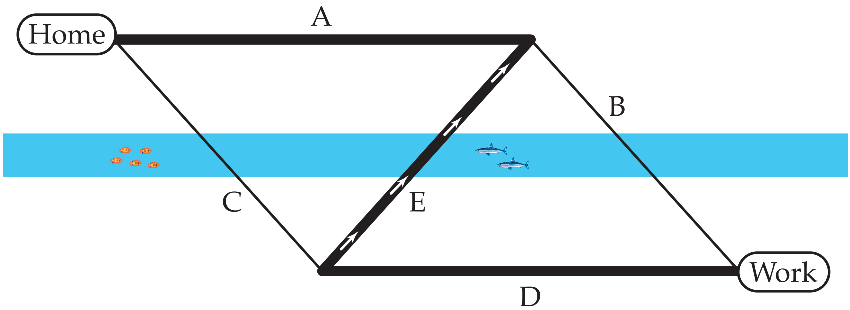


$$c_A(u) = c_D(u) = 4 + 20u$$

$$c_B(u) = c_C(u) = 2 + 30u^2$$

$$c_E(u) = 1 + 20u.$$

Equilibrium before link E is opened: $x^* = (.5, .5)$. Travel time: 23.5.



Equilibrium before link E is opened: $x^* = (.5, .5)$. Travel time: 23.5.

Equilibrium after link E is opened: $x^* = (.4616, .4616, .0768)$. Travel time: 23.93.

Adding a link to the network worsens performance!

This phenomenon is known as Braess's paradox.

A congestion game's inefficiency ratio (or "price of anarchy") is the ratio between the game's equilibrium social cost and its minimal feasible social cost $\bar{C}(x) = -\bar{F}(x)$.

Example. Two parallel links, $c_1(u) = 1$ and $c_2(u) = u$.

In the unique Nash equilibrium, all drivers travel on route 2, creating a social cost of 1.

The efficient state, which minimizes $\bar{C}(x) = x_1 + (x_2)^2$, is $x_{\text{min}} = (\frac{1}{2}, \frac{1}{2})$; it generates a social cost of $\bar{C}(x_{\text{min}}) = \frac{3}{4}$.

Thus, the inefficiency ratio in this game is $\frac{4}{3}$.

Remarkably, this example is the worst case for any network with affine costs.

Theorem.

Let C be a congestion game whose cost functions c_ℓ are nonnegative, nondecreasing, and affine: $c_\ell(u) = a_\ell + b_\ell u$ with $a_\ell, b_\ell \geq 0$.

If $x^* \in NE(C)$ and $x \in X$, then $\bar{C}(x^*) \leq \frac{4}{3}\bar{C}(x)$.

Before 6.5: Stability and recurrence for flows

Let $X \subset \mathbb{R}^n$ be compact.

Let $V\colon X \to \mathbb{R}^n$ be Lipschitz continuous with $V(x) \in TX(x)$ for all $x \in X$. Then

(D) $\quad \dot{x} = V(x)$

has a unique forward solution from every initial condition in X.

Let $\xi \in X$, and let $\{x_t\}_{[0,\infty}$ be the solution to (D) with $x_0 = \xi$.

The *ω-limit* $\omega(\xi)$ is the set of all points that the solution from ξ approaches arbitrarily closely infinitely often:

$$\omega(\xi) = \left\{ y \in X\colon \text{ there exists } \{t_k\}_{k=1}^{\infty} \text{ with } \lim_{k\to\infty} t_k = \infty \text{ such that } \lim_{k\to\infty} x_{t_k} = y \right\}.$$

We write $\Omega = \bigcup_{\xi \in X} \omega(\xi)$ and $\bar{\Omega} = \mathrm{cl}(\Omega)$.

Proposition. (*i*) *$\omega(\xi)$ is non-empty and connected.*

(*ii*) *$\omega(\xi)$ is closed. In fact, $\omega(\xi) = \bigcap_{t \geq 0} \mathrm{cl}(\{x_s : s \geq t\})$, where $\{x_t\}$ solves* (D) *with $x_0 = \xi$.*

(*iii*) *$\omega(\xi)$ is invariant under* (D).

$Y \subseteq X$ is forward invariant under (D) if for each solution $\{x_t\}$ of (D), $x_0 \in Y$ implies that $x_t \in Y$ for all $t > 0$.

Y is backward invariant if $x_0 \in Y$ implies that x_t exists and is in Y for all $t < 0$.

Y is invariant it is both forward and backward invariant.

Let $A \subseteq X$ be a closed set.

$O \subseteq X$ is a neighborhood of A if it is open relative to X and contains A.

A is Lyapunov stable under (D) if for every neighborhood O of A,
there exists a neighborhood O' of A such that every solution $\{x_t\}$
that starts in O' is contained in O (that is, $x_0 \in O' \Rightarrow x_t \in O$ for all $t \geq 0$).

A is attracting if there is a neighborhood Y of A such that every solution that starts in Y converges to A (that is, $x_0 \in Y \Rightarrow \omega(x_0) \subseteq A$).

A is globally attracting if it is attracting with $Y = X$.

A is asymptotically stable if it is Lyapunov stable and attracting.

A is globally asymptotically stable if it is Lyapunov stable and globally attracting.

Example: Attracting but not Lyapunov stable.

Example: Asymptotically stable but not backward invariant.

Before 6.5: Lyapunov functions

A Lyapunov function is a function whose value changes monotonically along solution trajectories of (D).

If monotonicity is strict whenever (D) is not at rest, it is a strict Lyapunov function.

Lemma.

Suppose that the function $L : Y \to \mathbb{R}$ and the trajectory $\{x_t\}_{t \geq 0}$ are Lipschitz continuous.

(*i*) *If $\dot{L}(x_t) \leq 0$ for almost all $t \geq 0$, then the map $t \mapsto L(x_t)$ is nonincreasing.*

(*ii*) *If in addition $\dot{L}(x_s) < 0$, then $L(x_t) < L(x_s)$ for all $t > s$.*

Call the (relatively) open set $Y \subset X$ inescapable if for each solution $\{x_t\}_{t \geq 0}$ with $x_0 \in Y$, we have that $\mathrm{cl}(\{x_t\}) \cap \mathrm{bd}(Y) = \varnothing$.

Theorem.

Let $Y \subset X$ be relatively open and inescapable under (D).

Let $L\colon Y \to \mathbb{R}$ be C^1, and suppose that $\dot{L}(x) \equiv \nabla L(x)' V(x) \leq 0$ for all $x \in Y$.

Then $\omega(\xi) \subseteq \{x \in Y\colon \dot{L}(x) = 0\}$ for all $\xi \in Y$.

Thus, if $\dot{L}(x) = 0$ implies that $V(x) = \mathbf{0}$, then $\omega(\xi) \subseteq RP(V) \cap Y$.

Theorem.

Let $A \subseteq X$ be closed, and let $Y \subseteq X$ be a neighborhood of A.
Let $L\colon Y \to \mathbb{R}_+$ be C^1 with $L^{-1}(0) = A$.

(*i*) *If $\dot{L}(x) \equiv \nabla L(x)' V(x) \leq 0$ for all $x \in Y - A$, then A is Lyapunov stable under* (D).
(*ii*) *If $\dot{L}(x) < 0$ for all $x \in Y - A$, then A is asymptotically stable under* (D).
(*iii*) *If in* (*ii*) *$Y = X$, then A is globally asymptotically stable under* (D).

With 6.5: Global convergence and local stability in potential games

Global convergence in potential games

Let F be a potential game with potential function f.

If V^F satisfies positive correlation (PC), then f is a strict Lyapunov function:

$$\dot{f}(x_t) = \nabla f(x_t)'\dot{x}_t = F(x_t)'V^F(x_t) \geq 0, \text{with equality only when } V^F(x) = \mathbf{0}.$$

Theorem. *Let F be a potential game, and let $\dot{x} = V^F(x)$ be a Lipschitz continuous evolutionary dynamic for F that satisfies* (PC). *Then $\Omega(V^F) = RP(V^F)$. In particular,*

(*i*) *If V^F is an imitative dynamic, then $\Omega(V^F) = RE(F)$.*

(*ii*) *If V^F is an excess payoff dynamic or a pairwise comparison dynamic, then $\Omega(V^F) = NE(F)$.*

An analogous result holds for the best response dynamic.

For perturbed best response dynamics we have

Theorem. *Let F be a potential game with potential function f, and let $\dot{x} = V^{F,v}(x)$ be the perturbed best response dynamic for F generated by the admissible deterministic perturbation v. Define the* perturbed potential function $\tilde{f}\colon \operatorname{int}(X) \to \mathbb{R}$ *by*

$$\tilde{f}(x) = f(x) - v(x).$$

Then $\tilde{f}$ is a strict Lyapunov function for $V^{F,v}$, and so $\Omega(V^{F,v}) = PE(F, v)$.

Local stability in potential games

Theorem. *Let F be a potential game with potential function f, let V^F be a Lipschitz continuous evolutionary dynamic for F.*

(*i*) *If $A \subseteq NE(F)$ is a local maximizer set of f, and V^F satisfies positive correlation* (PC)*, then A is Lyapunov stable. If in addition V^F satisfies Nash stationarity* (NS) *and A is isolated in $NE(F)$, then A is asymptotically stable.*

(*ii*) *Conversely, if V^F satisfies* (PC) *and* (NS) *and $A \subseteq NE(F)$ is a smoothly connected asymptotically stable set, then A is a local maximizer set of f and is isolated in $NE(F)$.*

With 6.6: Local stability of strict equilibrium

V^F satisfies strong positive correlation in $Y \subseteq X$ if

(SPC) There exists a $c > 0$ such that for all $x \in Y$,

$$V^F(x) \neq \mathbf{0} \text{ implies that } \operatorname{Corr}(V^F(x), F(x)) = \frac{V^F(x)'\Phi F(x)}{|V^F(x)||\Phi F(x)|} \geq c.$$

That is, the correlation between strategies' growth rates and payoffs, (= the cosine of the angle between the growth rate and excess payoff vectors) is be bounded away from zero on Y.

Theorem. *Let e_k be a strict equilibrium of F.*

Suppose that V^F satisfies strong positive correlation (SPC) *in some neighborhood of e_k in X.*

Define the function $L\colon X \to \mathbb{R}$ by

$$L(x) = (e_k - x)'F(e_k).$$

Then $L(x) \geq 0$, with equality only when $x = e_k$, and there is a neighborhood of e_k on which $\dot{L}(x) \leq 0$, with equality only when $V^F(x) = \mathbf{0}$.

Thus e_k is Lyapunov stable under V^F, and if e_k is an isolated rest point of V^F, e_k is asymptotically stable under V^F.

Explain the construction in a picture.

7. Evolutionarily Stable States and Contractive Games

7.1 Evolutionarily Stable States

Three equivalent definitions. First, an ESS is a (infinitesimal) local invader:

(34) There is a neighborhood O of x such that $(y-x)'F(y) < 0$ for all $y \in O \smallsetminus \{x\}$.

Second, an ESS has a uniform invasion barrier:

(35) There is an $\bar{\varepsilon} > 0$ such that $(y-x)'F(\varepsilon y + (1-\varepsilon)x) < 0$ for all $y \in X \smallsetminus \{x\}$ and $\varepsilon \in (0, \bar{\varepsilon})$.

The third definition is the original one of Maynard Smith and Price.

(36a) x is a Nash equilibrium: $(y-x)'F(x) \leq 0$ for all $y \in X$.

(36b) There is a neighborhood O of x such that for all $y \in O \smallsetminus \{x\}$, $(y-x)'F(x) = 0$ implies that $(y-x)'F(y) < 0$.

Theorem. *The following are equivalent:*

(i) *x satisfies condition* (34).

(ii) *x satisfies condition* (35).

(iii) *x satisfies condition* (36).

Proof.

(34) $\Leftrightarrow$ (35) is on the problem set.

(34) $\Rightarrow$ (36) will be done later.

(36) $\Rightarrow$ (34) is immediate if $x = e_i$ or $x \in \text{int}(X)$; otherwise it is complicated.
(But if $F(x) = Ax$, there is an easier proof that (36) $\Rightarrow$ (35).)

(34) There is a neighborhood O of x such that $(y - x)'F(y) < 0$ for all $y \in O \setminus \{x\}$.

(35) $$\text{There is an } \bar{\varepsilon} > 0 \text{ such that } (y - x)'F(\varepsilon y + (1 - \varepsilon)x) < 0 \quad \text{for all } y \in X \setminus \{x\} \text{ and } \varepsilon \in (0, \bar{\varepsilon}).$$

(36a) x is a Nash equilibrium: $(y - x)'F(x) \leq 0$ for all $y \in X$.

(36b) There is a neighborhood O of x such that for all $y \in O \setminus \{x\}$,
$$(y - x)'F(x) = 0 \text{ implies that } (y - x)'F(y) < 0.$$

We call state x a regular ESS if

(37a) x is a quasistrict equilibrium: $F_i(x) = \bar{F}(x) > F_j(x)$ whenever $x_i > 0$ and $x_j = 0$.

(37b) $z'DF(x)z < 0$ for all $z \in TX \setminus \{\mathbf{0}\}$ such that $z_i = 0$ whenever $x_i = 0$.

7.2 Contractive Games

The population game $F\colon X \to \mathbb{R}^n$ is a contractive game if

$$(y - x)'(F(y) - F(x)) \leq 0 \text{ for all } x, y \in X. \tag{38}$$

If the inequality in condition (38) holds strictly whenever $x \neq y$,
F is strictly contractive,

If this inequality always binds, F is null contractive.

(other names: stable game, negative semidefinite game, monotone vector field.)

$$(y - x)'(F(y) - F(x)) \leq 0 \text{ for all } x, y \in X. \tag{38}$$

Intuition 1: If F is a potential game, (38) says that f is concave.

Intuition 2: Consider the projection dynamic on int(X): $\dot{x} = \Phi F(x)$.

Run from two initial states x_0 and y_0:

$$\frac{\mathrm{d}}{\mathrm{d}t} \left| y_t - x_t \right|^2 = 2(y_t - x_t)'(\dot{y}_t - \dot{x}_t) = 2(y_t - x_t)'(F(y_t) - F(x_t)). \tag{39}$$

Intuition 3: Global version of ESS stability condition:

Proposition.

If F is continuously differentiable, it is contractive
if and only if it satisfies self-defeating externalities:

(40) $\quad DF(x)$ *is negative semidefinite with respect to* TX *for all* $x \in X$.

(40) can be rewritten as

$$\sum_{i \in S} z_i \frac{\partial F_i}{\partial z}(x) \leq 0 \text{ for all } z \in TX \text{ and } x \in X.$$

Improvements in the payoffs of strategies to which revising agents are switching are always exceeded by the improvements in the payoffs of strategies which revising agents are abandoning.

When $z = e_j - e_i$, this becomes $\dfrac{\partial F_j}{\partial (e_j - e_i)}(x) \leq \dfrac{\partial F_i}{\partial (e_j - e_i)}(x)$.

7.3 Examples

Example (Matching in symmetric zero-sum games).

$A \in \mathbb{R}^{n \times n}$ is symmetric zero-sum if A is skew-symmetric: $A_{ji} = -A_{ij}$ for all $i, j \in S$.

Then $F(x) = Ax$ satisfies $z'DF(x)z = z'Az = 0$ for all $z \in \mathbb{R}^n$, and so is null contractive.

Example (Matching in Rock-Paper-Scissors). Let $F(x) = Ax$ with

$$A = \begin{pmatrix} 0 & -l & w \\ w & 0 & -l \\ -l & w & 0 \end{pmatrix}.$$

Here $w > 0$ and $l > 0$ represent the benefit from a win and the cost of a loss.

$w = l$ is (standard) RPS, $w > l$ is good RPS, $w < l$ is bad RPS.

In all cases, the unique Nash equilibrium is $x^* = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

When is $F(x) = Ax$ contractive?

Example ((Perturbed) concave potential games).

If F is a potential game with a strictly concave potential function, $(y - x)'(F(y) - F(x)) \leq 0$ always holds strictly.

If we slightly perturb F, the result is unlikely to be a potential game, but it will still be strictly contractive.

7.4 Equilibrium in Contractive Games

We call x is a globally neutrally stable state of F ($x \in GNSS(F)$) if

$$(41) \qquad (y - x)' F(y) \leq 0 \text{ for all } y \in X.$$

We call x a globally evolutionarily stable state of F ($x \in GESS(F)$) if

$$(y - x)' F(y) < 0 \text{ for all } y \in X \smallsetminus \{x\}.$$

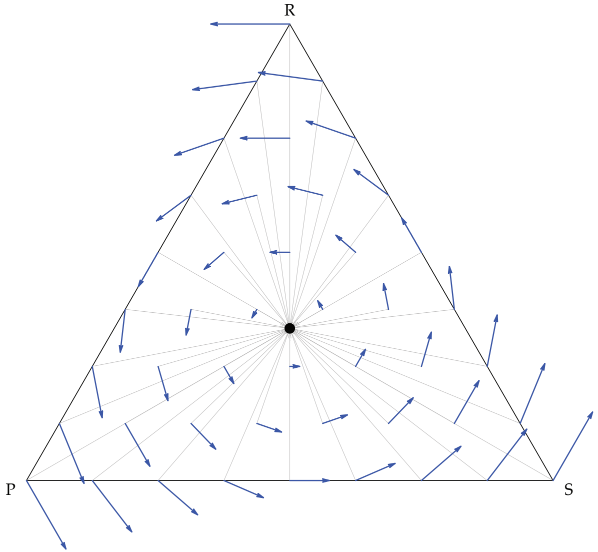
GESS is the global analogue of ESS (which used $y \in O \smallsetminus \{x\}$).

GNSS is the global analogue of NSS.

Geometric interpretation:

Moving from y in direction $F(y)$ moves the state (weakly) closer to x.

Example (GNSS in standard RPS).



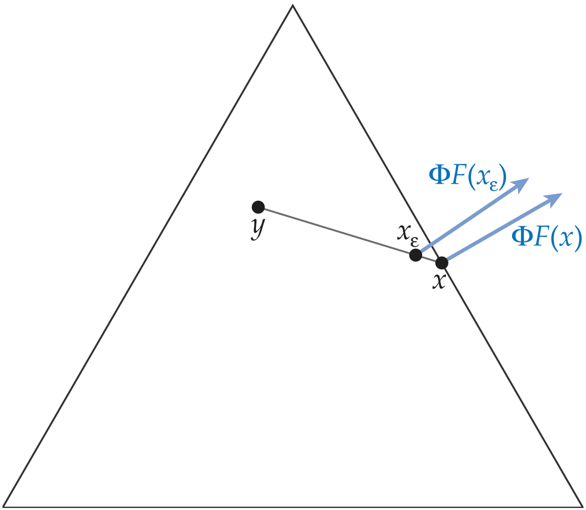
The GNSS of standard RPS.

GNSS: $(y - x)'F(y) \leq 0$ for all $y \in X$.

$GNSS(F)$ is an intersection of half spaces, and so is convex. Moreover,

Proposition. $GNSS(F) \subseteq NE(F)$.

Proof.



Theorem.

If F is a contractive game, then $NE(F) \subseteq GNSS(F)$.
Thus $NE(F) = GNSS(F)$, and so is a convex set.

Proof. Add.

Theorem.

If F is a strictly contractive game,
then $GNSS(F)$ is a singleton and coincides with $NE(F)$.

After 7.4: Existence of Nash equilibrium in contractive games

Proposition.

Let F be a contractive game, and let Y be a finite subset of X.

Then there exists a state $x^* \in \operatorname{conv}(Y)$ such that $(y - x^*)'F(y) \leq 0$ for all $y \in Y$.

Since F is contractive,

$$NE(F) = GNSS(F) = \bigcap_{y \in X} \{x \in X : (y - x)'F(y) \leq 0\},$$

so the proposition and the finite intersection property imply that $NE(F) \neq \varnothing$.

7.5 Global Convergence and Local Stability

7.5.1 Imitative dynamics

Theorem. *Let* $X_{x^*} = \{x \in X\colon\ \mathrm{supp}(x^*) \subseteq \mathrm{supp}(x)\}$.

Let F *be a strictly contractive game with unique Nash equilibrium* x^**, and let* $\dot{x} = V^F(x)$ *be the replicator dynamic for* F*.*

Define the function $H_{x^*} : X_{x^*} \to \mathbb{R}_+$ *by*

$$H_{x^*}(x) = \sum_{i \in \mathrm{supp}(x^*)} x_i^* \log \frac{x_i^*}{x_i},$$

Then $H_{x^*}^{-1}(0) = \{x^*\}$*, and* $H_{x^*}(x)$ *approaches infinity whenever* x *approaches* $X - X_{x^*}$*.*

Moreover, $\dot{H}_{x^*}(x) \le 0$*, with equality only when* $x = x^*$*.*

Therefore, x^* *is globally asymptotically stable with respect to* X_{x^*}*.*

A very similar argument shows that any ESS is locally asymptotically stable.

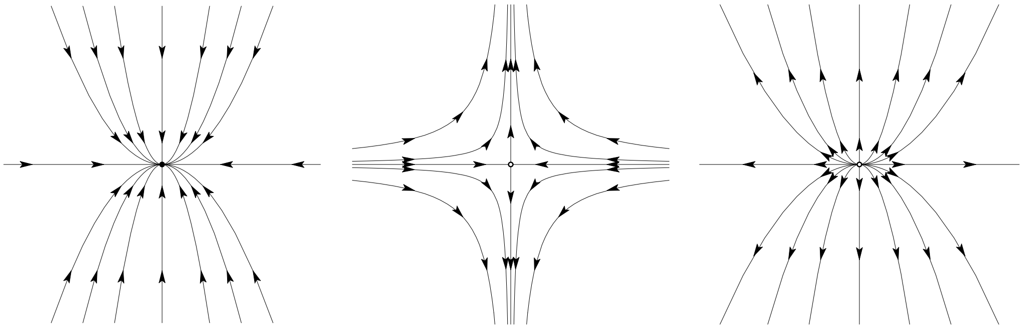
With 7.5: Linear differential equations

The simplest ordinary differential equations on $\mathbb{R}^n$ are linear differential equations:

$$\text{(L)} \qquad \dot{x} = Ax, \quad A \in \mathbb{R}^{n \times n}.$$

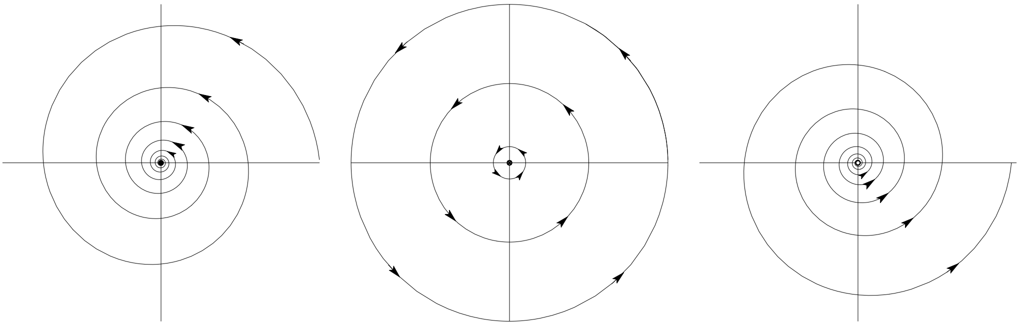
Example: If $n = 1$, so that $\dot{x} = ax$, we have $x_t = \xi \exp(at)$.
The flow of (L) is an contraction if $a < 0$, and an expansion if $a > 0$.

Example: If $n = 2$, the nature of the dynamics depends on the eigenvalues and diagonalizability of A.



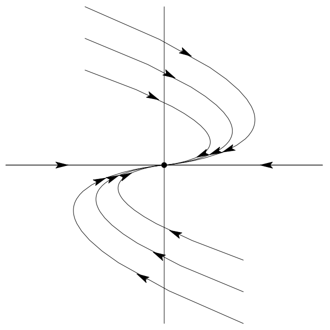
(i) stable node ($\mu < \lambda < 0$) (ii) saddle ($\mu > 0 > \lambda$) (iii) unstable node ($\mu > \lambda > 0$)

Linear dynamics on the plane: two real eigenvalues λ, μ.

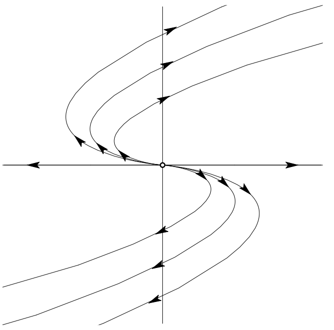


(i) stable spiral ($a < 0$) (ii) center ($a = 0$) (iii) unstable spiral ($a > 0$)

Linear dynamics on the plane: complex eigenvalues $a \pm \mathrm{i}\, b$, $b < 0$.



(i) stable improper node ($\lambda < 0$)



(ii) unstable improper node ($\lambda > 0$)

Linear dynamics on the plane: A not diagonalizable, one real eigenvalue λ.

Theorem. *Let* $\{x_t\}_{t\in(-\infty,\infty)}$ *be the solution to* (L) *from initial condition* x_0. *Then each coordinate of* x_t *is a linear combination of terms of the form* $t^k e^{at}\cos(bt)$ *and* $t^k e^{at}\sin(bt)$, *where* $a + \mathrm{i}\, b \in \mathbb{C}$ *is an eigenvalue of* A *and* $k \in \mathbf{Z}_+$ *is less than the algebraic multiplicity of this eigenvalue.*

Thus in generic case, the stability of the origin under (L) depends on the real parts a_i of the eigenvalues $\{a_1 + \mathrm{i}\, b_1, \ldots, a_n + \mathrm{i}\, b_n\}$ of A.

If each a_i is negative, all solutions converge to the origin; the origin is called a sink, and the flow of (L) is called a contraction.

Solutions to (L) converge to the origin at an exponential rate:
for any $a > 0$ satisfying $a < |a_i|$ for all $i \in \{1, \ldots n\}$, there is a $C = C(a) \geq 1$ such that

$$\mathbf{0} \text{ is a sink} \;\Leftrightarrow\; \left|\phi_t(\xi)\right| \leq C e^{-at}\,|\xi| \quad \text{for all } t \geq 0 \text{ and all } \xi \in \mathbb{R}^n.$$

If each a_i is positive, then all solutions besides the one at $\mathbf{0}$ move away from $\mathbf{0}$; the origin is called a source, and the flow of (L) is called an expansion.

More generally, the flow of (L) may be contracting in some directions and expanding in others.

When each real part a_i of an eigenvalue of A is nonzero, the differential equation $\dot{x} = Ax$, its rest point at the origin, and the flow of (L) are all said to be hyperbolic.

Hyperbolic linear flows come in three varieties: contractions, expansions, and saddles.

With 7.5: Linearization of nonlinear differential equations

Consider the C^1 differential equation

(D) $\qquad \dot{x} = V(x)$

with rest point x^*.

We can approximate the value of V in the neighborhood of x^* via

$$V(y) = \mathbf{0} + DV(x^*)(y - x^*) + o(|y - x^*|).$$

This suggests that the behavior of the dynamic (D) near x^*
can be approximated by the behavior near $\mathbf{0}$ of

(L) $\qquad \dot{z} = DV(x^*)z.$

We say that flows ϕ and ψ are topologically conjugate on X and Z if there is a homeomorphism $h : X \to Z$ such that $\phi_t(x_0) = h^{-1} \circ \psi_t \circ h\,(x_0)$.

Theorem (The Hartman-Grobman Theorem)**.**

Let ϕ and ψ be the flows of the C^1 equation (D) *and the linear equation* (L)*, where x^* is a hyperbolic rest point of* (D)*.*

Then there exist neighborhoods O_{x^*} of x^* and $O_{\mathbf{0}}$ of the origin $\mathbf{0}$ on which ϕ and ψ are topologically conjugate.

Corollary.

Let x^* be a hyperbolic rest point of (D).

Then x^* is asymptotically stable if all eigenvalues of $DV(x^*)$ have negative real parts, and x^* is unstable otherwise.

With 7.5: Local stability of ESS via linearization

We want to approximate the behavior of

(D) $\quad \dot{x} = V(x)$

near x^* using the linear dynamic

(L) $\quad \dot{z} = DV(x^*)z.$

Since $V\colon X \to TX$, $DV(x^*)$ maps TX into itself.

Therefore, we can (and should) think of (L) as a dynamic on TX.

So, rather than looking at all the eigenvalues of $DV(x^*)$, we should only consider those associated with the restricted map $DV(x^*)\colon TX \to TX$.

One way to do this is to compute the eigenvalues of $DV(x^*)\Phi$, and to ignore the eigenvalue 0 corresponding to the eigenvector $\mathbf{1}$ (which is mapped to $\mathbf{0}$ by Φ).

The following result, called Hines's lemma, is often useful.

Lemma. *Suppose that* $Q \in \mathbb{R}^{n \times n}$ *is symmetric, satisfies* $Q\mathbf{1} = \mathbf{0}$*, and is positive definite with respect to* TX*, and that* $A \in \mathbb{R}^{n \times n}$ *is negative definite with respect to* TX*. Then each eigenvalue of the linear map* $QA : TX \to TX$ *has negative real part.*

If we ignored the complications caused by the dynamics being defined on X, this would reduce to

Lemma. *If* Q *is symmetric positive definite and* A *is negative definite, then the eigenvalues of* QA *have negative real parts.*

The replicator dynamic

Theorem. *Let x^* be a regular ESS of F. Then x^* is linearly stable under the replicator dynamic.*

This condition is not necessary. (A counterexample is Zeeman's game.)

Proof for $x^* \in \mathrm{int}(X)$.

Other imitative dynamics

(R) $\dot{x}_i = x_i \hat{F}_i(x)$

(I) $\dot{x}_i = x_i G_i(x), \quad G_i(x) \geq G_j(x)$ if and only if $F_i(x) \geq F_j(x)$.

Theorem. *Assume that x^* is a hyperbolic rest point of both* (R) *and imitative dynamic* (I). *Then x^* is linearly stable under* (R) *if and only if it is linearly stable under* (I). *Thus, if x^* is a regular ESS that satisfies the hyperbolicity assumptions, it is linearly stable under* (I).

Assume $x^* \in \operatorname{int}(X)$. Write $V(x) = \operatorname{diag}(x)\hat{F}(x)$ and $W(x) = \operatorname{diag}(x)G(x)$.

$$DV(x^*)\Phi = Q(x^*)DF(x^*)\Phi = Q(x^*)\Phi DF(x^*)\Phi \text{ and}$$

$$DW(x^*)\Phi = Q(x^*)\Phi DG(x^*)\Phi.$$

Lemma. *Let x^* be an interior Nash equilibrium, and suppose that $\Phi DF(x^*)$ and $\Phi DG(x^*)$ define invertible maps from TX to itself. Then $\Phi DG(x^*)\Phi = c\,\Phi DF(x^*)\Phi$ for some $c > 0$.*

The logit dynamic

Recall:

$$\text{(R)} \qquad \dot{x}_i = V_i(x) = x_i \hat{F}_i(x)$$

$$DV(x^*) = Q(x^*)DF(x^*)$$

Consider the logit dynamic.

$$\dot{x}_i = V_i^\eta(x) = \frac{\exp(\eta^{-1}F_i(x))}{\sum_{j \in S} \exp(\eta^{-1}F_j(x))} - x_i.$$

One can show that at rest point $\tilde{x}^\eta$,

$$DV^\eta(\tilde{x}^\eta) = \eta^{-1}Q(\tilde{x}^\eta)DF(\tilde{x}^\eta) - I.$$

At rest points, the derivative matrices of the replicator and logit dynamics differ only by a positive affine transformation!

7.5.2 Target and pairwise comparison dynamics: Global convergence in contractive games

With 7.5: More on global convergence in contractive games

Integrable target dynamics

The BNN, best response, and logit dynamics can be expressed as target dynamics that condition on the vector of excess payoffs: $\sigma^F_{ij}(x) = \tau_j(\hat{F}(x))$.

In general, such dynamics need not converge in contractive games.

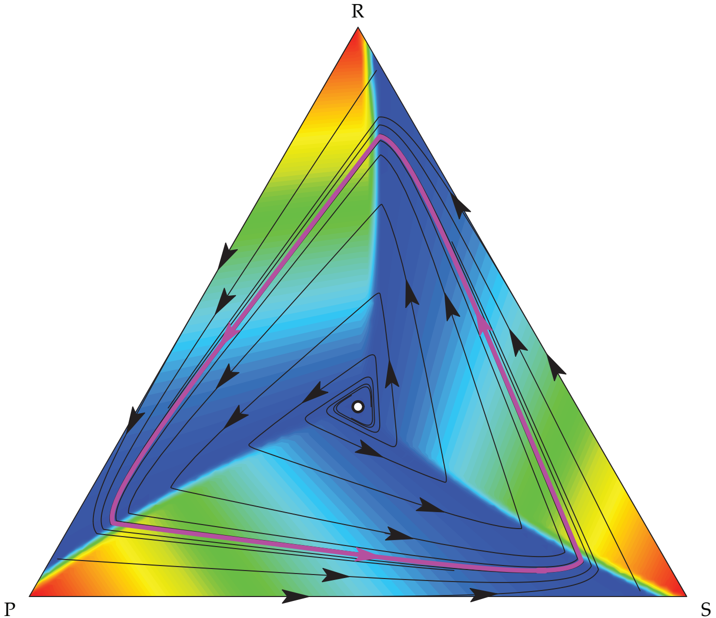
Example: In good RPS, consider the protocol

$$\begin{pmatrix} \tau_R(\hat{\pi}) \\ \tau_P(\hat{\pi}) \\ \tau_S(\hat{\pi}) \end{pmatrix} = \begin{pmatrix} [\hat{\pi}_R]_+ g^\varepsilon(\hat{\pi}_S) \\ [\hat{\pi}_P]_+ g^\varepsilon(\hat{\pi}_R) \\ [\hat{\pi}_S]_+ g^\varepsilon(\hat{\pi}_P) \end{pmatrix},$$

where $g^\varepsilon : \mathbb{R} \to \mathbb{R}$ equals 1 on $(-\infty, 0]$, equals ε^2 on $[\varepsilon, \infty)$, and is linear on $[0, \varepsilon]$.

The weight placed on a strategy is proportional to positive part of the strategy's excess payoff, but is only of order ε^2 if the strategy it beats in RPS has an excess payoff greater than ε.

This protocol satisfies acuteness, and so the corresponding dynamic satisfies (PC) and (NS).



An excess payoff dynamic in standard RPS.

But convergence can be ensured when τ is integrable:
there is a C^1 revision potential $\gamma \colon \mathbb{R}^n \to \mathbb{R}$ such that

$$\tau \equiv \nabla \gamma.$$

This is weaker than separability:

$$\tau_i(\hat{\pi}) \text{ is independent of } \hat{\pi}_{-i} \quad \Rightarrow \quad \gamma(\hat{\pi}) = \sum_{i \in S} \int_0^{\hat{\pi}_i} \tau_i(s)\, \mathrm{d}s.$$

Interpretation.

Practical consequence: a building block for constructing Lyapunov functions.

We first consider integrable excess payoff dynamics.

These include the BNN dynamic:

$$\tau_i(\hat{\pi}) = [\hat{\pi}_i]_+ \quad \Rightarrow \quad \gamma(\hat{\pi}) = \frac{1}{2}\sum_{i \in S} [\hat{\pi}_i]_+^2.$$

Theorem. *Let F be a C^1 contractive game, and let $\dot{x} = V^F(x)$ be the integrable excess payoff dynamic for F based on revision protocol τ with revision potential γ. Define the C^1 function $\Gamma : X \to \mathbb{R}$ by*

$$\Gamma(x) = \gamma(\hat{F}(x)). \tag{42}$$

Then Γ is a strict Lyapunov function for V_F, and $NE(F)$ is globally attracting.

In addition, if F admits a unique Nash equilibrium, or if τ is separable, then $NE(F)$ is globally asymptotically stable.

For the best response dynamic, the target protocol is the maximizer correspondence

$$M(\hat{\pi}) = \underset{y \in X}{\operatorname{argmax}}\, y'\hat{\pi}.$$

Consider the maximum function

$$\mu(\hat{\pi}) = \max_{y \in X} y'\hat{\pi} = \max_{i \in S} \hat{\pi}_i.$$

Then when the unique optimal strategy under π is i, we have $\mu(\hat{\pi}) = \hat{\pi}_i$, so at such states we have $\nabla\mu(\hat{\pi}) = e_i = M(\hat{\pi})$.

Theorem. *Let F be a C^1 contractive game, and let $\dot{x} \in V^F(x)$ be the best response dynamic for F. Define the Lipschitz continuous function $G\colon X \to \mathbb{R}_+$ by*

$$G(x) = \max_{y \in X} (y - x)'F(x) = \max_{i \in S} \hat{F}_i(x).$$

Then $G^{-1}(0) = NE(F)$. Moreover, if $\{x_t\}_{t \geq 0}$ is a solution to V^F, then for almost all $t \geq 0$ we have that $\dot{G}(x_t) \leq -G(x_t)$, and so $NE(F)$ is globally asymptotically stable under V^F.

For perturbed best response dynamics: the perturbed maximizer function

$$\tilde{M}(\hat{\pi}) = \underset{y \in \mathrm{int}(X)}{\mathrm{argmax}}\; y'\hat{\pi} - v(y),$$

where v is admissible.

The perturbed maximum function

$$\tilde{\mu}(\pi) = \max_{y \in \mathrm{int}(X)} y'\pi - v(y),$$

is a potential function for $\tilde{M}$.

Theorem. *Let F be a C^1 contractive game, and let $\dot{x} = V^{F,v}(x)$ be the perturbed best response dynamic for F generated by the admissible deterministic perturbation v. Define the function $\tilde{G}\colon \mathrm{int}(X) \to \mathbb{R}_+$ by*

$$\tilde{G}(x) = \tilde{\mu}(\hat{F}(x)) + v(x),$$

Then $G^{-1}(0) = PE(F, v)$, and this set is a singleton. Moreover, $\tilde{G}$ is a strict Lyapunov function for $V^{F,v}$, and so $PE(F, v)$ is globally asymptotically stable under $V^{F,v}$.

Impartial pairwise comparison dynamics

Pairwise comparison dynamics are defined using revision protocols $\rho_{ij}(\pi)$ that are sign preserving:

$$\operatorname{sgn}(\rho_{ij}(\pi)) = \operatorname{sgn}([\pi_j - \pi_i]_+) \quad \text{for all } i, j \in S.$$

To obtain a convergence result, we assume impartiality:

$$\rho_{ij}(\pi) = \phi_j(\pi_j - \pi_i) \quad \text{for some functions } \phi_j : \mathbb{R} \to \mathbb{R}_+.$$

Theorem. *Let F be a C^1 contractive game, and let $\dot{x} = V_F(x)$ be an impartial pairwise comparison dynamic for F. Define the Lipschitz continuous function $\Psi : X \to \mathbb{R}_+$ by*

$$\Psi(x) = \sum_{i \in S} \sum_{j \in S} x_i \psi_j(F_j(x) - F_i(x)), \quad \textit{where} \quad \psi_k(d) = \int_0^d \phi_k(s)\, \mathrm{d}s.$$

Then $\Psi^{-1}(0) = NE(F)$. Moreover, $\dot{\Psi}(x) \leq 0$ for all $x \in X$, with equality if and only if $x \in NE(F)$, and so $NE(F)$ is globally asymptotically stable.

Dynamic	Lyapunov function for contractive games
replicator	$H_{x^*}(x) = \sum_{i\in\text{supp}(x^*)} x_i^* \log \frac{x_i^*}{x_i}$
best response	$G(x) = \max\limits_{i\in S} \hat{F}_i(x)$
logit	$\tilde{G}(x) = \max\limits_{y\in\text{int}(X)} \left(y'\hat{F}(x) - \eta \sum_{i\in S} y_i \log y_i\right) + \eta \sum_{i\in S} x_i \log x_i$
BNN	$\Gamma(x) = \frac{1}{2} \sum_{i\in S} [\hat{F}_i(x)]_+^2$
Smith	$\Psi(x) = \frac{1}{2} \sum\limits_{i\in S} \sum\limits_{j\in S} x_i [F_j(x) - F_i(x)]_+^2$

Table 4: Lyapunov functions for five basic deterministic dynamics in contractive games.

7.5.3 Target and pairwise comparison dynamics: Local stability of regular ESS

With 7.5: Local stability of ESS via Lyapunov functions

Recall that $x \in X$ a regular ESS if

(37a) x is a quasistrict equilibrium: $F_i(x) = \bar{F}(x) > F_j(x)$ whenever $x_i > 0$ and $x_j = 0$.

(37b) $z' DF(x) z < 0$ for all $z \in TX \setminus \{\mathbf{0}\}$ such that $z_i = 0$ whenever $x_i = 0$.

Theorem. *Let x^* be a regular ESS of F. Then x^* is asymptotically stable under the replicator dynamic for F.*

Proof. Use the Lyapunov function H_{x^*}. ■

Theorem. *Let x^* be a regular ESS of F. Then x^* is asymptotically stable under*

(*i*) *any separable excess payoff dynamic for F;*

(*ii*) *the best response dynamic for F;*

(*iii*) *any impartial pairwise comparison dynamic for F.*

Idea of proof. Augment the Lyapunov functions Γ, G, and Ψ by adding

$$\Upsilon_{x^*}(x) = C \sum_{j:\, x_j^* = 0} x_j,$$

which is a multiple of the total mass of agents using strategies unused in x^*.

Then $\Gamma_{x^*} = \Gamma + \Upsilon_{x^*}$, $G_{x^*} = G + \Upsilon_{x^*}$, and $\Psi_{x^*} = \Psi + \Upsilon_{x^*}$
are strict local Lyapunov functions for x^*.

8. Iterative Solution Concepts, Supermodular Games, and Equilibrium Selection

8.1 Iterated Strict Dominance and Never-a-Best-Response

Strategy i is strictly dominated by strategy j if $F_j(x) > F_i(x)$ for all $x \in X$.

Theorem. *Let $\{x_t\}$ be an interior solution trajectory of an imitative dynamic in game F. If strategy $i \in S$ is strictly dominated in F, then* $\lim_{t \to \infty} (x_t)_i = 0$.

A continuity argument extends this conclusions to iteratively dominated strategies.

Under the best response dynamic, it is obvious that any strategy i
that is never a best response (for every $x \in X$ there is a $j \in S$ such that $F_j(x) > F_i(x)$)
vanishes at an exponential rate ($(x_t)_i = (x_0)_i \mathrm{e}^{-t}$).

This is true from all initial conditions.

By continuity, strategies eliminated by iterative removal of strategies
that are never a best response eventually vanish at an exponential rate.

8.2 Supermodular Games and Perturbed Best Response Dynamics

Define the stochastic dominance matrix $\Sigma \in \mathbb{R}^{(n-1)\times n}$ by

$$\Sigma = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Then:

$(\Sigma x)_i = \sum_{j=i+1}^{n} x_j$ equals the total mass on actions greater than i at state x,

$\Sigma y \geq \Sigma x$ if and only if y stochastically dominates x.

We call F a supermodular game if

$$\Sigma y \geq \Sigma x \text{ implies that } F_{i+1}(y) - F_i(y) \geq F_{i+1}(x) - F_i(x) \text{ for all } i < n. \tag{43}$$

If F is continuously differentiable, then it is supermodular if and only if

$$\frac{\partial(F_{i+1} - F_i)}{\partial(e_{j+1} - e_j)}(x) \geq 0 \text{ for all } i < n, j < n, \text{ and } x \in X. \tag{44}$$

In words, if some agents switch from strategy j to strategy $j + 1$,
the performance of strategy $i + 1$ improves relative to that of strategy i.

Conditions (43) and (44) are both called strategic complementarity.

Example (Search with positive externalities).

$$F_i(x) = m(i)\, b(a(x)) - c(i), \text{ where } a(x) = \textstyle\sum_{k=1}^{n} k x_k.$$

Since

$$\frac{\partial(F_{i+1} - F_i)}{\partial(e_{j+1} - e_j)}(x) = (m(i+1) - m(i))\, b'(a(x)) \geq 0,$$

F is a supermodular game.

Intuitively, supermodular games should have
increasing best response correspondences.

This has implications for the structure of the set of Nash equilibrium.

Let $\underline{B}(x) = \min B(x)$ and $\bar{B}(x) = \max B(x)$.

For states $\underline{x}, \bar{x} \in X$ satisfying $\Sigma\underline{x} \leq \Sigma\bar{x}$, we define $[\underline{x}, \bar{x}] = \{x \in X : \Sigma\underline{x} \leq \Sigma x \leq \Sigma\bar{x}\}$.

Theorem. *Suppose F is a supermodular game. Then*

(*i*) *$\underline{B}$ and $\bar{B}$ are increasing in the stochastic dominance order: if $\Sigma x \leq \Sigma y$, then $\Sigma\underline{B}(x) \leq \Sigma\underline{B}(y)$ and $\Sigma\bar{B}(x) \leq \Sigma\bar{B}(y)$.*

(*ii*) *The sequences of iterates $\{\underline{B}^k(e_1)\}_{k\geq 0}$ and $\{\bar{B}^k(e_n)\}_{k\geq 0}$ are monotone sequences of pure states, and so converge within n steps to their limits, $\underline{x}^*$ and $\bar{x}^*$.*

(*iii*) *$\underline{x}^* = \underline{B}(\underline{x}^*)$ and $\bar{x}^* = \bar{B}(\bar{x}^*)$, so $\underline{x}^*$ and $\bar{x}^*$ are pure Nash equilibria of F.*

(*iv*) *$NE(F) \subseteq [\underline{x}^*, \bar{x}^*]$. Thus if $\underline{x}^* = \bar{x}^*$, then this state is the unique Nash equilibrium of F.*

It is natural to look for convergence results for the best response dynamic.

It follows from previous results that all solutions converge to $[\underline{x}^*, \bar{x}^*]$.

But few results on convergence to equilibrium exist.

(Strongly) cooperative differential equations:

$$\dot{\chi} = \mathcal{V}(\chi), \quad \mathcal{V} \text{ is } C^1, \ \frac{\partial \mathcal{V}_i}{\partial \chi_j}(\chi) > 0 \text{ whenever } i \neq j.$$

Consider the stochastically perturbed best response dynamic

$$\text{(45)} \qquad \dot{x} = \tilde{M}^{\varepsilon}(F(x)) - x,$$

Perform a change of variable using the stochastic dominance operator:

Let $\mathcal{X} = \Sigma X \subset \mathbb{R}^{n-1}$ denote the image of X under Σ,
and let $\bar{\Sigma}\colon \mathcal{X} \to X$ denote the (affine) inverse of the map Σ.

Then the change of variable Σ converts (45) into the following dynamic on $\mathcal{X}$:

$$\text{(46)} \qquad \dot{\chi} = \Sigma \tilde{M}^{\varepsilon}(F(\bar{\Sigma}\chi)) - \chi.$$

By combining strategic complementarity (44) with the properties of $D\tilde{M}^{\varepsilon}(\pi)$, one can show that (46) is strongly cooperative.

Theorem. *Let F be a C^1 strictly supermodular game, and let $\dot{x} = V^{F,\varepsilon}(x)$ be a stochastically perturbed best response dynamic for F. Then*

(*i*) *States $\underline{x}^* \equiv \omega(\underline{x})$ and $\bar{x}^* \equiv \omega(\bar{x})$ exist and are the minimal and maximal elements of the set of perturbed equilibria. Moreover, $[\underline{x}^*, \bar{x}^*]$ contains all ω-limit points of $V^{F,\varepsilon}$ and is globally asymptotically stable.*

(*ii*) *Solutions to $\dot{x} = V^{F,\varepsilon}(x)$ from an open, dense, full measure set of initial conditions in X converge to perturbed equilibria.*

8.3 Iterated p-Dominance and Equilibrium Selection

So far we have considered dynamics based on reactive protocols $\rho^F(x) = \rho(F(x), x)$.

This formulation leads naturally to dynamics satisfying positive correlation (PC) and Nash stationarity (NS).

But it also imposes restrictions on what the dynamics can achieve.

For instance, in coordination games, all pure equilibria are locally stable, implying that predictions of play will depend on initial conditions.

Sampling best response dynamics

We consider an analog of the best response dynamic in which agents do not know the population state, instead basing their decisions on information from samples:

Revision opportunities arrive at a unit rate.

Each revising agent obtains information by drawing a sample of size k from the population.

The agent then plays a best response to the empirical distribution of strategies in his sample.

Note that this revision protocol is prospective:
an agent must consider the payoffs that would obtain
if his sample were representative of behavior in the population at large.

k-sampling best response dynamics

$\mathbb{Z}_+^{n,k} = \{z \in \mathbb{Z}_+^n \colon \sum_{i \in S} z_i = k\}$ possible outcomes of samples of size k.

Assume that each such outcome generates a unique best response.

Then we can define the k-sampling best response function $B^k \colon X \to X$ by

$$B_i^k(x) = \sum_{z \in \mathbb{Z}_+^{n,k} \colon b(\frac{1}{k}z) = \{i\}} \binom{k}{z_1 \;\; \ldots \;\; z_n} x_1^{z_1} \cdots x_n^{z_n}.$$

Assuming Poisson arrivals of revision opportunities, we obtain the k-sampling best response dynamic:

$$(\mathrm{S}_k) \qquad \dot{x} = B^k(x) - x.$$

Observation.

If strategy i is a strict equilibrium of F, then state e_i is a rest point of any sampling best response dynamic for F.

Theorem. *Consider a game with strategy set $S = \{0, 1\}$,*
and in which strategy 1 *is* $\frac{1}{k}$*-dominant, where* $k \geq 2$,
Then under the k-sampling best response dynamic,
if play begins at a state at which a positive mass of agents choose strategy 1,
it converges to the state where all play strategy 1.

Selection of iterated p-dominant equilibrium

To obtain more general results, we develop an iterated version of the analysis above.

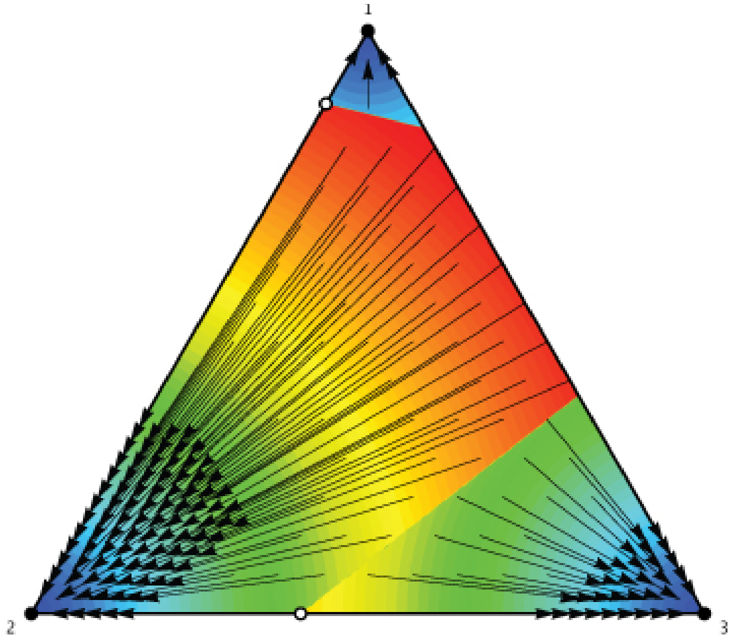
$S^* \subseteq S$ is a p-best response set of F (Tercieux (2006)) if $b(x) \subseteq S^*$ holds for all $x \in X$ with $\sum_{i \in S^*} x_i \geq p$.

$S^* \subseteq S$ is an iterated p-best response set of F if there exists a sequence $S^0, \ldots, S^m$ with $S = S^0 \supseteq \cdots \supseteq S^m = S^*$ such that S^ℓ is a p-best response set in the restricted game $F|_{S^{\ell-1}}$ for each $\ell = 1, \ldots, m$.

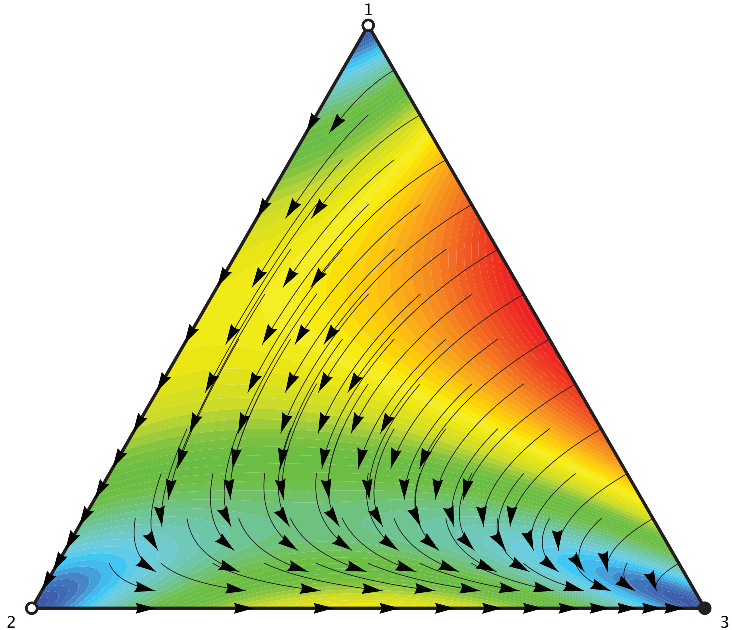
$a^* \in S$ is an iterated p-dominant equilibrium if $\{a^*\}$ is an iterated p-best response set.

Theorem. *Let S^* be an iterated $\frac{1}{k}$-best response set. Then $X_{S^*} = \{x \in X \colon \operatorname{supp}(x) \subseteq S^*\}$ is almost globally asymptotically stable under the k-sampling best response dynamic.*

In particular, if a^* is an iterated $\frac{1}{k}$-dominant equilibrium, then e_{a^*} is almost globally asymptotically stable.



The best response dynamic in Young's game, $A = \begin{pmatrix} 6 & 0 & 0 \\ 5 & 7 & 5 \\ 0 & 5 & 8 \end{pmatrix}$.



The 2-sampling best response dynamic in Young's game, $A = \begin{pmatrix} 6 & 0 & 0 \\ 5 & 7 & 5 \\ 0 & 5 & 8 \end{pmatrix}$.

A key element of the proof is the transitivity theorem for asymptotic stability (Conley (1978)):

> *If Y is asymptotically stable in X, and Z is asymptotically stable in Y, then Z is asymptotically stable in X.*

9. Nonconvergence of Evolutionary Dynamics

9.1 Examples

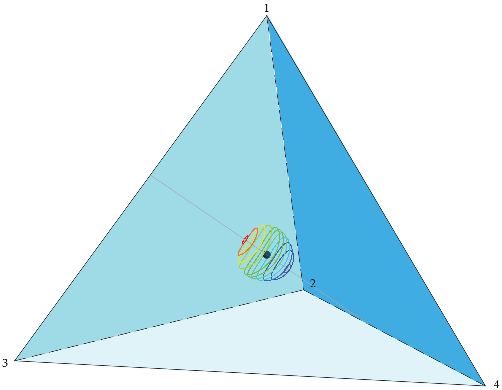
Example. In zero-sum games ($F(x) = Ax$, $A' = -A$), H_{x^*} is a constant of motion.

$$H_{x^*}(x) = \sum_{i \in \mathrm{supp}(x^*)} x_i^* \log \frac{x_i^*}{x_i},$$

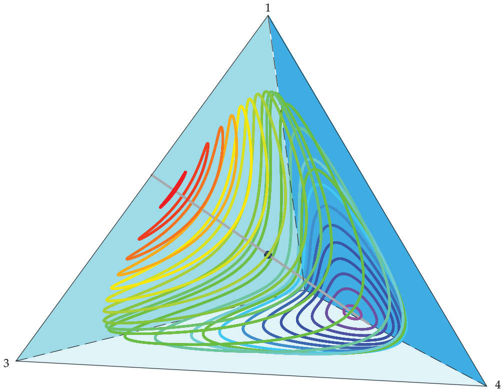
But time averaged solutions converge to Nash equilibrium.

For instance, consider

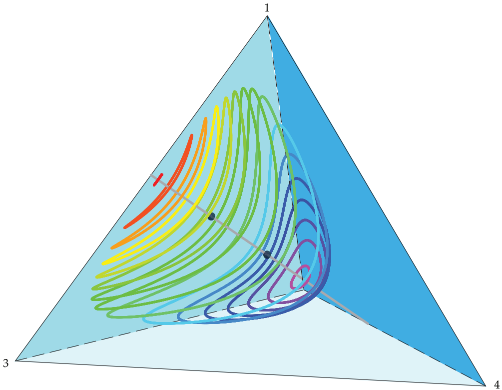
$$F(x) = Ax = \begin{pmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_4 - x_2 \\ x_1 - x_3 \\ x_2 - x_4 \\ x_3 - x_1 \end{pmatrix}.$$



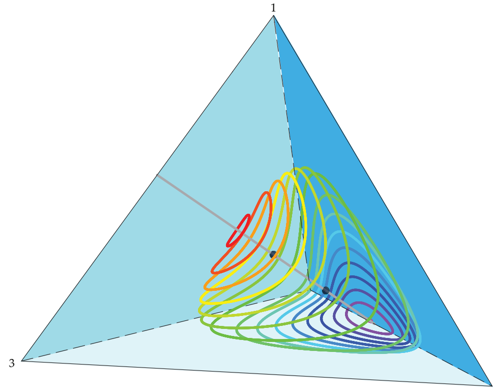
$x^* = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}),\ H_{x^*}(x) = .02$



$x^* = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}),\ H_{x^*}(x) = .58$



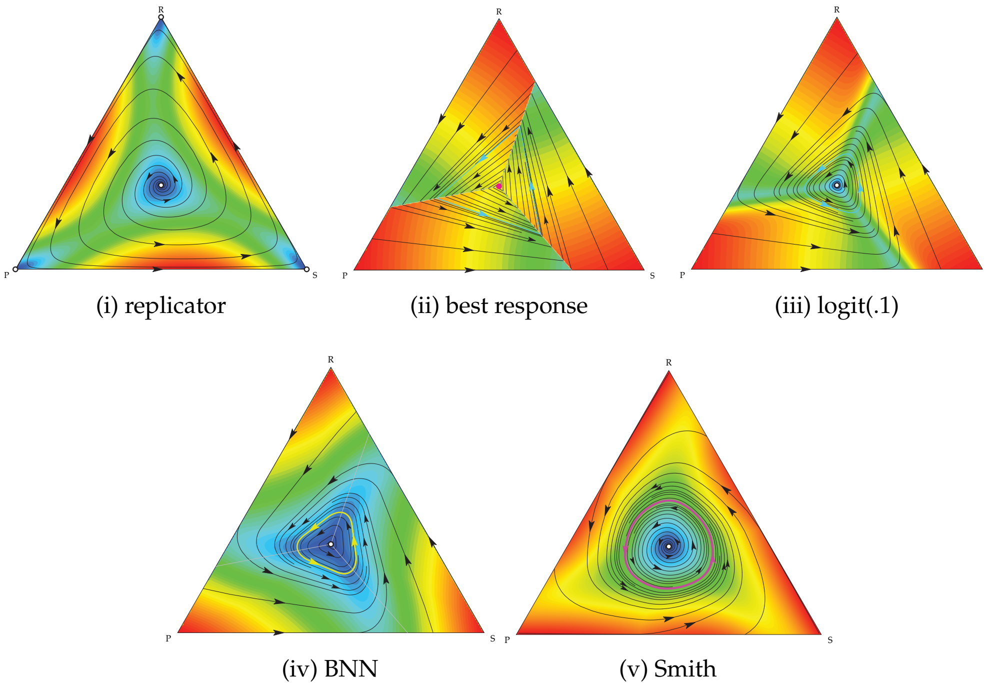
$x^* = (\frac{3}{8}, \frac{1}{8}, \frac{3}{8}, \frac{1}{8}),\ H_{x^*}(x) = .35$



$x^* = (\frac{1}{8}, \frac{3}{8}, \frac{1}{8}, \frac{3}{8}),\ H_{x^*}(x) = .35$

Example (Bad RPS).

$$F(x) = Ax, \text{ where } A = \begin{pmatrix} 0 & -l & w \\ w & 0 & -l \\ -l & w & 0 \end{pmatrix}, \; l > w > 0.$$



Five basic deterministic dynamics in bad Rock-Paper-Scissors ($l = 2$, $w = 1$).

With 9.1: More on games with nonconvergent dynamics

The hypercycle system

Consider the game

$$F(x) = Ax = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} x.$$

The replicator dynamic for this game is known as the hypercycle system.

The unique Nash equilibrium of F is the barycenter $x^* = \frac{1}{n}\mathbf{1}$.

Let $\dot{x} = R(x)$ denote the replicator dynamic for F.

Then the eigenvalue/eigenvector pairs of $DR(x^*)$ are

$$(\lambda_k, v_k) = \left(\frac{1}{n} \iota_n^{(n-1)k} - \frac{2}{n^2} \sum_{j=0}^{n-1} \iota_n^{jk},\ (1, \iota_n^k, \ldots, \iota_n^{(n-1)k})' \right), \quad k = 0, \ldots, n-1.$$

where $\iota_n = \exp(\frac{2\pi \mathrm{i}}{n}) = \cos(\frac{2\pi}{n}) + \mathrm{i} \sin(\frac{2\pi}{n})$ is the nth root of unity.

Eigenvalue $\lambda_0 = \frac{1}{n} - \frac{2}{n} = -\frac{1}{n}$ corresponds to eigenvector $v_0 = \mathbf{1}$.

For $k \geq 1$, the sum in the formula for λ_k vanishes, leaving us with

$$\lambda_k = \frac{1}{n} \iota_n^{(n-1)k} = \frac{1}{n} \iota_n^{-k}.$$

The stability of x^* therefore depends on whether any λ_k with $k > 0$
has positive real part.

This largest real part is negative when $n \leq 3$, zero when $n = 4$, and positive when $n \geq 5$. Thus x^* is asymptotically stable when $n \leq 3$, but unstable when $n \geq 5$.

With 9.1: Attractors and continuation

A set $\mathcal{A} \subseteq X$ is an attractor of the flow ϕ if it is nonempty, compact, and invariant under ϕ, and if there is a neighborhood U of $\mathcal{A}$ such that

$$\lim_{t \to \infty} \sup_{x \in U} \operatorname{dist}(\phi_t(x), \mathcal{A}) = 0.$$

Thus, attractors are asymptotically stable sets that are also invariant under the flow.

The set $B(\mathcal{A}) = \{x \in X : \omega(x) \subseteq \mathcal{A}\}$ is called the basin of $\mathcal{A}$.

A key property of attractors for the current context is known as continuation:

Let $\dot{x} = V^\varepsilon(x)$ be a family of differential equations on $\mathbb{R}^n$
with unique solutions $x_t = \phi_t^\varepsilon(x_0)$.

Suppose that $(\varepsilon, x) \mapsto V^\varepsilon(x)$ is continuous, and that $X \subset \mathbb{R}^n$ is compact and forward invariant under ϕ^ε.

Then as ε varies continuously from 0,
there exist attractors $\mathcal{A}^\varepsilon$ of the flows ϕ^ε that vary upper hemicontinuously from $\mathcal{A}^0$;
their basins $B(\mathcal{A}^\varepsilon)$ vary lower hemicontinuously from $B(\mathcal{A}^0)$.

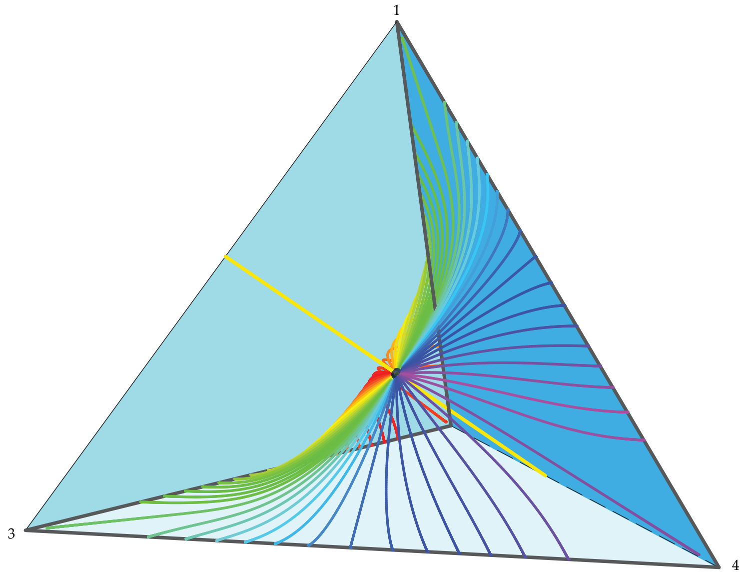
Thus, if we slightly change the parameter ε, the attractors that exist under ϕ^0 continue to exist, and they do not explode.

Example (The Hofbauer-Swinkels game).

$$F^{\varepsilon}(x) = A^{\varepsilon}x = \begin{pmatrix} 0 & 0 & -1 & \varepsilon \\ \varepsilon & 0 & 0 & -1 \\ -1 & \varepsilon & 0 & 0 \\ 0 & -1 & \varepsilon & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

When $\varepsilon = 0$, the payoff matrix $A^{\varepsilon} = A^0$ is symmetric,
so F^0 is a potential game with potential function $f(x) = \frac{1}{2}x'A^0x = -x_1x_3 - x_2x_4$.

The function f attains its minimum of $-\frac{1}{4}$ at states $v = (\frac{1}{2}, 0, \frac{1}{2}, 0)$ and $w = (0, \frac{1}{2}, 0, \frac{1}{2})$,
has a saddle point with value $-\frac{1}{8}$ at the Nash equilibrium $x^* = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$,
and attains its maximum of 0 along the closed path of Nash equilibria γ
consisting of edges $\overline{e_1e_2}$, $\overline{e_2e_3}$, $\overline{e_3e_4}$, and $\overline{e_4e_1}$.



The Smith dynamic in F^0.

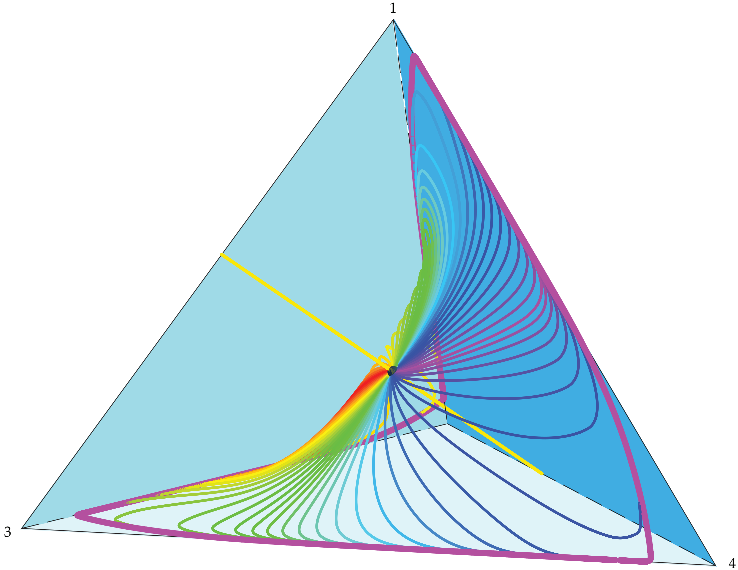
$$F^\varepsilon(x) = A^\varepsilon x = \begin{pmatrix} 0 & 0 & -1 & \varepsilon \\ \varepsilon & 0 & 0 & -1 \\ -1 & \varepsilon & 0 & 0 \\ 0 & -1 & \varepsilon & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

Now suppose that $\varepsilon > 0$.

Then the attractor γ of V^{F^0} continues to an attractor γ^ε of V^{F^ε}.

This attractor attracts solutions to V^{F^ε} from initial conditions x with $f(x) > -\frac{1}{8} + \delta$.

But the unique Nash equilibrium is now x^*!



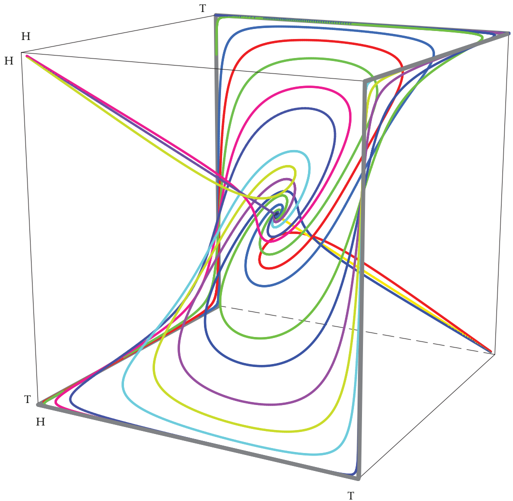
The Smith dynamic in F^{ε}, $\varepsilon = \frac{1}{10}$.

Example. Mismatching Pennies is a three-player normal form game.

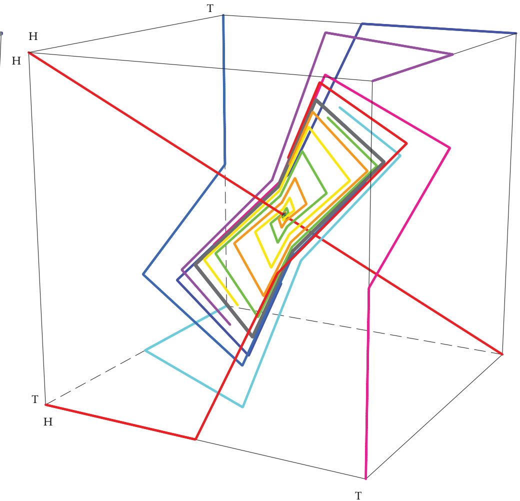
Each player has two strategies, Heads and Tails.

Player p receives a payoff of 1 for choosing a different strategy than player $p+1$ and a payoff of 0 otherwise, where players are indexed modulo 3.

The unique Nash equilibrium of this game has each player play each of his strategies with equal probability.



replicator



best response

Proposition.

Let $V^{(\cdot)}$ be an evolutionary dynamic that is generated by a C^1 *revision protocol* ρ *and that satisfies Nash stationarity* (NS).

Let F be Mismatching Pennies, and suppose that the unique Nash equilibrium x^* of F is a hyperbolic rest point of $\dot{x} = V^F(x)$.

Then x^* is unstable under V^F, and there is an open, dense, full measure set of initial conditions from which solutions to V^F do not converge.

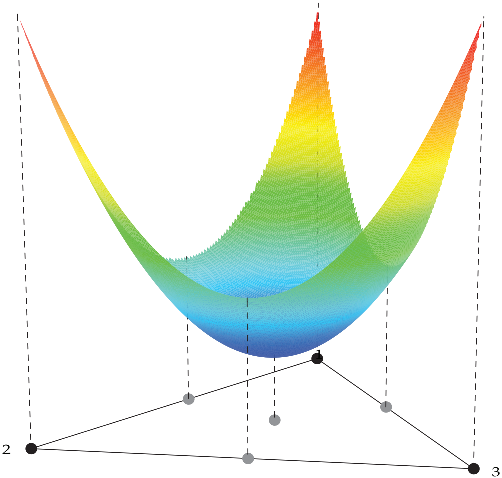
Example (The hypnodisk game).

Hypnodisk is a three-strategy game with nonlinear payoffs
and unique Nash equilibrium $x^* = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. (We define this game shortly.)

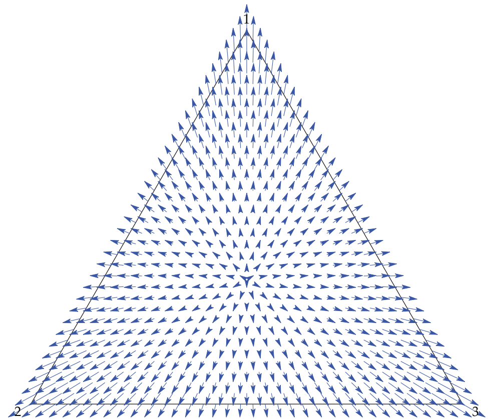
Proposition.

Let $V^{(\cdot)}$ be a Lipschitz continuous evolutionary dynamic that satisfies (NS) *and* (PC)*, and let H be the hypnodisk game.*

Then every solution to $\dot{x} = V^H(x)$ other than the stationary solution at x^* converges to a cycle.

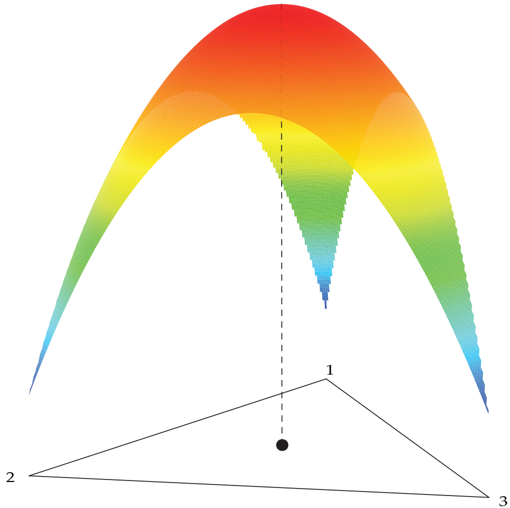


(i) The potential function

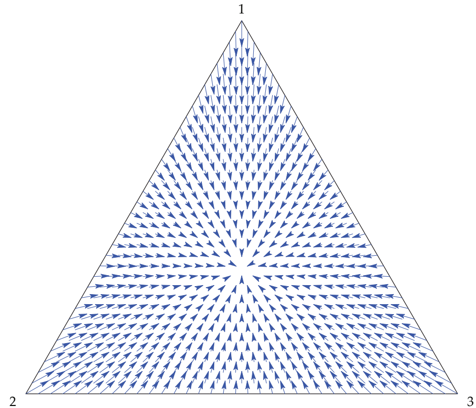


(ii) The projected payoff vector field

The coordination game $F^C(x) = Cx = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$

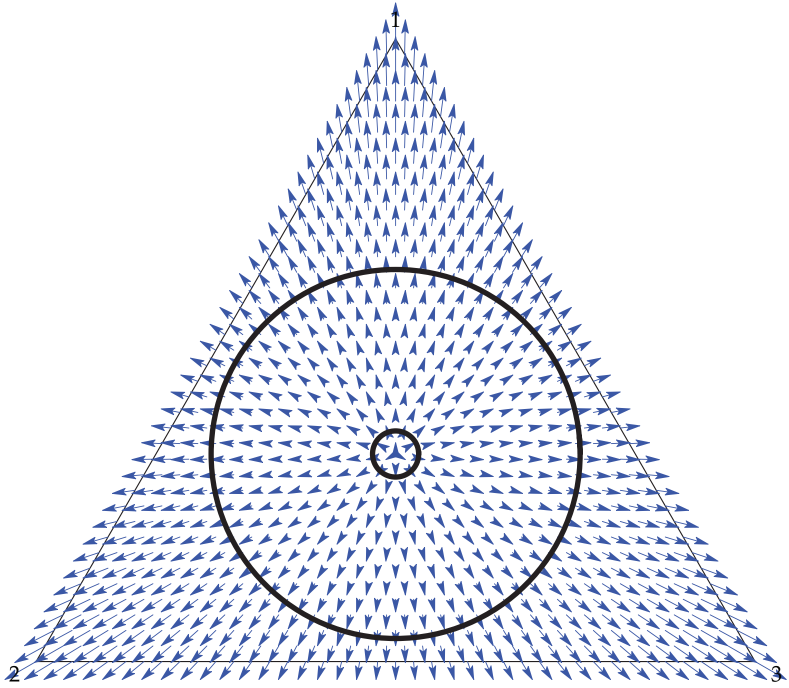


(i) The potential function

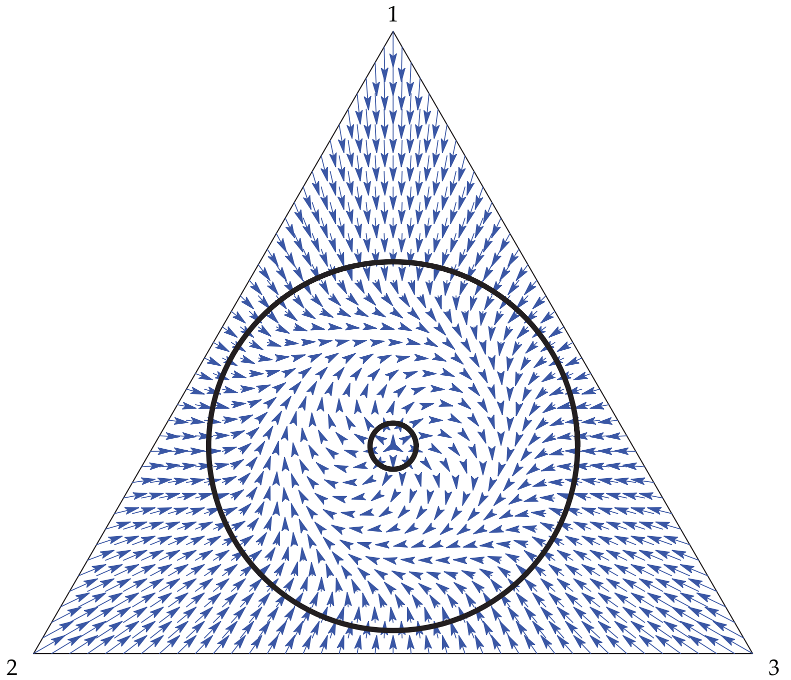


(ii) The projected payoff vector field

The anticoordination game $F^{-C}(x) = -Cx = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_1 \\ -x_2 \\ -x_3 \end{pmatrix}.$



Projected payoff vector field for the coordination game.



Projected payoffs in the hypnodisk game.

The claim of convergence to a closed orbit follows from this basic result for dynamics on the plane.

Theorem (The Poincaré-Bendixson Theorem)**.**

Let $V\colon \mathbb{R}^2 \to \mathbb{R}^2$ *be Lipschitz continuous, and consider the differential equation* $\dot{x} = V(x)$*.*

(i) *Let* $x \in \mathbb{R}^2$*. If* $\omega(x)$ *is compact, nonempty, and contains no rest points, then it is a periodic orbit.*

(ii) *Let* $Y \subset \mathbb{R}^2$*. If* Y *is nonempty, compact, forward invariant, and contains no rest points, then it contains a periodic orbit.*

Thus in planar systems, the only possible ω-limit sets are rest points, sequences of trajectories leading from one rest point to another (heteroclinic cycles and homoclinic orbits), and periodic orbits.

Example (Chaotic dynamics).

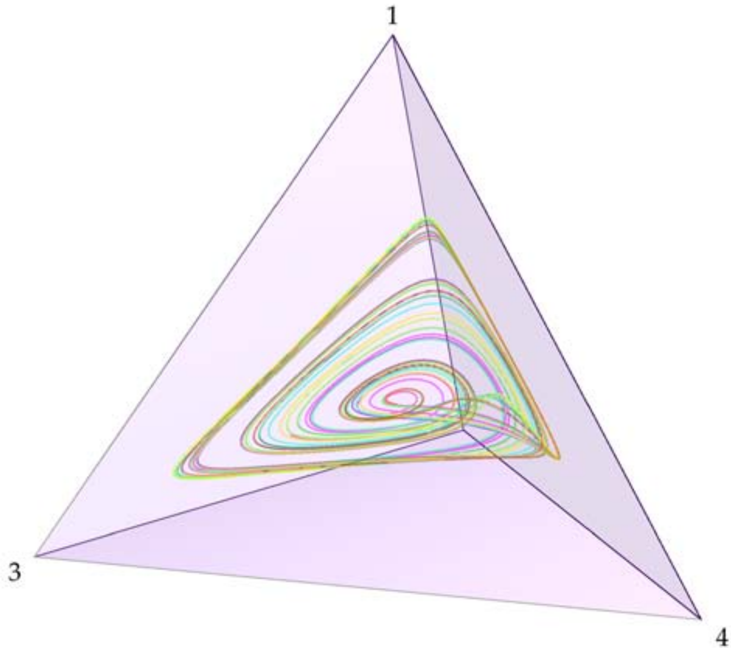
In population games with four or more strategies, and hence state spaces with three or more dimensions, solutions of game dynamics can converge to chaotic attractors.

Central to most definitions of chaos is sensitive dependence on initial conditions: solution trajectories starting from close together points on the attractor move apart at an exponential rate.

Chaotic attractors can be recognized by their intricate appearance.

Consider the replicator dynamic in

$$F(x) = Ax = \begin{pmatrix} 0 & -12 & 0 & 22 \\ 20 & 0 & 0 & -10 \\ -21 & -4 & 0 & 35 \\ 10 & -2 & 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$



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3
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9.2 Survival of Strictly Dominated Strategies

We saw earlier that strictly dominated strategies are eliminated under imitative dynamics and the best response dynamic.

How robust are these results?

Consider the Smith dynamic for "bad RPS with a twin":

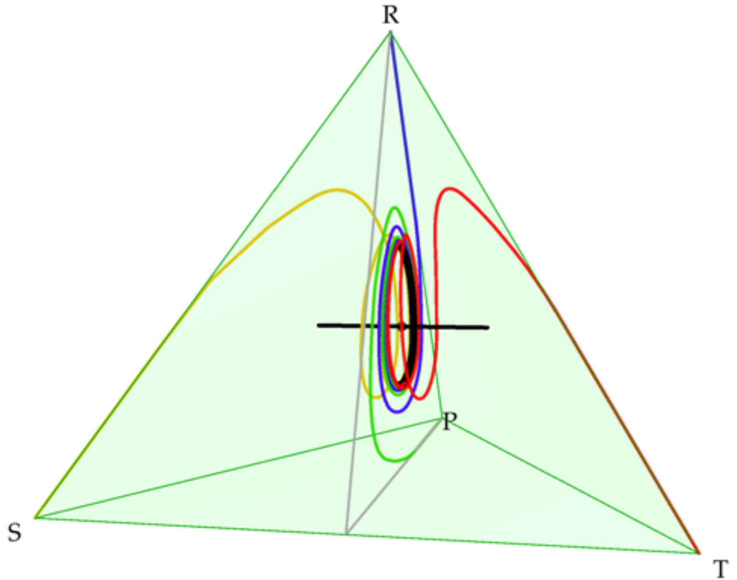
$$F(x) = Ax = \begin{pmatrix} 0 & -2 & 1 & 1 \\ 1 & 0 & -2 & -2 \\ -2 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_R \\ x_P \\ x_S \\ x_T \end{pmatrix}.$$

The Nash equilibria of F are the states on line segment $\{x^* \in X \colon x^* = (\frac{1}{3}, \frac{1}{3}, c, \frac{1}{3} - c)\}$, which is a repellor under the Smith dynamic.

Away from Nash equilibrium, strategies gain players at rates that depend on their payoffs, but lose players at rates proportional to their current usage levels.

The proportions of players choosing the twin strategies are therefore equalized, with the state approaching the plane $\mathcal{P} = \{x \in X : x_S = x_T\}$.

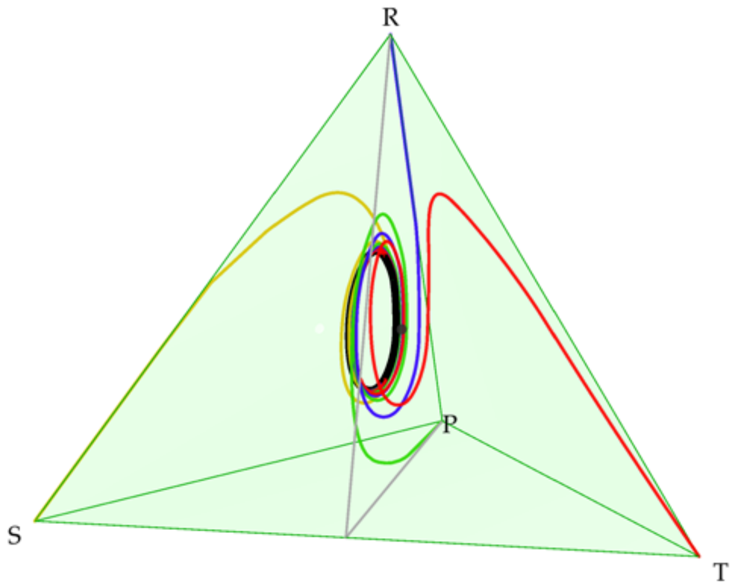
Since F is based on bad RPS, solutions on plane $\mathcal{P}$ approach a closed orbit away from any Nash equilibrium.



The Smith dynamic in bad RPS with a twin.

Now consider the Smith dynamic in “bad RPS with a feeble twin”,

$$F^d(x) = A^d x = \begin{pmatrix} 0 & -2 & 1 & 1 \\ 1 & 0 & -2 & -2 \\ -2 & 1 & 0 & 0 \\ -2-d & 1-d & -d & -d \end{pmatrix} \begin{pmatrix} x_R \\ x_P \\ x_S \\ x_T \end{pmatrix},$$



The Smith dynamic in bad RPS with a feeble twin, $d = \frac{1}{10}$.

To obtain a general survival result, we:

(i) start the analysis with the hypnodisk game.

(ii) introduce a new condition called innovation:

(IN) If $x \notin NE(F)$, $x_i = 0$, and $i \in \underset{j \in S}{\operatorname{argmax}} F_j(x)$, then $(V_F)_i(x) > 0$.

In words: when a non-Nash population state includes an unused optimal strategy, this strategy's growth rate must be positive.

Theorem.

Suppose the Lipschitz continuous evolutionary dynamic $V^{(\cdot)}$ satisfies (NS), (PC), *and* (IN).

Then there is a game F such that under $\dot{x} = V^F(x)$, along solutions from most initial conditions, there is a strictly dominated strategy played by a fraction of the population bounded away from 0.

Remarks:

1. The theorem covers hybrid dynamics that combine imitation with a small probability of direct choice of alternatives. Thus the elimination result for imitative dynamics is not robust.

2. The best response dynamic is not subject to the theorem because it is discontinuous. But one can show numerically that dominated strategies can survive in economically significant proportions under the logit(η) dynamic when η is small.

3. It is important that agents base their decisions on the strategies' present payoffs, and not on general knowledge about payoff functions.

4. The possibility that dynamics do not converge is crucial to the survival result.

10. Connections and Further Developments

10.1 Connections with Stochastic Stability Theory

10.2 Connections with Models of Heuristic Learning

10.3 Games with Continuous Strategy Sets

10.4 Extensive Form Games and Set-Valued Solution Concepts

10.5 Applications

Probability Models and their Interpretation

Countable Probability Models

$(\Omega, \mathbb{P})$	countable probability model
Ω	sample space (finite or countable)
$\mathbb{P} : 2^{\Omega} \to [0, 1]$	probability measure

$\mathbb{P}$ must satisfy $\mathbb{P}(\varnothing) = 0$, $\mathbb{P}(\Omega) = 1$, and *countable additivity*:

if $\{A_k\}$ is a finite or countable collection of disjoint *events* (i.e., subsets of Ω), then $\mathbb{P}(\bigcup_k A_k) = \sum_k \mathbb{P}(A_k)$.

A random variable X is a function whose domain is Ω.

The distribution of X is defined by $\mathbb{P}(X \in B) = \mathbb{P}(\omega \in \Omega : X(\omega) \in B)$ for all subsets B of the range of X.

To define a finite collection of discrete random variables $\{X_k\}_{k=1}^n$, we specify a probability model $(\Omega, \mathbb{P})$ and then define the random variables as functions on Ω.

To interpret this construction, imagine picking an ω at random from the sample space Ω according to the probability distribution $\mathbb{P}$.

The value of ω so selected determines the realizations $X_1(\omega), X_2(\omega), \ldots, X_n(\omega)$ of the entire sequence of random variables $X_1, X_2, \ldots, X_n$.

Example (Repeated rolls of a fair die).

$$R = \{1, 2, 3, 4, 5, 6\}$$

$$\Omega = R^n$$

$$\mathbb{P}(\{\omega\}) = (\tfrac{1}{6})^n \text{ for all } \omega \in \Omega$$

$$X_k(\omega) = \omega_k$$

Then

$$\mathbb{P}(X_k = x_k) = \mathbb{P}(\omega \in \Omega : X_k(\omega) = x_k) = \mathbb{P}(\omega \in \Omega : \omega_k = x_k) = \frac{1}{6}$$

$$\mathbb{P}\left(\bigcap_{k=1}^{n} \{X_k \in A_k\}\right) = \prod_{k=1}^{n} \mathbb{P}(X_k \in A_k).$$

The expected value of a random variable is its integral with respect to the probability measure $\mathbb{P}$.

In the case of the kth roll of a fair die,

$$\mathbb{E}X_k = \int_\Omega X_k(\omega)\,\mathrm{d}\mathbb{P}(\omega) = \sum_{\omega\in\Omega} \omega_k\,\mathbb{P}(\{\omega\}) = \sum_{\omega_k\in R} \omega_k \left(\sum_{\omega_{-k}} \mathbb{P}(\{(\omega_k, \omega_{-k})\}) \right) = \sum_{i=1}^{6} i \times \tfrac{1}{6} = 3\tfrac{1}{2}.$$

We can create new random variables out of old ones using functional operations.

$$S_n = \sum_{k=1}^{n} X_k$$

More explicitly,

$$S_n(\omega) = \sum_{k=1}^{n} X_k(\omega) \text{ for all } \omega \in \Omega.$$

Uncountable Probability Models and Measure Theory

Suppose we want to construct a random variable representing a uniform draw from the unit interval.

It is natural to choose $\Omega = [0,1]$ as our sample space and to define our random variable as the identity function on Ω: that is, $X(\omega) = \omega$.

But it is impossible to define a countably additive probability measure $\mathbb{P}$ that specifies the probability of *every* subset of Ω!

To resolve this problem, one chooses a set of subsets $\mathcal{F} \subseteq 2^{\Omega}$ whose probabilities will be specified, and then introduces corresponding restrictions on the definition of a random variable.

A random variable satisfying these restrictions is said to be measurable.

In summary, an uncountable probability model consists of a triple $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is a sample space, $\mathcal{F} \subseteq 2^{\Omega}$ is a collection (more specifically, a σ-algebra) of subsets of Ω, and $\mathbb{P} : \mathcal{F} \to [0,1]$ is a countably additive probability measure.

Distributional Properties and Sample Path Properties

Why bother with the explicit construction of random variables?

Why not just work with the joint distributions?

This is fine for distributional properties of the random variables.

But many key results in probability theory concern not the distributional properties of random variables, but rather their sample path properties.

These are properties of realization sequences: i.e., the sequences of values $X_1(\omega), X_2(\omega), X_3(\omega), \ldots$ that arise for each choice of $\omega \in \Omega$.

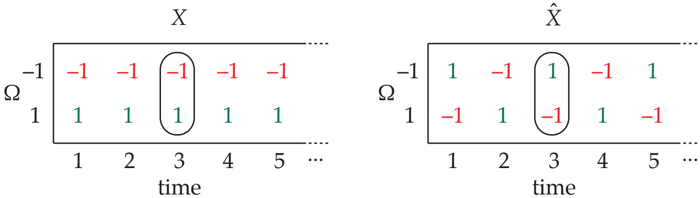
Example. Consider the probability model $(\Omega, \mathbb{P})$ with sample space $\Omega = \{-1, 1\}$ and probability measure $\mathbb{P}(\{-1\}) = \mathbb{P}(\{1\}) = \frac{1}{2}$.

Define the sequences of random variables $\{X_t\}_{t=1}^{\infty}$ and $\{\hat{X}_t\}_{t=1}^{\infty}$ as follows:

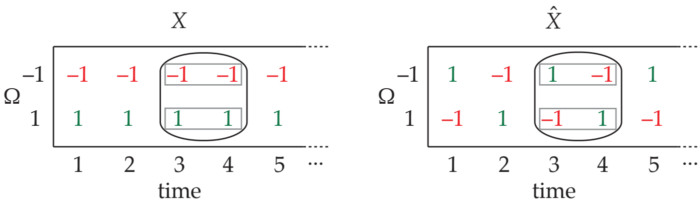
$$
\begin{aligned}
X_t(\omega) &= \omega; \\
\hat{X}_t(\omega) &= \begin{cases} -\omega & \text{if } t \text{ is odd,} \\ \omega & \text{if } t \text{ is even.} \end{cases}
\end{aligned}
$$

The time t marginal distributions, $\{X_t\}_{t=1}^{\infty}$ and $\{\hat{X}_t\}_{t=1}^{\infty}$ look identical.

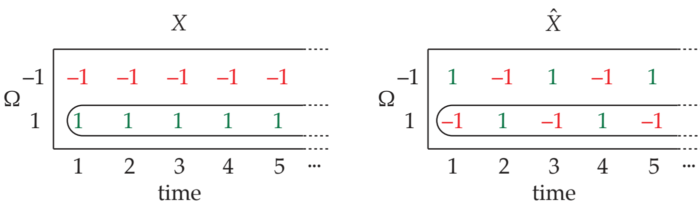
But from the sample path point of view, the two sequences are different.



(i) Time t marginal distributions of X and $\hat{X}$.



(ii) Time $(t, t+1)$ joint distributions of X and $\hat{X}$.



(iii) Sample paths of X and $\hat{X}$

Distributional properties are only meaningful from an ex ante point of view, before the realization of ω (and hence of the random variables) is known.

Theorems on distributional properties constrain the probabilities of certain events, often in the limit as some parameter (e.g., the number of trials) grows large.

Theorems on sample path properties typically state that with probability one, the infinite sequence of realizations of a process must satisfy certain properties.

These theorems can be interpreted as ex post statements about the random variables, since they provide information about the infinite sequence of realizations $\{X_t(\omega)\}_{t=1}^{\infty}$ that we actually observe.

(To be precise, this is "only" true for a set of ωs that has probability one: we cannot completely avoid referring to the *ex ante* point of view.)

Example (Properties of i.i.d. random variables).

The Weak Law of Large Numbers: For all $\varepsilon > 0$, $\lim_{n\to\infty} \mathbb{P}\left(\omega \in \Omega : \overline{X}_n(\omega) \in [-\varepsilon, \varepsilon]\right) = 1.$

The Strong Law of Large Numbers: $\mathbb{P}\left(\omega \in \Omega : \lim_{n\to\infty} \overline{X}_n(\omega) = 0\right) = 1.$

Distributional and sample path results can be distinguished by the order in which $\mathbb{P}$ and lim appear:

The WLLN concerns a "limit of probabilities".

The SLLN concerns a "probability of a limit".

Results about variation of sums of i.i.d. random variables:

The Central Limit Theorem: $$\lim_{n\to\infty} \mathbb{P}\left(\omega \in \Omega : \frac{S_n(\omega)}{\sqrt{n}} \in [a,b]\right) = \frac{1}{\sqrt{2\pi}} \int_a^b \mathrm{e}^{-x^2/2}\,\mathrm{d}x.$$

The Law of the Iterated Logarithm: $$\mathbb{P}\left(\omega \in \Omega : \limsup_{n\to\infty} \frac{S_n(\omega)}{\sqrt{2n\log\log n}} = 1\right) = 1.$$

Example (Kurtz's Theorem).

$$\lim_{N\to\infty} \mathbb{P}\left(\omega \in \Omega : \sup_{t\in[0,T]} \left|X_t^N(\omega) - x_t\right| < \varepsilon\right) = 1. \tag{47}$$

Kurtz's Theorem concerns a distributional property.

It is a statement about a limit of probabilities.

The "random variables" $\{X^N\}_{t\in[0,T]}$ whose distributions are at issue are infinite-dimensional, taking values in the set of step functions from the time interval $[0, T]$ to the simplex X.

Countable State Markov Chains and Processes

Markov chains and Markov processes are collections of random variables $\{X_t\}_{t \in T}$ with the property that "the future only depends on the past through the present".

We focus on settings where these random variables take values in some finite or countable state space $\mathcal{X}$.

We use the terms "Markov chain" and "Markov process" to distinguish between the discrete time ($T = \{0, 1, \ldots\}$) and continuous time ($T = [0, \infty)$) frameworks.

Countable State Markov Chains

The sequence of random variables $\{X_t\} = \{X_t\}_{t=0}^{\infty}$ is a Markov chain if it satisfies the Markov property:

$$\mathbb{P}\left(X_{t+1} = x_{t+1} \,|\, X_0 = x_0, \ldots, X_t = x_t\right) = \mathbb{P}\left(X_{t+1} = x_{t+1} \,|\, X_t = x_t\right)$$

for all times $t \in \{0, 1, \ldots\}$ and all collections of states $x_0, \ldots, x_{t+1} \in \mathcal{X}$ for which the conditional expectations are well defined.

We only consider temporally homogeneous Markov chains, which are Markov chains whose one-step transition probabilities are independent of time:

$$\mathbb{P}\left(X_{t+1} = y \,|\, X_t = x\right) = P_{xy}.$$

We call the matrix $P \in \mathbb{R}_+^{\mathcal{X} \times \mathcal{X}}$ the transition matrix for the Markov chain $\{X_t\}$.

The vector $\pi \in \mathbb{R}_+^{\mathcal{X}}$ defined by $\mathbb{P}(X_0 = x) = \pi_x$ is the initial distribution of $\{X_t\}$.

The vector π and the matrix P fully determine the joint distributions of $\{X_t\}$ via

$$\mathbb{P}\left(X_0 = x_0, \ldots, X_t = x_t\right) = \pi_{x_0} \prod_{s=1}^{t} P_{x_{s-1} x_s}.$$

Since certain properties of Markov chains do not depend on the initial distribution π, it is sometimes left unspecified.

By definition, the one-step transition probabilities of the Markov chain $\{X_t\}$ are the elements of the matrix P:

$$\mathbb{P}(X_1 = y \mid X_0 = x) = P_{xy}.$$

The two-step transition probabilities of $\{X_t\}$ are obtained by multiplying P by itself

$$\begin{aligned}\mathbb{P}(X_2 = y \mid X_0 = x) &= \sum_{z \in \mathcal{X}} \mathbb{P}(X_2 = y, X_1 = z \mid X_0 = x) \\ &= \sum_{z \in \mathcal{X}} \mathbb{P}(X_1 = z \mid X_0 = x)\, \mathbb{P}(X_2 = y \mid X_1 = z, X_0 = x) \\ &= \sum_{z \in \mathcal{X}} P_{xz} P_{zy} \\ &= (P^2)_{xy}.\end{aligned}$$

Thus the t step transition probabilities of $\{X_t\}$ are given by the entries of the tth power of the transition matrix:

$$\mathbb{P}(X_t = y \mid X_0 = x) = (P^t)_{xy}.$$

Countable State Markov Processes: Definition and Construction

A (temporally homogeneous) Markov process on the countable state space $\mathcal{X}$ is a collection of random variables $\{X_t\} = \{X_t\}_{t \geq 0}$ with continuous time index t. This collection must satisfy the following three properties:

(MP) The (continuous time) Markov property:

$\mathbb{P}\left(X_{t_{k+1}} = x_{t_{k+1}} \middle| X_{t_0} = x_{t_0}, \ldots, X_{t_k} = x_{t_k}\right) = \mathbb{P}\left(X_{t_{k+1}} = x_{t_{k+1}} \middle| X_{t_k} = x_{t_k}\right)$ for all $0 \leq t_0 < \ldots < t_{k+1}$ and $x_{t_0}, \ldots, x_{t_{k+1}} \in \mathcal{X}$ with $\mathbb{P}\left(X_{t_0} = x_{t_0}, \ldots, X_{t_k} = x_{t_k}\right) > 0$.

(TH) Temporal homogeneity:

$\mathbb{P}\left(X_{t+u} = y \,|\, X_t = x\right) = P_{xy}(u)$ for all $t, u \geq 0$.

(RCLL) Right continuity and left limits:

For every $\omega \in \Omega$, the sample path $\{X_t(\omega)\}_{t \geq 0}$ is continuous from the right and has left limits. That is, $\lim_{s \downarrow t} X_s(\omega) = X_t(\omega)$ for all $t \in [0, \infty)$, and $\lim_{s \uparrow t} X_s(\omega)$ exists for all $t \in (0, \infty)$.

While conditions (MP) and (TH) are restrictions on the (joint) distributions of $\{X_t\}$, condition (RCLL) is a restriction on the sample paths of $\{X_t\}$.

Long Run Behavior of Markov Chains (and Processes)

Communication, Recurrence, and Irreducibility

State y is accessible from state x, denoted $x \rightsquigarrow y$, if for some $n \geq 0$ there is a sequence of states $x = x_0, x_1, \ldots, x_n = y$ such that $P_{x_{i-1},x_i} > 0$ for all $i \in \{1, \ldots, n\}$.

We allow $n = 0$ to ensure that each state is accessible from itself.

We write $x \leftrightsquigarrow y$ to indicate that x and y are mutually accessible.

Accessibility defines a partial order on the set $\mathcal{X}$.

The equivalence classes under this order, referred to as communication classes, are the maximal sets of (pairwise) mutually accessible states.

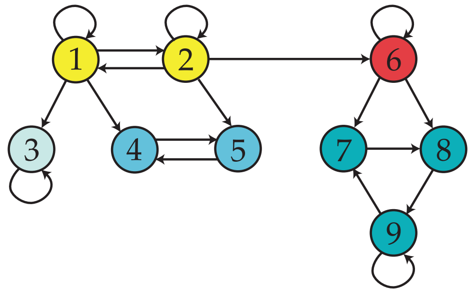
A set of states $R \subseteq \mathcal{X}$ is closed if the process cannot leave it: $[x \in R, x \rightsquigarrow y] \Rightarrow y \in R$.

If R is a communciation class, then R is closed if and only if it is minimal under $\rightsquigarrow$.

Once $\{X_t\}$ enters a closed communication class, it remains in the class forever.

Example. $\mathcal{X} = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$$P = \begin{pmatrix} .4 & .3 & .2 & .1 & 0 & 0 & 0 & 0 & 0 \\ .5 & .1 & 0 & 0 & .3 & .1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & .8 & .1 & .1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & .9 & 0 & .1 \end{pmatrix}$$



State x is transient if $\mathbb{P}_x(\{t : X_t = x\}$ is unbounded$) = 0$.

State x recurrent if $\mathbb{P}_x(\{t : X_t = x\}$ is unbounded$) = 1$.

Theorem. *Let $\{X_t\}$ be a Markov chain or Markov process on a finite set $\mathcal{X}$. Then*

(*i*) *Every state in $\mathcal{X}$ is either transient or recurrent.*

(*ii*) *A state is recurrent if and only if it is a member of a closed communication class.*

A closed communication class R is commonly called a recurrent class.

When all of $\mathcal{X}$ forms a single recurrent class, $\{X_t\}$ is said to be irreducible.

ex.: stochastic evolutionary processes with full support revision protocols.

Example (Birth and death chains).

$$\mathcal{X} = \{0, 1, \ldots, N\}$$

$$P_{ij} = \begin{cases} p_i & \text{if } j = i + 1, \\ q_i & \text{if } j = i - 1, \\ 1 - p_i - q_i & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

Irreducible if and only if $p_k > 0$ for $k < N$ and $q_k > 0$ for $k > 0$.

Periodicity

In the discrete-time case, the behavior of the Markov chain $\{X_t\}_{t=0}^{\infty}$ within a recurrent class depends on the period structure of that class.

The period of the recurrent state x of the Markov chain $\{X_t\}_{t=0}^{\infty}$ is

$$\gcd(\{t \geq 1 : \mathbb{P}_x(X_t = x) > 0\}).$$

It is the greatest common divisor of the set of times at which the chain can revisit x if it is run from x.

If the Markov chain $\{X_t\}$ is irreducible, then all of its states are of the same period.

If this common period is greater than 1, $\{X_t\}$ is periodic.

If it is 1, $\{X_t\}$ is aperiodic.

If $P_{xx} > 0$ for some $x \in \mathcal{X}$, then $\{X_t\}$ is aperiodic.

Hitting Times and Hitting Probabilities

Proposition. *Let $\{X_t\}$ be an irreducible Markov chain or process, let $Z \subseteq X$, and let $\{w_x\}_{x \notin Z}$ be the collection of expected times to hit Z starting from states outside Z.*

If $\{X_t\}_{t=0}^{\infty}$ is a Markov chain with transition matrix P, then $\{w_x\}_{x \notin Z}$ is the unique solution to the linear equations

$$w_x = 1 + \sum_{y \notin Z} P_{xy}\, w_y \quad \text{for all } x \notin Z. \tag{48}$$

The Perron-Frobenius Theorem

Call a transition matrix $P \in \mathbb{R}^{\mathcal{X} \times \mathcal{X}}$ (i.e., a nonnegative row matrix with row sums equal to 1) as a stochastic matrix.

Call such a matrix irreducible or aperiodic according to whether the induced Markov chain has these properties.

Theorem (Perron-Frobenius).

Suppose that the matrix $P \in \mathbb{R}^{\mathcal{X}\times\mathcal{X}}$ is stochastic and irreducible. Then:

(*i*) 1 *is an eigenvalue of P of algebraic multiplicity* 1, *and no eigenvalue of P has modulus greater than* 1.

(*ii*) *The vector* $\mathbf{1}$ *is a right eigenvector of P corresponding to eigenvalue* 1. *That is,* $P\mathbf{1} = \mathbf{1}$.

(*iii*) *There is a probability vector μ with positive components that is a left eigenvector of P corresponding to eigenvalue* 1; *thus,* $\mu' P = \mu'$.

Suppose in addition that P is aperiodic. Then:

(*iv*) *All eigenvalues of P other than* 1 *have modulus less than* 1.

(*v*) *The matrix powers P^t converge to the matrix $\mathbf{1}\mu'$ as t approaches infinity.*

(*vi*) *Indeed, let λ_2 be the eigenvalue of P with the second-largest modulus, and let $r \in (|\lambda_2|, 1)$. Then for some $c > 0$, we have that*

$$\max_{ij} \left|(P^t - \mathbf{1}\mu')_{ij}\right| \leq c\, r^t \text{ for all } t \geq 1.$$

If P is (*real or complex*) *diagonalizable, this statement remains true when $r = |\lambda_2|$.*

Stationary Distributions for Markov Chains

A probability distribution $\mu \in \mathbb{R}^{\mathcal{X}}$ a stationary distribution of $\{X_t\}_{t=0}^{\infty}$ if

$$\mu' P = \mu'. \tag{49}$$

More explicitly, μ is a stationary distribution if

$$\sum_{x \in \mathcal{X}} \mu_x P_{xy} = \mu_y \quad \text{for all } y \in \mathcal{X}. \tag{50}$$

Decompose the probability of the chain being at state y at time 1 as follows:

$$\sum_{x \in \mathcal{X}} \mathbb{P}(X_0 = x)\, \mathbb{P}(X_1 = y \,|\, X_0 = x) = \mathbb{P}(X_1 = y).$$

Thus if X_0 is distributed according to the stationary distribution μ, then X_1 is also distributed according to μ; by the Markov property, so is every subsequent X_t.

Theorem. *If the Markov chain $\{X_t\}$ is irreducible, it has a unique stationary distribution.*

Example.

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mu = (\tfrac{1}{2}, \tfrac{1}{2}).$$

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ .9 & 0 & .1 \end{pmatrix}, \quad \mu = (\tfrac{9}{28}, \tfrac{9}{28}, \tfrac{10}{28}).$$

The hitting time of state x under the Markov chain $\{X_t\}$ is the random variable $T_x = \inf\{t \geq 1 : X_t = x\}$.

When $\{X_t\}$ is run from initial condition x, T_x is called the return time of state x.

Proposition. *If the Markov chain $\{X_t\}$ is irreducible with stationary distribution μ, then $\mathbb{E}_x T_x = \mu_x^{-1}$ for all $x \in \mathcal{X}$.*

The higher is the weight on x in the stationary distribution, the less time we expect will pass before a chain starting at x returns to this state.

Reversible Markov Chains

Markov chain $\{X_t\}$ is reversible if it admits a reversible distribution: a probability distribution μ that satisfies the detailed balance conditions

(51) $\quad \mu_x P_{xy} = \mu_y P_{yx}$ for all $x, y \in \mathcal{X}$.

Summing over $x \in \mathcal{X}$ shows that μ is also a stationary distribution.

Why the name?

Example (Birth and death chains: Reversibility and the stationary distribution). The detailed balance conditions (51) reduce to $\mu_k q_k = \mu_{k-1} p_{k-1}$ for $k \in \{1, \ldots, N\}$. Applying this formula inductively and normalizing yields

$$\mu_k = \mu_0 \prod_{i=1}^{k} \frac{p_{i-1}}{q_i} \text{ for } k \in \{1, \ldots, N\}, \text{ and } \mu_0 = \left(\sum_{k=0}^{N} \prod_{i=1}^{k} \frac{p_{i-1}}{q_i} \right)^{-1}.$$

Convergence in Distribution

Theorem (Convergence in distribution). *Suppose that* $\{X_t\}$ *is an irreducible aperiodic Markov chain with stationary distribution is* μ. *Then for any initial distribution* π,

$$\lim_{t\to\infty} \mathbb{P}_\pi(X_t = x) = \mu_x \text{ for all } x \in \mathcal{X}.$$

This is part (v) of Perron-Frobenius:

$$\lim_{t\to\infty} \mathbb{P}_\pi(X_t = x) = \lim_{t\to\infty} (\pi' P^t)_x = (\pi' \mathbf{1} \mu')_x = \mu_x.$$

Part (vi) of Perron-Frobenius shows that the rate of convergence to the stationary distribution is determined by the second-largest eigenvalue modulus of P.

Ergodicity

Having considered distributional properties, we turn to sample path properties.

Theorem (Ergodicity)**.**

Suppose that $\{X_t\}_{t=0}^{\infty}$ is an irreducible Markov chain with stationary distribution μ. Then for any initial distribution π,

$$\mathbb{P}_\pi\left(\lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1}1_{\{X_t=x\}}=\mu_x\right)=1 \;\; \textit{for all } x\in\mathcal{X}.$$

For almost all realizations of $\omega \in \Omega$, the proportion of time that the sample path $\{X_t(\omega)\}$ spends in each state x converges; the limit is the s.d. weight μ_x.

If $\{X_t\}_{t=0}^{\infty}$ is an i.i.d. sequence, with each X_t having distribution μ, this conclusion can be obtained by applying the SLLN to the sequence of indicator RVs $\{1_{\{X_t=x\}}\}_{t=0}^{\infty}$.

The theorem shows that Markov dependence is enough to reach this conclusion.

While irreducible, aperiodic Markov chains converge in distribution and are ergodic, the latter two properties are distinct in general.

A Markov chain that is irreducible but not aperiodic will still be ergodic, but will not converge in distribution.

Conversely, consider a Markov chain whose transition matrix is the identity matrix. This chain (trivially) converges in distribution to its initial distribution, but the chain is not ergodic.

Stationary Distributions and Infinite Horizon Behavior

Irreducibile Evolutionary Processes

Full Support Revision Protocols

The Markov chain $\{X_t^N\}_{t \in \{0, \frac{1}{N}, \frac{2}{N} \dots\}}$:

State space: $\mathcal{X}^N = X \cap \frac{1}{N}\mathbb{Z}^n = \{x \in X : Nx \in \mathbb{Z}^n\}$

Transition probabilities:

$$(52) \qquad P_{xy}^N = \begin{cases} \dfrac{x_i \rho_{ij}(F(x), x)}{R} & \text{if } y = x + \frac{1}{N}(e_j - e_i),\, j \neq i, \\ 1 - \displaystyle\sum_{i \in S} \sum_{j \neq i} \frac{x_i \rho_{ij}(F(x), x)}{R} & \text{if } y = x, \\ 0 & \text{otherwise.} \end{cases}$$

Revision protocol ρ has full support if for some $\underline{R} > 0$,

$$(53) \qquad \rho_{ij}(F(x), x) \geq \underline{R} \;\text{ for all } i, j \in S \text{ and } x \in X.$$

Example (Best response with mutations, mutation rate $\varepsilon > 0$).

A revising agent switches to his current best response with probability $1 - \varepsilon$, but chooses a strategy uniformly at random (or mutates) with probability $\varepsilon > 0$.

Must specify what happens when there are multiple best responses.

Example (Logit choice, noise level $\eta > 0$).

$$\begin{aligned}\rho_{ij}(\pi) &= \frac{\exp(\eta^{-1}\pi_j)}{\sum_{k \in S} \exp(\eta^{-1}\pi_k)} \\ &= \frac{\exp(\eta^{-1}(\pi_j - \pi_{k^*}))}{\sum_{k \in S} \exp(\eta^{-1}(\pi_k - \pi_{k^*}))}, \quad \text{where } k^* \text{ is optimal under } \pi.\end{aligned}$$

Stationary Distributions for Two-Strategy Games

Birth and death chains:

$$P^N_{\chi y} \equiv \begin{cases} p^N_\chi & \text{if } y = \chi + \frac{1}{N}, \\ q^N_\chi & \text{if } y = \chi - \frac{1}{N}, \\ 1 - p^N_\chi - q^N_\chi & \text{if } y = \chi, \\ 0 & \text{otherwise.} \end{cases}$$

For irreducibility: $p^N_\chi > 0$ for $\chi < 1$ and $q^N_\chi > 0$ for $\chi > 0$.

Stationary distribution:

$$\text{(54)} \qquad \frac{\mu^N_\chi}{\mu^N_0} = \prod_{j=1}^{N\chi} \frac{p^N_{(j-1)/N}}{q^N_{j/N}} \;\; \text{for } \chi \in \{\tfrac{1}{N}, \ldots, 1\}, \quad \mu^N_0 = \left(\sum_{i=0}^{N} \prod_{j=1}^{i} \frac{p^N_{(j-1)/N}}{q^N_{j/N}} \right)^{-1}.$$

For two-strategy games, let $S = \{0, 1\}$, and write $\chi = x_1$.

Example (Toss and switch).

When an agent's clock rings, he flips a fair coin and switches strategies after Heads.

$p^N_\chi = \frac{1}{2}(1 - \chi)$

$q^N_\chi = \frac{1}{2}\chi.$

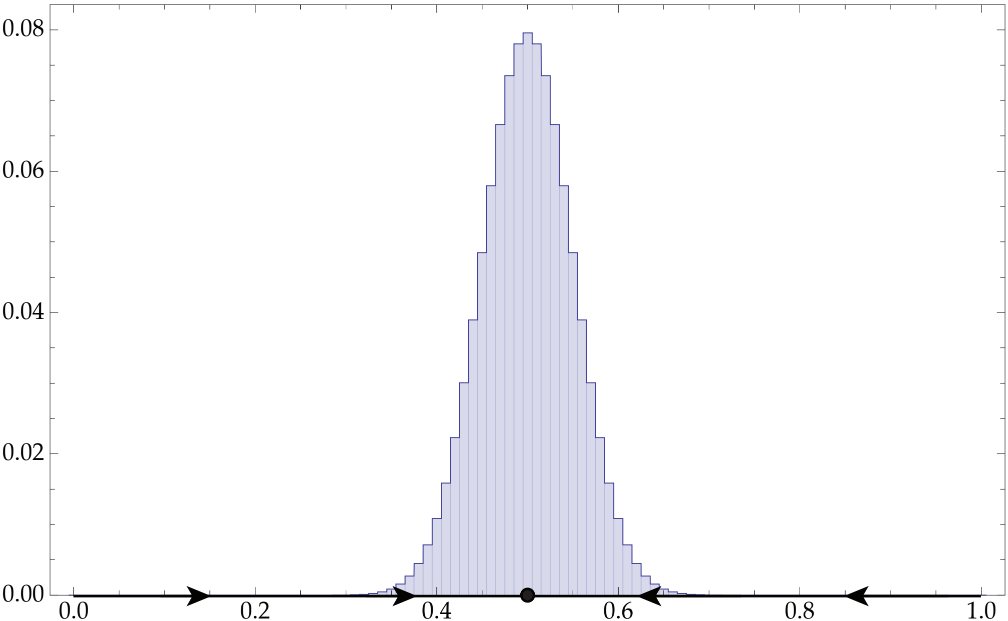
Mean dynamic: $\dot{\chi} = \frac{1}{2} - \chi.$

Solutions: $\chi_t = \frac{1}{2} + (\chi_0 - \frac{1}{2})\, \mathrm{e}^{-t}.$

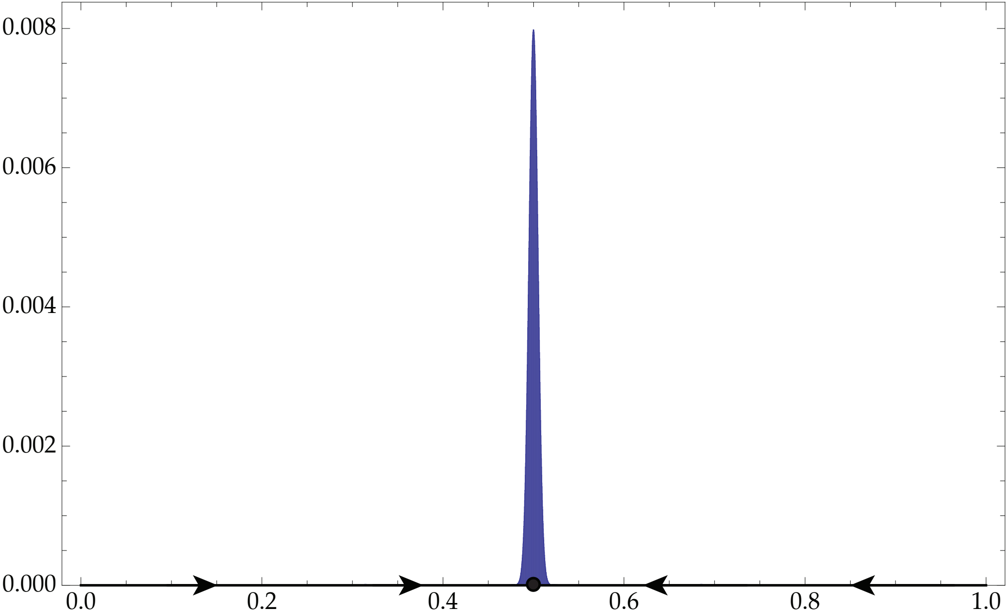
Stationary distribution:

$$\mu^N_\chi = \frac{1}{2^N}\binom{N}{N\chi} \text{ for all } \chi \in \mathcal{X}^N = \{0, \tfrac{1}{N}, \ldots, 1\}.$$

= a "binomial distribution" with parameters N and $\frac{1}{2}$, but with outcomes in $\mathcal{X}^N$.



$N = 100$



$N = 10{,}000$

Fix two-strategy game F, full support revision protocol ρ, and population size N.

$S = \{0, 1\}$, $\chi = x_1$.

For $\chi \in \{\frac{1}{N}, \frac{2}{N}, \ldots, 1\}$,

$$\frac{\mu_\chi^N}{\mu_0^N} = \prod_{j=1}^{N\chi} \frac{p_{(j-1)/N}^N}{q_{j/N}^N} = \prod_{j=1}^{N\chi} \frac{(1 - \frac{j-1}{N})}{\frac{j}{N}} \cdot \frac{\frac{1}{R}\,\rho_{01}(F(\frac{j-1}{N}), \frac{j-1}{N})}{\frac{1}{R}\,\rho_{10}(F(\frac{j}{N}), \frac{j}{N})}.$$

Theorem. *The stationary distribution for the evolutionary process $\{X_t^N\}$ on $\mathcal{X}^N$ is*

$$\frac{\mu_\chi^N}{\mu_0^N} = \prod_{j=1}^{N\chi} \frac{(N - j + 1)}{j} \cdot \frac{\rho_{01}(F(\frac{j-1}{N}), \frac{j-1}{N})}{\rho_{10}(F(\frac{j}{N}), \frac{j}{N})} \quad \textit{for } \chi \in \{\tfrac{1}{N}, \tfrac{2}{N}, \ldots, 1\},$$

with μ_0^N determined by the requirement that $\sum_{\chi \in \mathcal{X}^N} \mu_\chi^N = 1$.

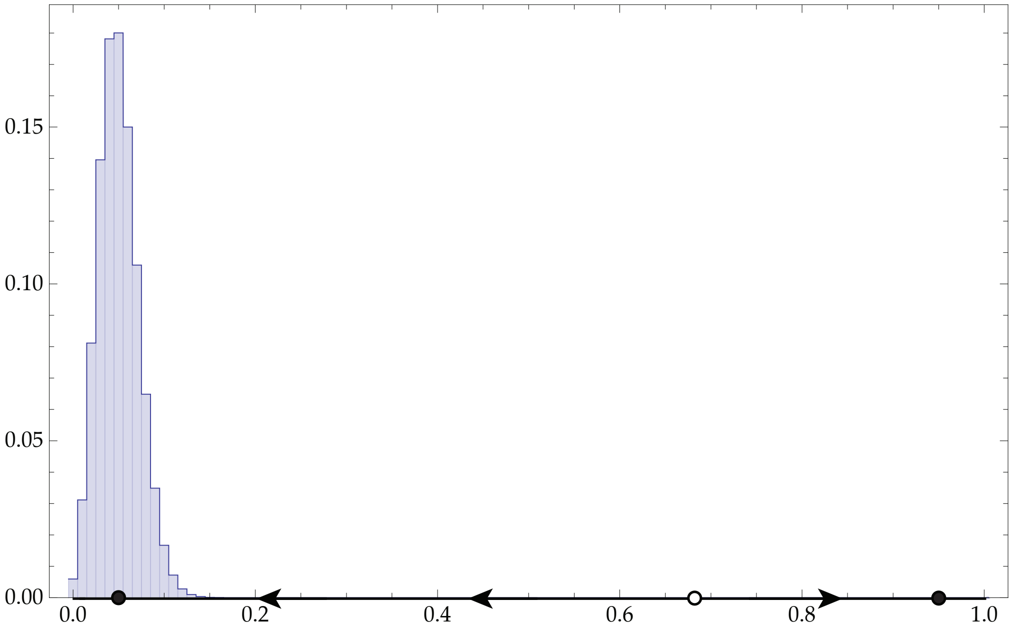
Example (Stag Hunt).

$$A = \begin{pmatrix} h & h \\ 0 & s \end{pmatrix} \quad s > h > 0.$$

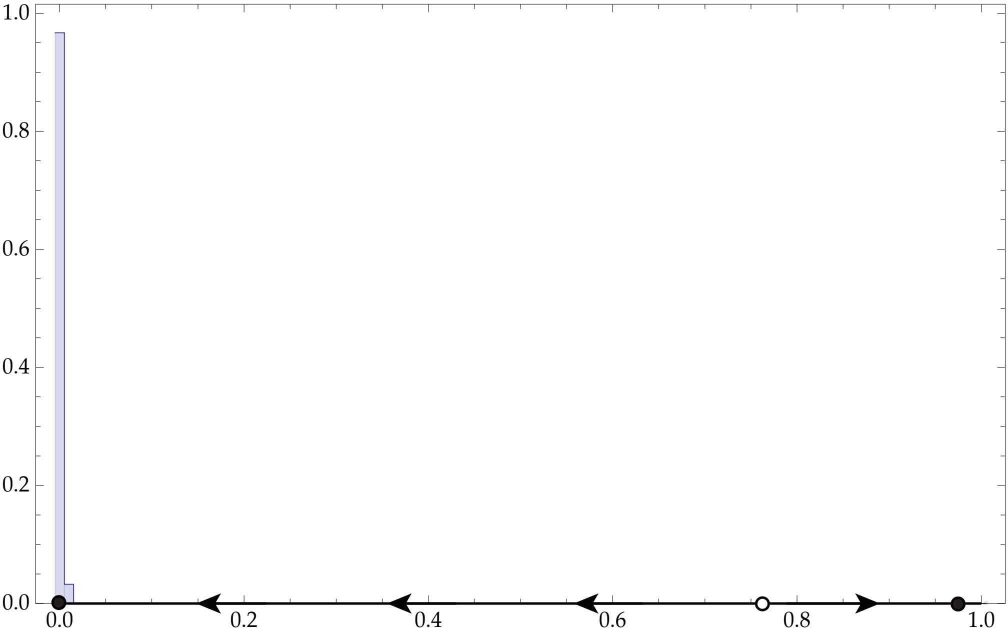
$F_H(\chi) = h$ and $F_S(\chi) = s\chi$.

Three Nash equilibria: $\chi = 1$, $\chi = 0$, $\chi^* = \frac{h}{s}$.

Let $h = 2$ and $s = 3$, so that $\chi^* = \frac{2}{3}$.



best response with mutations (ε = .10), N = 100



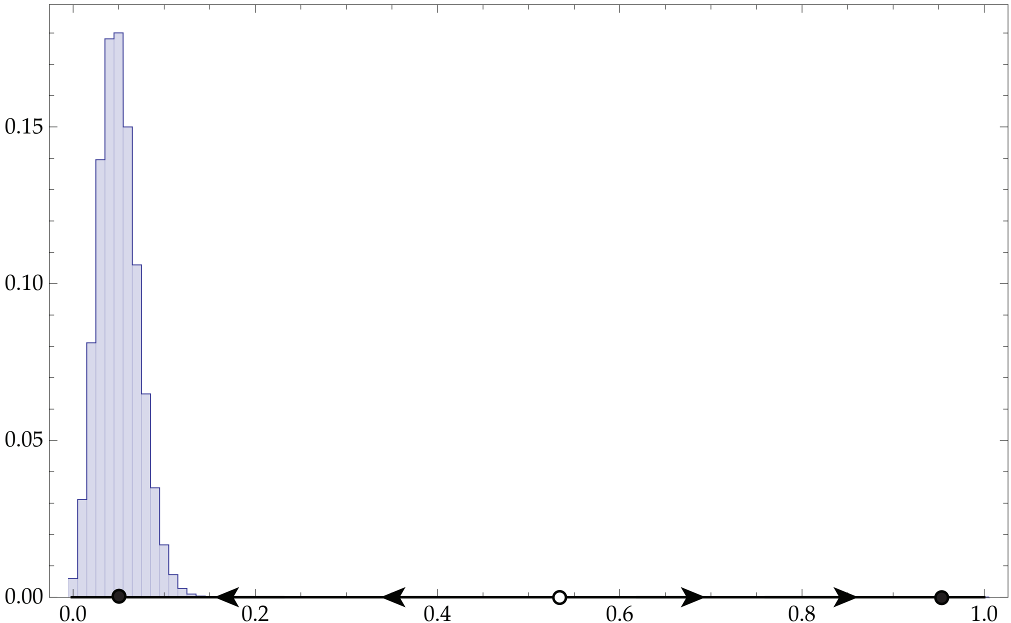
logit (η = .25), N = 100

Example (A nonlinear Stag Hunt).

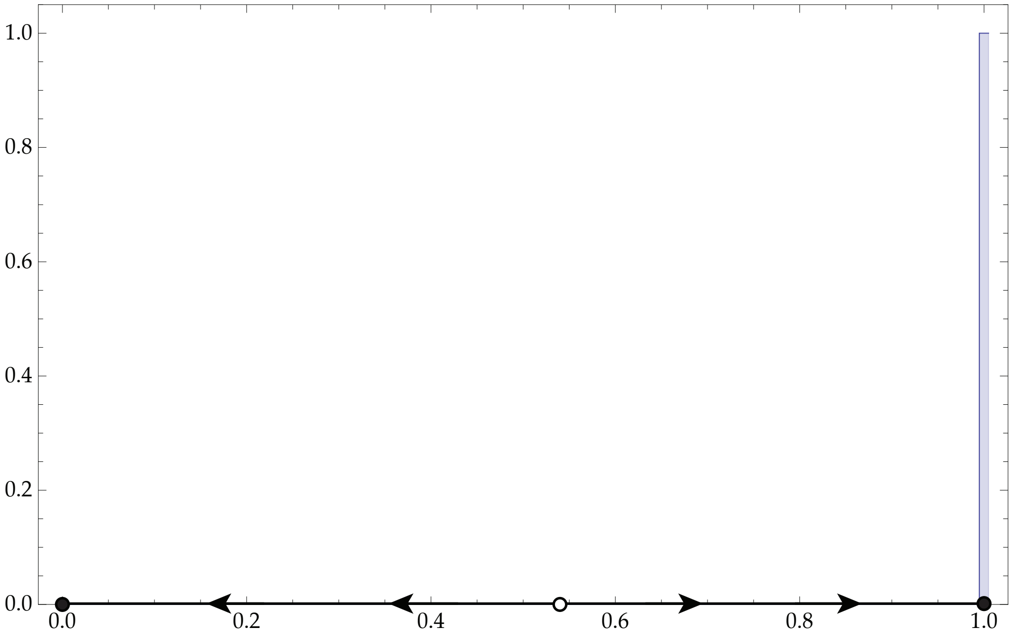
$F_H(\chi) = h$ and $F_S(\chi) = s\chi^2$,

Three Nash equilibria: $\chi = 0$, $\chi = 1$, $\chi^* = \sqrt{h/s}$.

Let $h = 2$ and $s = 7$, so that $\chi^* = \sqrt{2/7} \approx .5345$.



best response with mutations (ε = .10), N = 100



logit (η = .25), N = 100

Waiting Times and Infinite Horizon Prediction

Example (Toss and Switch once more).

$$p^N_\chi = \tfrac{1}{2}(1-\chi), \quad q^N_\chi = \tfrac{1}{2}\chi.$$

	$N = 100$	$N =$1,000	$N =$10,000
$x = .45$	2.0389	2.2588	2.2977
$x = .50$	2.9378	4.0891	5.2403

Table 5: Expected wait to reach state x from state 0 in Toss and Switch.

Mean dynamic: $\quad \dot{\chi} = \frac{1}{2} - \chi$

Solutions: $\quad \chi_t = \frac{1}{2} + (\chi_0 - \frac{1}{2})\,\mathrm{e}^{-t}$

Time for mean dynamic to go from $\chi = 0$ to $\chi = .45$: $\quad T = \log 10 \approx 2.3026.$

Time for mean dynamic to go from $\chi = 0$ to $\chi = .50$: $\quad T = \infty.$

Example (Stag Hunt once more).

$$A = \begin{pmatrix} 2 & 2 \\ 0 & 3 \end{pmatrix} \quad \chi^* = \tfrac{2}{3} \tag{55}$$

Under BRM(.10) protocol, stationary distribution puts most weight near $\chi = .05$.

	$N = 100$	$N = 1{,}000$	$N = 10{,}000$
$\varepsilon = .1$	3.03×10^{17}	7.33×10^{171}	1.59×10^{1720}
$\varepsilon = .01$	1.23×10^{50}	2.60×10^{492}	1.43×10^{4920}

Table 6: Expected wait to go from $\chi = .95$ to $\chi^* = \frac{2}{3}$ under BRM(ε).

Estimated age of the universe: 4.33×10^{17} seconds.

Discussion: Waiting times and infinite-horizon prediction

1. Relevance for economic modeling

2. Relevance for biological modeling

3. Alternative models

    (i) local interactions

    (ii) aggregate noise

Model Adjustments for Finite Populations

Finite-Population Games

$S = \{1, \ldots, n\}$	set of strategies
$\mathcal{X}^N = \{x \in X : Nx \in \mathbb{Z}^n\}$	N agent state space
$F^N : \mathcal{X}^N \to \mathbb{R}^n$	N agent finite-population game
$F_i^N(x) \in \mathbb{R}$	payoff to strategy i at $x \in \mathcal{X}^N$
$\mathcal{X}_i^N = \{x \in \mathcal{X}^N : x_i > 0\}$	where F_i^N actually needs to be defined

$x \in \mathcal{X}^N$ is a Nash equilibrium of F^N if

$$[x_i > 0 \Rightarrow F_i^N(x) \geq F_j^N(x + \tfrac{1}{N}(e_j - e_i))] \text{ for all } i, j \in S.$$

A sequence of finite-population games $\{F^N\}_{N=N_0}^{\infty}$ converges uniformly to $F : X \to \mathbb{R}^n$ if

$$\lim_{N\to\infty} \max_{x \in \mathcal{X}^N} \left| F^N(x) - F(x) \right| = 0. \tag{56}$$

Clever Payoff Evaluation

If a population has N members, an agent who switches from i to j when the state is x changes the state to $x + \frac{1}{N}(e_j - e_i)$.

If this agent wants to compare his current payoff $F_i^N(x)$ to the payoff he will obtain after switching, the relevant comparison is not to $F_j^N(x)$, but rather to $F_j^N(x+\frac{1}{N}(e_j-e_i))$.

Agents who account for this when deciding whether to switch strategies use clever payoff evaluation.

Those who do not use simple payoff evaluation.

$\mathcal{X}^N_- = \{z \in \mathbb{R}^n_+ : \sum_{i \in S} z_i = \frac{N-1}{N} \text{ and } Nz \in \mathbb{Z}^n\}$ set of diminished population states

Given a game $F^N : \mathcal{X}^N \to \mathbb{R}^n$, define the clever payoff function $\check{F}^N : \mathcal{X}^N_- \to \mathbb{R}^n$ by

$$(57) \qquad \check{F}^N_k(z) = F^N_k(z + \tfrac{1}{N}e_k).$$

$\check{F}^N(z)$ describes the current payoff opportunities of an agent whose *opponents'* behavior distribution is $z \in \mathcal{X}^N_-$.

An agent using revision protocol $\rho = \rho(\pi, x)$ in game F^N is clever
if at state x, his conditional switch rate from i to j is not $\rho_{ij}(F^N(x), x)$, but

$$(58) \qquad \rho_{ij}(\check{F}^N(x - \tfrac{1}{N}e_i), x).$$

Committed Agents and Imitative Protocols

Imitative processes are not irreducible.

To make them so, we can add one (or more) committed agent for each strategy in $S = \{1, \ldots, n\}$.

Example. Suppose that a standard agent who receives a revision opportunity picks an opponent at random and imitates him: that is, let $r_{ij} \equiv 1$, so that $\rho_{ij}(\pi, x) = x_j$.

Without committed agents, the resulting stochastic evolutionary process converges with probability 1 to one of the n pure states $e_1, \ldots, e_n$.

But with a single committed agent for each strategy, the process $\{X_t^N\}$ is irreducible.

In fact, the process is reversible, and its stationary distribution is the uniform distribution on $\mathcal{X}^N$.

Proof. Let $x \in \mathcal{X}^N$ and $y = x + \frac{1}{N}(e_j - e_i)$.

Then $z = x - \frac{1}{N}e_i = y - \frac{1}{N}e_j \in \mathcal{X}^N_-$ represents the behavior of the revising player's opponents.

$$
\begin{aligned}
P^N_{xy} &= x_i \cdot \frac{Nx_j + 1}{N + n - 1} \\
&= \frac{Nz_i + 1}{N} \cdot \frac{Nz_j + 1}{N + n - 1} \\
&= \frac{Nz_j + 1}{N} \cdot \frac{Nz_i + 1}{N + n - 1} \\
&= y_j \cdot \frac{Ny_i + 1}{N + n - 1} \\
&= P^N_{yx}.
\end{aligned}
$$

Exponential Protocols and Potential Games

Finite-Population Potential Games

A finite-population game $F^N : \mathcal{X}^N \to \mathbb{R}^n$ is a potential game if it admits a full potential function: a function $f^N : \mathcal{X}^N \cup \mathcal{X}_-^N \to \mathbb{R}$ such that

$$(59) \qquad F_i^N(x) = f^N(x) - f^N(x - \tfrac{1}{N} e_i) \text{ for all } x \in \mathcal{X}^N \text{ and } i \in S.$$

Thus the payoff to i is the ith "discrete partial derivative" of $\frac{1}{N} f^N$.

(There is an equivalent definition using potential functions defined only on $\mathcal{X}^N$.)

Exponential Revision Protocols

Definition. *We call $\rho : \mathbb{R}^n \to \mathbb{R}_+^{n\times n}$ a direct exponential protocol with noise level η if*

$$\rho_{ij}(\pi) = \frac{\exp(\eta^{-1}\psi(\pi_i, \pi_j))}{d_{ij}(\pi)}, \tag{60}$$

where the functions $\psi : \mathbb{R}^2 \to \mathbb{R}$ and $d : \mathbb{R}^n \to (0,\infty)^{n\times n}$ satisfy

$$\psi(\pi_i, \pi_j) - \psi(\pi_j, \pi_i) = \pi_j - \pi_i, \text{ and} \tag{61}$$

$$d_{ij}(\pi) = d_{ji}(\pi). \tag{62}$$

positive dependence on candidate payoff:	$\psi(\pi_i, \pi_j) = \pi_j$
negative dependence on current payoff:	$\psi(\pi_i, \pi_j) = -\pi_i$
positive dependence on payoff difference:	$\psi(\pi_i, \pi_j) = \frac{1}{2}(\pi_j - \pi_i)$
positive dependence on positive payoff difference:	$\psi(\pi_i, \pi_j) = [\pi_j - \pi_i]_+$
negative dependence on negative payoff difference:	$\psi(\pi_i, \pi_j) = -[\pi_j - \pi_i]_-$

Different choices of the function d can be used to reflect different reference groups that agents employ when considering a switch.

$d_{ij}(\pi) = d_{ji}(\pi)$ says that when an i player considers switches to strategy j, he employs the same comparison group as a j player who considers switching to i.

Suppose $\psi(\pi_i, \pi_j) = \pi_j$.

If agents use the full set of strategies as the comparison group: logit protocol

$$d_{ij}(\pi) = \sum_{k \in S} \exp(\eta^{-1}\pi_k) \quad \Rightarrow \quad \rho_{ij}(\pi) = \frac{\exp(\eta^{-1}\pi_j)}{\sum_{k \in S} \exp(\eta^{-1}\pi_k)}.$$

If the comparison group only contains the current and candidate strategies: pairwise logit protocol

$$d_{ij}(\pi) = \exp(\eta^{-1}\pi_i) + \exp(\eta^{-1}\pi_j) \quad \Rightarrow \quad \rho_{ij}(\pi) = \frac{\exp(\eta^{-1}\pi_j)}{\exp(\eta^{-1}\pi_i) + \exp(\eta^{-1}\pi_j)}.$$

Exercise. Show that ρ is a direct exponential protocol if and only if

$$\eta \log \frac{\rho_{ij}(\pi)}{\rho_{ji}(\pi)} = \pi_j - \pi_i \text{ for all } \pi \in \mathbb{R}^n. \tag{63}$$

Definition. $\rho : \mathbb{R}^n \times X \to \mathbb{R}^{n\times n}_+$ *an imitative exponential protocol with noise level* η *if*

$$\rho_{ij}(\pi, x) = x_j \, \frac{\exp(\eta^{-1}\psi(\pi_i, \pi_j))}{d_{ij}(\pi, x)}, \tag{64}$$

where the functions $\psi : \mathbb{R}^2 \to \mathbb{R}$ *and* $d : \mathbb{R}^n \times X \to (0, \infty)^{n\times n}$ *satisfy conditions* (61) *and*

$$d_{ij}(\pi, x) = d_{ji}(\pi, x). \tag{65}$$

Reversibility and Stationary Distributions

Theorem. *Let F^N be a finite population potential game with potential function f^N, and suppose that agents are clever and follow a direct exponential protocol with noise level η. Then the stochastic evolutionary process $\{X_t^N\}$ is reversible with stationary distribution*

$$(66) \qquad \mu_x^N = \frac{1}{K^N} \frac{N!}{\prod_{k \in S} (N x_k)!} \exp(\eta^{-1} f^N(x))$$

for $x \in \mathcal{X}^N$, where K^N is determined by the requirement that $\sum_{x \in \mathcal{X}^N} \mu_x^N = 1$.

Proof. Let $x \in \mathcal{X}^N$ and $y = x + \frac{1}{N}(e_j - e_i)$.

Let $z = x - \frac{1}{N}e_i = y - \frac{1}{N}e_j \in \mathcal{X}^N_-$.

Then z is both the distribution of opponents for an i player at state x, and the distribution of opponents of a j player at state y.

Thus, in both cases, a clever player who is revising will consider the payoff vector $\check{F}^N(z)$ defined by $\check{F}^N_k(z) = F^N_k(z + \frac{1}{N}e_k)$.

Write $\check{\pi} = \check{F}^N(z)$.

By the definition of the potential function f^N,

$$f^N(y) - f^N(x) = F_j^N(y) - F_i^N(x) = \check{F}_j^N(z) - \check{F}_i^N(z) = \check{\pi}_j - \check{\pi}_i. \tag{67}$$

So:

$$\begin{aligned}
\mu_x^N P_{xy}^N &= \mu_x^N \cdot x_i \, \rho_{ij}(\check{\pi}) \\
&= \frac{1}{K^N} \frac{N!}{\prod\limits_{k \in S} (Nx_k)!} \exp\left(\eta^{-1} f^N(x)\right) \cdot x_i \; \frac{\exp\left(\eta^{-1} \psi(\check{\pi}_i, \check{\pi}_j)\right)}{d_{ij}(\check{\pi})} \\
&= \frac{1}{K^N} \frac{(N-1)!}{\prod\limits_{k \in S} (Nz_k)!} \; \exp\left(\eta^{-1} (f^N(y) - \check{\pi}_j + \check{\pi}_i)\right) \; \frac{\exp\left(\eta^{-1} (\psi(\check{\pi}_j, \check{\pi}_i) + \check{\pi}_j - \check{\pi}_i)\right)}{d_{ji}(\check{\pi})} \\
&= \frac{1}{K^N} \frac{N!}{\prod\limits_{k \in S} (Ny_k)!} \exp\left(\eta^{-1} f^N(y)\right) \cdot y_j \; \frac{\exp\left(\eta^{-1} \psi(\check{\pi}_j, \check{\pi}_i)\right)}{d_{ji}(\check{\pi})} \\
&= \mu_y^N \cdot y_j \, \rho_{ji}(\check{\pi}) \\
&= \mu_y^N P_{yx}^N.
\end{aligned}$$

Theorem. *Let F^N be a finite population potential game with potential function f^N. Suppose that there are N clever agents who follow an imitative exponential protocol with noise level η, and that there is one committed agent for each strategy. Then the stochastic evolutionary process $\{X_t^N\}$ is reversible with stationary distribution*

$$\mu_x^N = \frac{1}{\kappa^N} \exp(\eta^{-1} f^N(x)) \tag{68}$$

for $x \in \mathcal{X}^N$, where κ^N is determined by the requirement that $\sum_{x \in \mathcal{X}^N} \mu_x^N = 1$.

Compare to the case of direct exponential protocols:

$$\mu_x^N = \frac{1}{K^N} \frac{N!}{\prod\limits_{k \in S} (Nx_k)!} \exp(\eta^{-1} f^N(x)) \tag{79}$$

Limiting Stationary Distributions and Stochastic Stability

Idea: For tractable analyses and clean equilibrium selection results, study stationary distributions $\mu^{N,\eta}$ as N and/or η are taken to their limiting values.

Two questions:

Stochastic stability (which states are such that all neighborhoods retain mass?)

Asymptotics of stationary distribution (at *all* states).

Definitions of Stochastic Stability

Small Noise Limits

Fix the population size N, and take the noise level η to zero.

Because N is fixed, each stationary distribution in the collection $\{\mu^{N,\eta}\}_{\eta \in (0,\bar{\eta}]}$ is a probability measure on the same finite state space, $\mathcal{X}^N$.

We call state $x \in \mathcal{X}^N$ stochastically stable in the small noise limit if

$$\lim_{\eta \to 0} \mu_x^{N,\eta} > 0. \tag{69}$$

A less demanding (but more useful) notion of stochastic stability:

Suppose that for some $r_x^N \geq 0$,

$$(70) \qquad -\lim_{\eta \to 0} \eta \log \mu_x^{N,\eta} = r_x^N \quad \Big(\Leftrightarrow \quad \mu_x^{N,\eta} = \exp(-\eta^{-1}(r_x^N + o(1))) \Big).$$

In words, r_x^N is the exponential rate of decay of $\mu_x^{N,\eta}$ as η^{-1} approaches infinity.

(Note: the book leaves off the minus sign in (70), so that $r_x^N \leq 0$.)

State $x \in \mathcal{X}^N$ weakly stochastically stable in the small noise limit if

$$(71) \qquad -\lim_{\eta \to 0} \eta \log \mu_x^{N,\eta} = 0.$$

In words, as η^{-1} approaches infinity, $\mu_x^{N,\eta}$ does not vanish at an exponential rate.

Proposition. *Every stochastically stable state is weakly stochastically stable.*

Large Population Limits

As the population size N grows, the state spaces $\mathcal{X}^N$ vary!

We call state $x \in X$ is stochastically stable in the large population limit if for every open set $O \subseteq X$ containing x, we have

$$\lim_{N \to \infty} \mu^{N,\eta}(O) > 0. \tag{72}$$

A less demanding (but more useful) notion of stochastic stability.

Suppose that for some continuous function $r^\eta \colon X \to \mathbb{R}_+$

$$(73) \qquad \lim_{N\to\infty} \max_{x\in\mathcal{X}^N} \left| -\tfrac{\eta}{N} \log \mu_x^{N,\eta} - r^\eta(x) \right| = 0.$$

Equivalently,

$$(74) \qquad \mu_x^{N,\eta} = \exp\left(-\eta^{-1} N \left(r^\eta(x) + o(1)\right)\right) \text{ uniformly in } x \in \mathcal{X}^N,$$

If $r^\eta(x) = 0$, we call state x weakly stochastically stable in the large population limit.

Proposition. *Suppose that condition* (73) *holds for some continuous function r^η. Then every stochastically stable state is weakly stochastically stable.*

Double Limits

We call state $x \in X$ stochastically stable in the small noise double limit if for every open set $O \subseteq X$ containing x, we have

$$\lim_{N\to\infty} \lim_{\eta\to 0} \mu^{N,\eta}(O) > 0. \tag{75}$$

We call state $x \in X$ stochastically stable in the large population double limit if for every open set $O \subseteq X$ containing x, we have

$$\lim_{\eta\to 0} \lim_{N\to\infty} \mu^{N,\eta}(O) > 0. \tag{76}$$

The less demanding notions.

Suppose that for some continuous function $r\colon X \to \mathbb{R}_+$ we have

$$\lim_{N\to\infty} \lim_{\eta\to 0} \max_{x\in\mathcal{X}^N} \left| -\tfrac{\eta}{N} \log \mu_x^{N,\eta} - r(x) \right| = 0. \tag{77}$$

This describes the small noise double limit: for N large, the exponential rate of decay of $\mu_x^{N,\eta}$ as η^{-1} approaches infinity is approximately $Nr(x)$.

Similarly, if there is a continuous function $\hat{r}\colon X \to \mathbb{R}_+$ such that

$$\lim_{\eta\to 0} \lim_{N\to\infty} \max_{x\in\mathcal{X}^N} \left| -\tfrac{\eta}{N} \log \mu_x^{N,\eta} - \hat{r}(x) \right| = 0, \tag{78}$$

This is the large population double limit: for small η, as N approaches infinity, the exponential rate of decay of the stationary distributions weights on states near x is approximately $\eta^{-1}\hat{r}(x)$.

For either double limit, we say that state x is weakly stochastically stable if its limiting rate of decay, $r(x)$ or $\hat{r}(x)$, is equal to zero.

$$\lim_{N\to\infty}\lim_{\eta\to 0}\max_{x\in\mathcal{X}^N}\left|-\frac{\eta}{N}\log\mu_x^{N,\eta}-r(x)\right|=0. \tag{77}$$

$$\lim_{\eta\to 0}\lim_{N\to\infty}\max_{x\in\mathcal{X}^N}\left|-\frac{\eta}{N}\log\mu_x^{N,\eta}-\hat{r}(x)\right|=0, \tag{78}$$

If $r(\cdot)$ and $\hat{r}(\cdot)$ are identical, then the asymptotics of $\mu^{N,\eta}$ in the two double limits agree in a very strong sense.

Double Limits: A Counterexample

The game: Hawk-Dove (two strategies; the unique Nash equlibrium is mixed)

The revision protocol: Imitation with mutations

The SNDL (77) selects a pure state.

The LPDL (78) selects the Nash equilibrium.

Exponential Protocols and Potential Games

Theorem. *Let F^N be a finite population potential game with potential function f^N, and suppose that agents are clever and follow a direct exponential protocol with noise level η. Then the stochastic evolutionary process $\{X_t^N\}$ is reversible with stationary distribution*

$$\mu_x^{N,\eta} = \frac{1}{K^N} \frac{N!}{\prod_{k\in S} (Nx_k)!} \exp(\eta^{-1} f^N(x)) \tag{79}$$

for $x \in \mathcal{X}^N$, where K^N is determined by the requirement that $\sum_{x\in\mathcal{X}^N} \mu_x^N = 1$.

What are the asymptotics of $\mu^{N,\eta}$ in the various limits?

Definition: for $g\colon C \to \mathbb{R}$, define $\Delta g\colon C \to \mathbb{R}_-$ by

$$\Delta g(x) = g(x) - \max_{y\in C} g(y).$$

The Small Noise Limit

$$\frac{\mu_x^{N,\eta}}{\mu_y^{N,\eta}} = \frac{\prod_{k\in S}(Ny_k)!}{\prod_{k\in S}(Nx_k)!} \exp\left(\eta^{-1}\left(f^N(x) - f^N(y)\right)\right) \tag{80}$$

$$\begin{aligned} \lim_{\eta\to 0} \eta \log \frac{\mu_x^{N,\eta}}{\mu_y^{N,\eta}} &= \lim_{\eta\to 0}\left(\left(f^N(x) - f^N(y)\right) + \eta \log\left(\frac{\prod_{k\in S}(Ny_k)!}{\prod_{k\in S}(Nx_k)!}\right)\right) \\ &= f^N(x) - f^N(y). \end{aligned} \tag{81}$$

Theorem.

$$\lim_{\eta\to 0} \eta \log \mu_x^{N,\eta} = \Delta f^N(x) \ \ \textit{for all } x \in \mathcal{X}^N. \tag{82}$$

The Large Population Limit

We need to consider a sequence of potential games $\{F^N\}_{N=N_0}^{\infty}$ that "settles down".

We suppose that the rescaled potential functions $\{\frac{1}{N} f^N\}_{N=N_0}^{\infty}$ converge uniformly to a limit function $f \colon X \to \mathbb{R}$.

This is necessary but not sufficient for uniform convergence of $\{F^N\}_{N=N_0}^{\infty}$

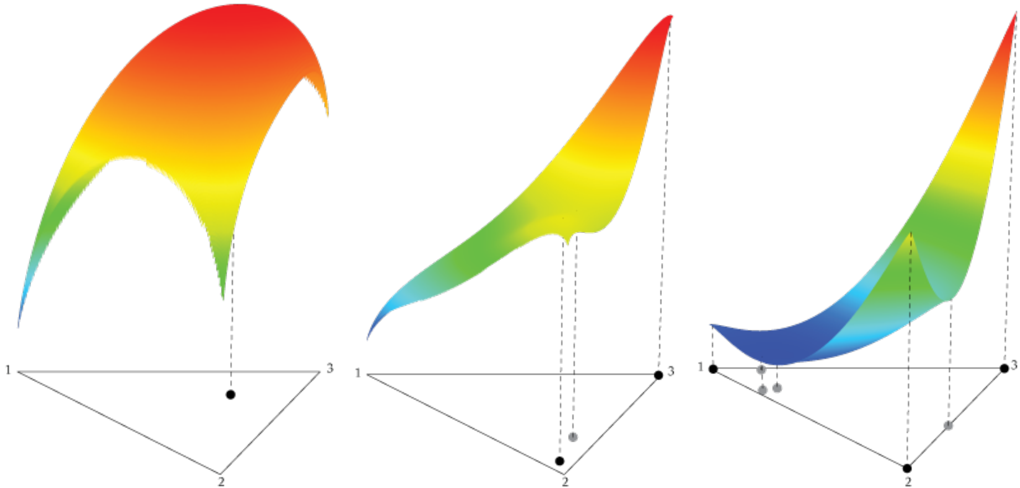
To account for the multinomial term in $\mu^{N,\eta}$, we need the logit potential function

$$f^\eta(x) = f(x) - \eta \sum_{i \in S} x_i \log x_i$$

$h(x) = -\sum_{i \in S} x_i \log x_i$ is the entropy function,
which measures the "randomness" of a probability distribution x. (Set $0 \log 0 \equiv 0$.)

Theorem.

$$\lim_{N \to \infty} \max_{x \in \mathcal{X}^N} \left| \frac{\eta}{N} \log \mu_x^{N,\eta} - \Delta f^\eta(x) \right| = 0. \tag{83}$$



f^{η} in 123 Coordination with $\eta = 1.5$, $\eta = .6$, $\eta = .2$

$$(84)\qquad \frac{\mu_x^{N,\eta}}{\mu_{e_1}^{N,\eta}} = \frac{N!}{\prod\limits_{i\in S}(Nx_i)!}\exp\left(\eta^{-1}\left(f^N(x) - f^N(e_1)\right)\right).$$

Stirling's formula: $N! \approx \sqrt{2\pi N}\, N^N \exp(-N)$.

$$(\Rightarrow)\qquad \frac{\mu_x^{N,\eta}}{\mu_{e_1}^{N,\eta}} \approx \exp\left(\eta^{-1}\left(f^N(x) - f^N(e_1)\right)\right)\frac{\sqrt{2\pi N}\, N^N \exp(-N)}{\prod\limits_{i\in S}\sqrt{2\pi N x_i}\,(Nx_i)^{Nx_i}\exp(-Nx_i)}$$

$$= \exp\left(\eta^{-1}\left(f^N(x) - f^N(e_1)\right)\right)\left(\prod_{i\in S} x_i^{-(Nx_i+1/2)}\right)\cdot\frac{1}{(2\pi N)^{(n-1)/2}}$$

$$(\Rightarrow)\qquad \frac{\eta}{N}\log\frac{\mu_x^{N,\eta}}{\mu_{e_1}^{N,\eta}} \approx \frac{1}{N}f^N(x) - \frac{1}{N}f^N(e_1) - \eta\sum_{i\in S} x_i\log x_i - \frac{\eta}{N}\frac{n-1}{2}\log 2\pi N.$$

$$(\Rightarrow)\qquad \lim_{N\to\infty}\max_{x\in\mathcal{X}^N}\left|\frac{\eta}{N}\log\frac{\mu_x^{N,\eta}}{\mu_{e_1}^{N,\eta}} - \left(f^\eta(x) - f^\eta(e_1)\right)\right| = 0.$$

Double Limits

$$\lim_{\eta\to 0} \eta \log \mu_x^{N,\eta} = \Delta f^N(x) \;\text{ for all } x \in \mathcal{X}^N. \tag{82}$$

$$\lim_{N\to\infty} \max_{x\in\mathcal{X}^N} \left| \frac{\eta}{N} \log \mu_x^{N,\eta} - \Delta f^\eta(x) \right| = 0. \tag{83}$$

The double limits agree:

Corollary.

(*i*) $$\lim_{N\to\infty} \lim_{\eta\to 0} \max_{x\in\mathcal{X}^N} \left| \frac{\eta}{N} \log \mu_x^{N,\eta} - \Delta f(x) \right| = 0 \;\; \textit{and}$$

(*ii*) $$\lim_{\eta\to 0} \lim_{N\to\infty} \max_{x\in\mathcal{X}^N} \left| \frac{\eta}{N} \log \mu_x^{N,\eta} - \Delta f(x) \right| = 0.$$

Noisy Best Response Protocols in Two-Strategy Games

Notation for two-strategy games:

$\chi \equiv x_1$

$\mathcal{X}^N = \{0, \frac{1}{N}, \dots, 1\} \subset [0, 1]$.

$F(\chi)$ (vs. $F(x)$);

$\rho(\pi, \chi)$ (vs. $\rho(\pi, x)$)

Noisy best response protocols:

(85) $\rho_{ij}^{\eta}(\pi) = \sigma^{\eta}(\pi_j - \pi_i),$

Basic requirement:

$$\lim_{\eta \to 0} \sigma^{\eta}(a) = \begin{cases} 1 & \text{if } a > 0, \\ 0 & \text{if } a < 0. \end{cases}$$

Additional structure:

The unlikelihood (or cost) of switching to a strategy with payoff *disadvantage d* is

(86) $\kappa(d) = -\lim_{\eta \to 0} \eta \log \sigma^{\eta}(-d).$

Equivalently:

$$\sigma^{\eta}(-d) = \exp\left(-\eta^{-1}(\kappa(d) + o(1))\right).$$

$$(86) \qquad \kappa(d) = -\lim_{\eta \to 0} \eta \log \sigma^{\eta}(-d).$$

$$(\Leftrightarrow) \qquad \sigma^{\eta}(-d) = \exp\left(-\eta^{-1}(\kappa(d) + o(1))\right).$$

Definition. *A noisy best response protocol is* regular *if*

(*i*) *the limit in* (86) *exists for all* $d \in \mathbb{R}$, *with convergence uniform on compact intervals;*

(*ii*) κ *is nondecreasing;*

(*iii*) $\kappa(d) = 0$ *whenever* $d < 0$*;*

(*iv*) $\kappa(d) > 0$ *whenever* $d > 0$.

Example (Best response with mutations).

$$\sigma^\eta(a) = \begin{cases} 1 - \exp(-\eta^{-1}) & \text{if } a > 0, \\ \exp(-\eta^{-1}) & \text{if } a \le 0. \end{cases} \quad (\eta = -(\log \varepsilon)^{-1})$$

If $d \ge 0$, then $-\eta \log \sigma^\eta(-d) = 1$. Thus

$$\kappa(d) = \begin{cases} 1 & \text{if } d \ge 0, \\ 0 & \text{if } d < 0. \end{cases}$$

Example (Logit choice).

$$\sigma^\eta(a) = \frac{\exp(\eta^{-1}a)}{\exp(\eta^{-1}a) + 1}.$$

If $d \ge 0$, then $-\eta \log \sigma^\eta(-d) = d + \eta \log(\exp(-\eta^{-1}d) + 1)$. Thus

$$\kappa(d) = \begin{cases} d & \text{if } d > 0, \\ 0 & \text{if } d \le 0. \end{cases}$$

Example (Probit choice).

$$\sigma^\eta(a) = \mathbb{P}(\sqrt{\eta}\, Z + a > \sqrt{\eta}\, Z'),$$

where Z and Z' are independent and standard normal. Then

$$\sigma^\eta(a) = \Phi\left(\frac{a}{\sqrt{2\eta}}\right), \tag{87}$$

where Φ is the standard normal distribution function.

Well known approximation: for $z < 0$, $\Phi(z)$ is of order $\exp(\frac{-z^2}{2})$.

Thus $-\eta \log \sigma^\eta(-d) = -\eta \log \Phi\left(\frac{-d}{\sqrt{2\eta}}\right) \approx -\eta \cdot \frac{-d^2}{4\eta} = \frac{1}{4}d^2$, so

$$\kappa(d) = \begin{cases} \frac{1}{4}d^2 & \text{if } d > 0, \\ 0 & \text{if } d \leq 0. \end{cases}$$

The Small Noise Limit

For convenience, we again assume clever payoff evaluation.

Define the relative unlikelihood function

$$\tilde{\kappa}(d) = \kappa(d) - \kappa(-d) = \lim_{\eta \to 0} \Big(-\eta \log \sigma^{\eta}(-d) + \eta \log \sigma^{\eta}(d) \Big).$$

Define $I^N : \mathcal{X}^N \to \mathbb{R}$ by

$$I^N(\chi) = \sum_{j=1}^{N\chi} \tilde{\kappa}\Big(F_1^N(\tfrac{j}{N}) - F_0^N(\tfrac{j-1}{N}) \Big), \tag{88}$$

Theorem. *If agents are clever and employ a regular noisy best response protocol with unlikelihood function* κ*, then*

$$\lim_{\eta \to 0} \eta \log \mu_{\chi}^{N,\eta} = \Delta I^N(\chi) \quad \textit{for all } \chi \in \mathcal{X}^N. \tag{89}$$

Idea of proof:

By the birth and death chain formula:

$$\eta \log \frac{\mu_x^{N,\eta}}{\mu_0^{N,\eta}} = \eta \log \left(\prod_{j=1}^{Nx} \frac{(N-j+1)}{j} \cdot \frac{\rho_{01}^{\eta}(\check{F}^N(\frac{j-1}{N}), \frac{j-1}{N})}{\rho_{10}^{\eta}(\check{F}^N(\frac{j-1}{N}), \frac{j}{N})} \right) \tag{90}$$

$$= \sum_{j=1}^{Nx} \left(-\eta \log \sigma^{\eta} \left(F_0^N(\tfrac{j-1}{N}) - F_1^N(\tfrac{j}{N}) \right) + \eta \log \sigma^{\eta} \left(F_1^N(\tfrac{j}{N}) - F_0^N(\tfrac{j-1}{N}) \right) + \eta \log \frac{N-j+1}{j} \right).$$

Thus

$$\begin{aligned}
\lim_{\eta \to 0} \eta \log \frac{\mu_x^{N,\eta}}{\mu_0^{N,\eta}} &= \sum_{j=1}^{Nx} \left(\kappa \left(F_1^N(\tfrac{j}{N}) - F_0^N(\tfrac{j-1}{N}) \right) - \kappa \left(F_0^N(\tfrac{j}{N}) - F_1^N(\tfrac{j-1}{N}) \right) \right) \\
&= \sum_{j=1}^{Nx} \tilde{\kappa} \left(F_1^N(\tfrac{j}{N}) - F_0^N(\tfrac{j-1}{N}) \right) \\
&= I^N(x).
\end{aligned}$$

Suppose $\{F^N\}_{N=N_0}^{\infty}$ converges uniformly to F, where $F\colon [0,1] \to \mathbb{R}^2$ is continuous.

Let $F_\Delta(\chi) \equiv F_1(\chi) - F_0(\chi)$.

Let $\tilde{\sigma}^\eta(a) = \dfrac{\sigma^\eta(a)}{\sigma^\eta(-a)}$. (relative choice probability function)

Define $I^\eta\colon [0,1] \to \mathbb{R}$ by

$$I^\eta(\chi) = \int_0^\chi \eta \log \tilde{\sigma}^\eta(F_\Delta(y))\,\mathrm{d}y - \eta\left(\chi \log \chi + (1-\chi)\log(1-\chi)\right). \tag{91}$$

Theorem. *If agents are clever and employ regular noisy best response protocol σ^η, then*

$$\lim_{N\to\infty} \max_{\chi \in \mathcal{X}^N} \left| \tfrac{\eta}{N} \log \mu_\chi^{N,\eta} - \Delta I^\eta(\chi) \right| = 0.$$

$$F_\Delta(x) \equiv F_1(x) - F_0(x); \quad \tilde{\sigma}^\eta(a) = \frac{\sigma^\eta(a)}{\sigma^\eta(-a)}.$$

Idea of proof:

$$\eta \log \frac{\mu_x^{N,\eta}}{\mu_0^{N,\eta}} = \sum_{j=1}^{Nx} \left(-\eta \log \sigma^\eta \left(F_0^N(\tfrac{j-1}{N}) - F_1^N(\tfrac{j}{N}) \right) + \eta \log \sigma^\eta \left(F_1^N(\tfrac{j}{N}) - F_0^N(\tfrac{j-1}{N}) \right) + \eta \log \tfrac{N-j+1}{j} \right)$$

$$= \frac{\eta}{N} \sum_{j=1}^{Nx} \left(\log \tilde{\sigma}^\eta \left(F_1^N(\tfrac{j}{N}) - F_0^N(\tfrac{j-1}{N}) \right) + \log \tfrac{N-j+1}{N} - \log \tfrac{j}{N} \right).$$

By the dominated convergence theorem,

$$\begin{aligned}
\lim_{N\to\infty} \frac{\eta}{N} \log \frac{\mu_x^{N,\eta}}{\mu_0^{N,\eta}} &= \int_0^x \eta \left(\log \tilde{\sigma}^\eta \left(F_1(y) - F_0(y) \right) + \log(1-y) - \log(y) \right) \mathrm{d}y \\
&= \int_0^x \eta \log \tilde{\sigma}^\eta \left(F_\Delta(y) \right) \mathrm{d}y - \eta \left(x \log x + (1-x) \log(1-x) \right) \\
&= I^\eta(x)
\end{aligned}$$

Double Limits

$$\tilde{\sigma}^{\eta}(a) = \frac{\sigma^{\eta}(a)}{\sigma^{\eta}(-a)} \qquad \tilde{\kappa}(a) = \lim_{\eta \to 0} \eta \log \tilde{\sigma}^{\eta}(a)$$

SNL: $\quad I^N(\chi) = \sum_{j=1}^{N\chi} \tilde{\kappa}\left(F_{\Delta}^{N}(\tfrac{j}{N})\right)$

LPL: $\quad I^{\eta}(\chi) = \int_0^{\chi} \eta \log \tilde{\sigma}^{\eta}(F_{\Delta}(y))\,\mathrm{d}y - \eta\left(\chi \log \chi + (1-\chi)\log(1-\chi)\right).$

Define the ordinal potential function $I(\chi) = \int_0^{\chi} \tilde{\kappa}(F_{\Delta}(y))\,\mathrm{d}y$

Theorem.

(*i*) $\quad \lim_{N\to\infty} \lim_{\eta\to 0} \max_{\chi \in \mathcal{X}^N} \left| \frac{\eta}{N} \log \mu_{\chi}^{N,\eta} - \Delta I(\chi) \right| = 0$ *and*

(*ii*) $\quad \lim_{\eta\to 0} \lim_{N\to\infty} \max_{\chi \in \mathcal{X}^N} \left| \frac{\eta}{N} \log \mu_{\chi}^{N,\eta} - \Delta I(\chi) \right| = 0.$

Example (Best response with mutations).

$$\kappa(d) = 1_{d \geq 0} \quad \Rightarrow \quad I_{\mathrm{sgn}}(\chi) = \int_0^{\chi} \mathrm{sgn}\left(F_\Delta(y)\right) \mathrm{d}y \qquad \text{signum potential function}$$

Example (Logit choice).

$$\kappa(d) = [d]_+ \quad \Rightarrow \quad I_1(\chi) = \int_0^{\chi} F_\Delta(y) \, \mathrm{d}y \qquad \text{(standard) potential function}$$

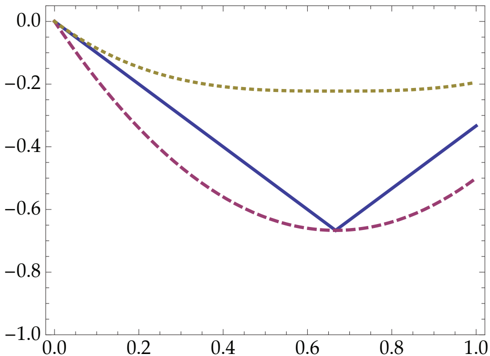
Example (probit choice).

$$\kappa(d) = \tfrac{1}{4}[d]_+^2 \quad \Rightarrow \quad I_2(\chi) = \int_0^{\chi} \tfrac{1}{4} \left\langle F_\Delta(y) \right\rangle^2 \mathrm{d}y \qquad \text{quadratic potential function}$$

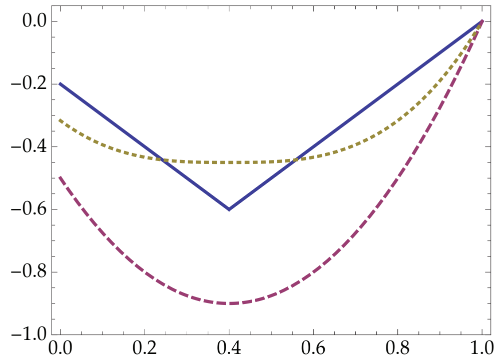
where $\langle a \rangle^2 = \mathrm{sgn}(a) \, a^2$.

Example (Stag Hunt revisited).

$$A = \begin{pmatrix} h & h \\ 0 & s \end{pmatrix}$$



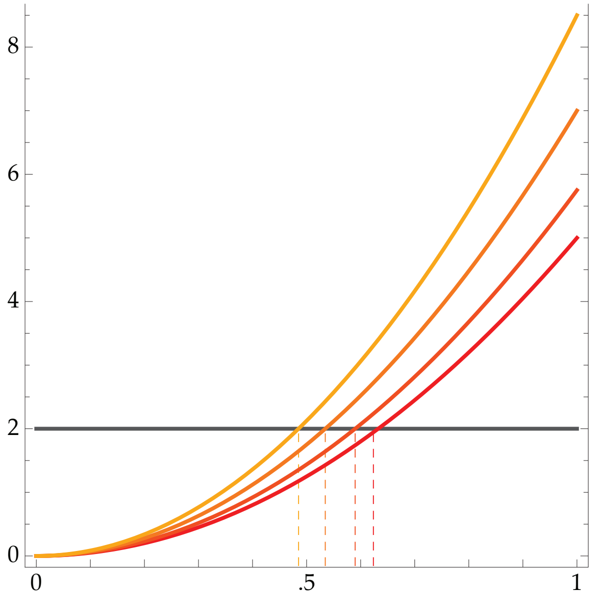
$h = 2, s = 3$



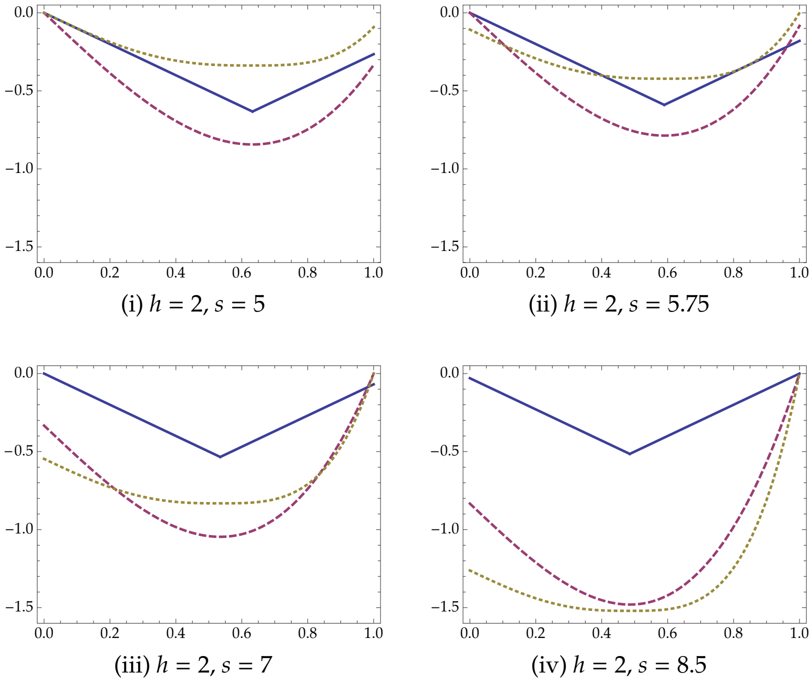
$h = 2, s = 5$

The ordinal potentials ΔI_{sgn} (blue), ΔI_1 (purple), and ΔI_2 (yellow).

Example (Nonlinear Stag Hunt revisited). $F_H(\chi) = h$, $F_S(\chi) = s\chi^2$, $\chi^* = \sqrt{h/s}$.



Payoffs and mixed equilibria when $h = 2$ and $s = 5, 5.75, 7,$ and 8.5.



(i) $h = 2$, $s = 5$

(ii) $h = 2$, $s = 5.75$

(iii) $h = 2$, $s = 7$

(iv) $h = 2$, $s = 8.5$

Figure 2: The ordinal potentials ΔI_{sgn} (blue), ΔI_1 (purple), and ΔI_2 (yellow).

Risk Dominance, Stochastic Dominance, and Stochastic Stability

Suppose that F is a coordination game: there is a state $\chi^* \in (0, 1)$ such that

$$\mathrm{sgn}(\Delta F(\chi)) = \mathrm{sgn}(\chi - \chi^*) \text{ for all } \chi \neq \chi^*.$$

Are there simple conditions that characterize stochastic stability?

Strategy i is risk dominant if its basin of attraction is bigger than that of strategy j.

Corollary. *Suppose that F is linear. Then state e_i is stochastically stable under every noisy best response protocol if and only if strategy i is risk dominant in F.*

Corollary. *Suppose that σ^η is the BRM rule. Then state e_i is stochastically stable under every noisy best response protocol if and only if strategy i is risk dominant in F.*

What about games with nonlinear payoffs?

Strategy i stochastically dominant if for every $a \geq 0$,
the set of states where i has a payoff advantage of at least a is larger than
the set of states where j has a payoff advantage of at least a.

Theorem. *State e_i is stochastically stable under every noisy best response protocol if and only if strategy i is stochastically dominant in F.*

Trees and Stochastic Stability

We now develop tools for analyzing nonreversible cases.

The Markov Chain Tree Theorem

$\{X_t\}$	irreducible Markov chain
$\mathcal{X}$	finite state space
$P \in \mathbb{R}_+^{\mathcal{X}\times\mathcal{X}}$	transition matrix
$\mu \in \mathbb{R}_+^{\mathcal{X}}$	unique stationary distribution: $\mu' P = \mu'$.

View $\mathcal{X}$ as a set of nodes to be connected with directed edges $(x, y) \in \mathcal{X} \times \mathcal{X}$, $x \neq y$.

Then a directed graph g on $\mathcal{X}$ can be identified with a set of directed edges.

Four special types of directed graphs:

A walk from x to y is a directed graph $\{(x, x_1), (x_1, x_2), \ldots, (x_{l-1}, y)\}$ whose directed edges traverse a route connecting x to y.

A path from x to $y \neq x$ is a walk from x to y with no repeated nodes.

A cycle is a walk from x to itself that contains no other repeated nodes.

A tree with root x (or an x-tree) is a directed graph with no outgoing edges from x, exactly one outgoing edge from each $y \neq x$, and a unique path from each $y \neq x$ to x.

Let T_x denote the set of x-trees on $\mathcal{X}$, and define the vector $\nu \in \mathbb{R}_+^{\mathcal{X}}$ by

$$\nu_x = \sum_{\tau_x \in T_x} \prod_{(y,z) \in \tau_x} P_{yz}. \tag{92}$$

Theorem (The Markov Chain Tree Theorem). $\nu \propto \mu$.

$$\nu_x = \sum_{\tau_x \in T_x} \prod_{(y,z)\in\tau_x} P_{yz}.$$

Proof. Let G_x be the set of directed graphs γ_x on $\mathcal{X}$ such that

(i) each $y \in \mathcal{X}$ has exactly one outgoing edge in γ_x;

(ii) γ_x contains a unique cycle; and

(iii) the unique cycle contains x.

Two representations of G_x:

(93) $$G_x = \bigcup_{y \neq x} \bigcup_{\tau_y \in T_y} \tau_y \cup \{(y,x)\};$$

(For each τ_y tree with $y \neq x$, add an edge from y to x.)

(94) $$G_x = \bigcup_{\tau_x \in T_x} \bigcup_{y \neq x} \tau_x \cup \{(x,y)\}.$$

(For each τ_x tree and each $y \neq x$, add an edge from x to y.)

$$G_x = \bigcup_{y \neq x} \bigcup_{\tau_y \in T_y} \tau_y \cup \{(y, x)\}; \tag{93}$$

$$G_x = \bigcup_{\tau_x \in T_x} \bigcup_{y \neq x} \tau_x \cup \{(x, y)\}. \tag{94}$$

Recall that $\nu_x = \sum_{\tau_x \in T_x} \prod_{(y,z) \in \tau_x} P_{yz}$. Define $\psi_x = \sum_{\gamma_x \in G_x} \prod_{(y,z) \in \gamma_x} P_{yz}$.

Then by (93) and (94), $\sum_{y \neq x} \nu_y P_{yx} = \psi_x = \nu_x \sum_{y \neq x} P_{xy}$.

Add $\nu_x P_{xx}$ to each side: $\sum_{y \in \mathcal{X}} \nu_y P_{yx} = \nu_x$.

Put differently, $\nu' P = \nu'$.

Stationary Distribution Asymptotics via Trees (Freidlin-Wentzell)

$\eta \in (0, \bar{\eta}]$	noise level
$\{X_t^\eta\}$	irreducible Markov chain
$\mathcal{X}$	common finite state space
$P^\eta \in \mathbb{R}_+^{\mathcal{X}\times\mathcal{X}}$	transition matrix
$\mu^\eta \in \mathbb{R}_+^{\mathcal{X}}$	unique stationary distribution: $(\mu^\eta)'P = (\mu^\eta)'$.

Assume that transition probabilities P_{xy}^η have well defined rates of decay, or costs:

$$c_{xy} = -\lim_{\eta\to 0} \eta \log P_{xy}^\eta \qquad (\Leftrightarrow \; P_{xy}^\eta = \exp(-\eta^{-1}(c_{xy} + o(1))) \,)$$

What can we say about the rates of decay of the stationary distribution weights μ_x^η?

$c_{xy} = -\lim_{\eta \to 0} \eta \log P^{\eta}_{xy}$ cost of a step from x to y

$C(\tau_x) = \sum_{(y,z) \in \tau_x} c_{yz}$ cost of tree τ_x

$C_x = \min_{\tau_x \in T_x} C(\tau_x)$ minimal cost of an x-tree

$\underline{C}_x = C_x - \min_{y \in \mathcal{X}} C_y$ normalization: $\min_{x \in \mathcal{X}} \underline{C}_x = 0$

$C^* = \min_{x \in \mathcal{X}} C_x$

Theorem. $-\lim_{\eta \to 0} \eta \log \mu^{\eta}_x = \underline{C}_x$.

States x with $\underline{C}_x = 0$ are said to be weakly stochastically stable.

If x is the unique weakly stochastically stable state, then $\lim_{\eta \to 0} \mu^{\eta}_x = 1$.

If $\limsup_{\eta \to 0} \mu^{\eta}_x > 0$, then x is weakly stochastically stable.

Sketch of proof. Recall that $P^{\eta}_{xy} = \exp(-\eta^{-1}(c_{xy} + o(1)))$.

$$\begin{aligned}\prod_{(y,z)\in\tau_x} P^{\eta}_{yz} &= \prod_{(y,z)\in\tau_x} \exp(-\eta^{-1}(c_{xy} + o(1))) \\ &= \exp(-\eta^{-1}(C(\tau_x) + o(1))).\end{aligned}$$

Thus

$$\begin{aligned}\nu^{\eta}_x &= \sum_{\tau_x\in T_x} \prod_{(y,z)\in\tau_x} P^{\eta}_{yz} \\ &= \sum_{\tau_x\in T_x} \prod_{(y,z)\in\tau_x} \exp(-\eta^{-1}(C(\tau_x) + o(1))) \\ &= \exp(-\eta^{-1}(C_x + o(1))). \qquad \text{(exercise!)}\end{aligned}$$

Since $\nu \propto \mu$ by the Markov chain tree theorem,

$$-\lim_{\eta\to 0} \eta \log \frac{\mu^{\eta}_x}{\mu^{\eta}_y} = -\lim_{\eta\to 0} \eta \log \left(\frac{\exp(-\eta^{-1}(C_x + o(1)))}{\exp(-\eta^{-1}(C_y + o(1)))} \right) = C_x - C_y.$$

Since $\nu \propto \mu$ by the Markov chain tree theorem,

$$-\lim_{\eta \to 0} \eta \log \frac{\mu_x^\eta}{\mu_y^\eta} = -\lim_{\eta \to 0} \eta \log \left(\frac{\exp(-\eta^{-1}(C_x + o(1)))}{\exp(-\eta^{-1}(C_y + o(1)))} \right) = C_x - C_y.$$

When $C_y = C^*$, we have $-\lim_{\eta \to 0} \eta \log \mu_y^\eta = 0$ (since the mass must go somewhere).

Fixing such a y, we have

$$\begin{aligned} -\lim_{\eta \to 0} \eta \log \mu_x^\eta &= -\lim_{\eta \to 0} \left(\eta \log \frac{\mu_x^\eta}{\mu_y^\eta} + \eta \log \mu_y^\eta \right) \\ &= C_x - C^* + 0 \\ &= \underline{C}_x. \end{aligned}$$

Two-Strategy Games Revisited

Notation for two-strategy games: $\chi \equiv x_1$; $\mathcal{X}^N = \{0, \frac{1}{N}, \ldots, 1\} \subset [0,1]$; $F(\chi)$; $\rho(\pi, \chi)$.

Noisy best response protocols and unlikelihood functions:

$$\rho_{ij}^{\eta}(\pi) = \sigma^{\eta}(\pi_j - \pi_i),$$

$$\kappa(d) = -\lim_{\eta \to 0} \eta \log \sigma^{\eta}(-d).$$

Example (Best response with mutations).

$$\sigma^{\eta}(a) = \begin{cases} 1 - \exp(-\eta^{-1}) & \text{if } a > 0, \\ \exp(-\eta^{-1}) & \text{if } a \leq 0. \end{cases} \quad \Rightarrow \quad \kappa(d) = \begin{cases} 1 & \text{if } d \geq 0, \\ 0 & \text{if } d < 0. \end{cases}$$

Example (Logit choice).

$$\sigma^{\eta}(a) = \frac{\exp(\eta^{-1} a)}{\exp(\eta^{-1} a) + 1} \quad \Rightarrow \quad \kappa(d) = \begin{cases} d & \text{if } d > 0, \\ 0 & \text{if } d \leq 0. \end{cases}$$

The small noise limit

Assume agents are clever.

$$\tilde{\kappa}(d) = \kappa(d) - \kappa(-d),$$

$$I^N(\chi) = \sum_{j=1}^{N\chi} \tilde{\kappa}\Big(F_1^N(\tfrac{j}{N}) - F_0^N(\tfrac{j-1}{N})\Big),$$

$$-\Delta I^N(\chi) = \max_{y \in \mathcal{X}^N} I^N(y) - I^N(\chi) \geq 0.$$

Theorem. $-\lim_{\eta \to 0} \eta \log \mu_\chi^{N,\eta} = -\Delta I^N(\chi)$.

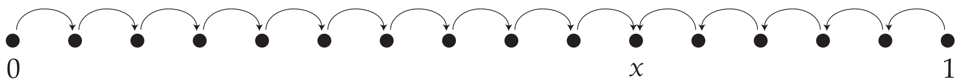
Analysis via trees:

Let C_χ be the minimal cost of an χ-tree.

Let $\underline{C}_\chi = C_\chi - \min_{y \in \mathcal{X}} C_y$.

Then by the Freidlin-Wentzell theorem, $-\lim_{\eta \to 0} \eta \log \mu_\chi^\eta = \underline{C}_\chi$.

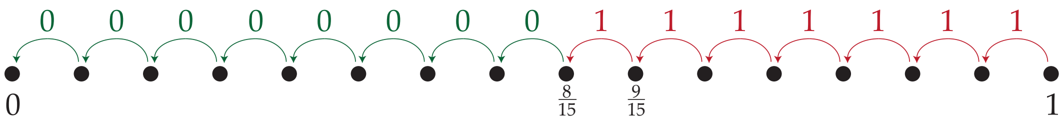
The only finite cost χ-tree is



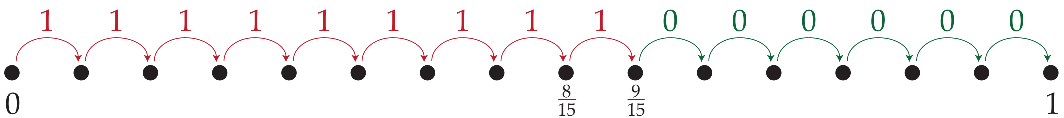
$$
\begin{aligned}
C_\chi = C(\tau_\chi) &= \sum_{j=0}^{N\chi-1} c_{j/N,(j+1)/N} + \sum_{j=N\chi+1}^{N} c_{j/N,(j-1)/N} \\
&= \sum_{j=0}^{N\chi-1} \kappa\left(F_0^N(\tfrac{j}{N}) - F_1(\tfrac{j+1}{N})\right) + \sum_{j=N\chi+1}^{N} \kappa\left(F_1^N(\tfrac{j}{N}) - F_0^N(\tfrac{j-1}{N})\right).
\end{aligned}
$$

Example: Stag Hunt, BRM protocol. (Assume indifferent agents do not switch.)

$$A = \begin{pmatrix} h & h \\ 0 & s \end{pmatrix}. \qquad h = 4, s = 7 \Rightarrow x^* = \tfrac{4}{7}.$$



$C(\tau_0) = 7$

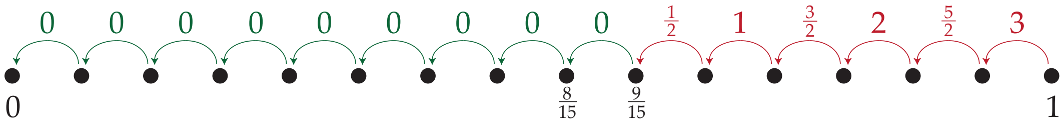


$C(\tau_1) = 9$

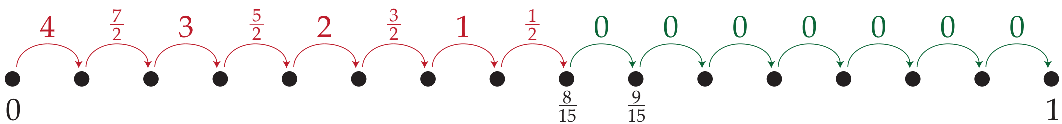
$\therefore$ state 0 is stochastically stable

Example: Stag Hunt, logit protocol

$$A = \begin{pmatrix} h & h \\ 0 & s \end{pmatrix}. \qquad h = 4, s = 7 \;\Rightarrow\; x^* = \tfrac{4}{7}.$$



$$C(\tau_0) = 10\tfrac{1}{2}$$



$$C(\tau_1) = 18$$

∴ state 0 is stochastically stable

Do the analyses agree? It's enough to show that $\underline{C}_\chi = -\Delta I^N(\chi)$.

$$C_\chi = C(\tau_\chi) = \sum_{j=0}^{N\chi-1} \kappa\left(F_0^N(\tfrac{j}{N}) - F_1(\tfrac{j+1}{N})\right) + \sum_{j=N\chi+1}^{N} \kappa\left(F_1^N(\tfrac{j}{N}) - F_0^N(\tfrac{j-1}{N})\right).$$

$$C_y = C(\tau_y) = \sum_{k=0}^{Ny-1} \kappa\left(F_0^N(\tfrac{k}{N}) - F_1(\tfrac{k+1}{N})\right) + \sum_{k=Ny+1}^{N} \kappa\left(F_1^N(\tfrac{k}{N}) - F_0^N(\tfrac{k-1}{N})\right).$$

$$(\Rightarrow) \qquad C_\chi - C_y = \sum_{j=N\chi+1}^{Ny} \kappa\left(F_1^N(\tfrac{j}{N}) - F_0^N(\tfrac{j-1}{N})\right) - \sum_{k=N\chi}^{Ny-1} \kappa\left(F_0^N(\tfrac{k}{N}) - F_1(\tfrac{k+1}{N})\right)$$

$$= \sum_{j=N\chi+1}^{y} \tilde{\kappa}\left(F_1^N(\tfrac{j}{N}) - F_0^N(\tfrac{j-1}{N})\right)$$

$$= I^N(y) - I^N(\chi).$$

$$(\Rightarrow) \qquad \underline{C}_\chi = -\Delta I^N(\chi).$$

Stationary Distribution Asymptotics via Trees on Recurrent Classes

Let P^* be a transition matrix satisfying $P^*_{xy} > 0 \iff c_{xy} = 0$.

Let $\mathcal{X}^* \subseteq \mathcal{X}$ be the set of recurrent states associated with P^*.

We assume for convenience that all recurrent classes are singletons.

By definition, there is a zero-cost path from every state in $\mathcal{X}$ to a state in $\mathcal{X}^*$.

This suggest thats we can conduct the tree analysis using trees on $\mathcal{X}^*$.

This can be very useful, since $\mathcal{X}^*$ is typically much smaller than $\mathcal{X}$.

$\phi = (\phi_0, \dots \phi_\ell) \in \mathcal{X}^\ell$	a path through $\mathcal{X}$
$c(\phi) = \sum_{k=0}^{\ell-1} c_{\phi_k, \phi_{k+1}}$	cost of a path
$\Phi(x, y)$	set of paths from $x \in \mathcal{X}$ to $y \in X$
$C(x, y) = \min\{c(\phi) \colon \phi \in \Phi(x, y)\}$	cost of a transition form x to y

For $x^* \in \mathcal{X}^*$, let $\mathcal{T}_{x^*}$ be the set of x^*-trees on $\mathcal{X}^*$.

$$C(\tau_{x^*}) = \sum_{(y^*, z^*) \in \tau_x^*} C(y^*, z^*)$$ cost of tree τ_{x^*}

Define $C \colon \mathcal{X} \to \mathbb{R}_+$ as follows:

$$C_{x^*} = \min_{\tau_{x^*} \in \mathcal{T}_{x^*}} C(\tau_{x^*})$$

$$C_x = \min_{x^* \in \mathcal{X}^*} (C_{x^*} + C(x^*, x))$$

Let $\underline{C}(x) = C_x - \min_{x \in \mathcal{X}} C_x$

Theorem. $-\lim_{\eta \to 0} \eta \log \mu_x^\eta = \underline{C}_x$. (cf. Catoni (1999))

Radius-Coradius Theorems

There are sufficient conditions for stochastic stability that do not require trees.

The radius of $x^* \in \mathcal{X}^*$ is the difficulty of going from x^* to another recurrent class:

$$\mathrm{rad}(x^*) = \min_{y^* \neq x^*} C(y^*, z^*).$$

Let $\Phi^*(y^*, z^*)$ be the set of paths through $\mathcal{X}^*$ from y^* to z^*.

The coradius is the maximal difficulty of getting to x^* from another recurrent class:

$$\mathrm{corad}(x^*) = \max_{y^* \neq x^*} \min_{\phi^* \in \Phi^*(y^*, x^*)} C(\phi^*).$$

$\Phi^*(x^*, y^*)$	set of paths through $\mathcal{X}^*$ from x^* to y^*
$C(\phi^*) = \sum_{k=0}^{\ell-1} C(\phi_k^*, \phi_{k+1}^*)$	cost of a path through $\mathcal{X}^*$

Theorem. *If* $\mathrm{rad}(x^*) > \mathrm{corad}(x^*)$, *then* x^* *is uniquely stochastically stable.*

A variety of weaker sufficient conditions exist.

Half-Dominance and BRM

Roughly speaking, strategy i is strongly half-dominant if for some $\alpha < \frac{1}{2}$, i is the unique best response whenever $x_i \geq \alpha$.

Theorem. *Suppose clever agents play a finite-population game F^N using the BRM rule. If strategy i is strongly half-dominant in F^N, then state e_i is uniquely stochastically stable.*

Idea of proof.

$\text{rad}(e_i) < N(1 - \alpha)$.

$\text{corad}(e_i) > N\alpha$.

Thus $\text{rad}(e_i) > \text{corad}(e_i)$, so e_i is uniquely stochastically stable.