The Projection Dynamic and the Replicator Dynamic*

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February 1, 2008

Abstract

We investigate a variety of connections between the projection dynamic and the replicator dynamic. At interior population states, the standard microfoundations for the replicator dynamic can be converted into foundations for the projection dynamic by replacing imitation of opponents with "revision driven by insecurity" and direct choice of alternative strategies. Both dynamics satisfy a condition called inflow-outflow symmetry, which causes them to select against strictly dominated strategies at interior states; still, because it is discontinuous at the boundary of the state space, the projection dynamic allows strictly dominated strategies to survive in perpetuity. The two dynamics exhibit qualitatively similar behavior in strictly stable and null stable games. Finally, the projection and replicator dynamics both can be viewed as gradient systems in potential games, the latter after an appropriate transformation of the state space. *JEL classification*: C72, C73.

1. Introduction

The projection dynamic is an evolutionary game dynamic introduced in the transportation science literature by Nagurney and Zhang (1996). Microfoundations for this dynamic

*We thank two anonymous referees, an anonymous Associate Editor, and many seminar audiences for helpful comments. Financial support from NSF Grants SES-0092145 and SES-0617753 is gratefully acknowledged.

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are provided in Lahkar and Sandholm (2008): there the dynamic is derived from a model of "revision driven by insecurity", in which each agent considers switching strategies at a rate inversely proportional to his current strategy's popularity. Although it is discontinuous at the boundary of the state space, the projection dynamic admits unique forward solution trajectories; its rest points are the Nash equilibria of the underlying game, and it converges to equilibrium from all initial conditions in potential games and in stable games.

In this paper, we investigate the many connections between the projection dynamic and the replicator dynamic of Taylor and Jonker (1978). Our companion paper, Lahkar and Sandholm (2008), established one link between the dynamics' foundations. In general, one provides foundations for an evolutionary dynamic by showing that it is the mean dynamic corresponding to a particular revision protocol—that is, a particular rule used by agents to decide when and how to choose new strategies.[1] The companion paper finds a new revision protocol that generates the replicator dynamic, and shows that a suitable modification of this protocol generates the projection dynamic.

In economic contexts, the replicator dynamic is best understood as a model of imitation. In Lahkar and Sandholm (2008), a principal step in changing the replicator protocol into a projection protocol is to eliminate the former's use of imitation, replacing this with "revision driven by insecurity" and direct (nonimitative) selection of alternative strategies. But if we look only at behavior at interior population states—that is, at states where all strategies are in use—then this one step becomes sufficient for converting replicator protocols into projection protocols. To be more precise, we show in Section 3 that for each of the three standard foundations for the replicator dynamic (due to Schlag (1998), Björnerstedt and Weibull (1996), and Hofbauer (1995)), replacing "imitation" with "revision driven by insecurity" yields a foundation for the projection dynamic valid at interior population states.

Both "imitation" and "revision driven by insecurity" are captured formally by directly including components of the population state in the outputs of revision protocols. We show that the precise ways in which these arguments appear lead the replicator and projection dynamics to satisfy a property called *inflow-outflow symmetry* at interior population states. Inflow-outflow symmetry implies that any strictly dominated strategy must always be "losing ground" to its dominating strategy, pushing the weight on the dominated strategy toward zero. Akin (1980) uses this observation to show that the replicator dynamic eliminates strictly dominated strategies along all interior solution trajectories.

This elimination result *does not* extend to the projection dynamic. Using the fact that

[1] For formal statements of this idea, see Benaïm and Weibull (2003) and Sandholm (2003).

solutions of the projection dynamic can enter and leave the boundary of the state space, we construct an example in which a strictly dominated strategy appears and then disappears from the population in perpetuity. Because the projection dynamic is discontinuous, the fact that the dominated strategy is losing ground to the dominating strategy at all interior population states is not enough to ensure its eventual elimination.[2]

The final section of the paper compares the global behavior of the two dynamics in two important classes of games: stable games and potential games. In the former case, the properties of the two dynamics mirror one another: one can establish (interior) global asymptotic stability for both dynamics in strictly stable games, and the existence of constants of motion in null stable games, including zero-sum games.[3]

The connection between the projection and replicator dynamics in potential games is particularly striking. On the interior of the state space, the projection dynamic for a potential game is the gradient system generated by the game's potential function: interior solutions of the dynamic always ascend potential in the most direct fashion. The link with the replicator dynamic arises by way of a result of Akin (1979). Building on work of Kimura (1958) and Shahshahani (1979), Akin (1979) shows that the replicator dynamic for a potential game is also a gradient system defined by the game's potential function; however, this is true only after the state space has been transformed by a nonlinear change of variable, one that causes greater importance to be attached to changes in the use of rare strategies. We conclude the paper with a direct proof of Akin's (1979) result: unlike Akin's (1979) original proof, ours does not require the introduction of tools from differential geometry.

In summary, this paper argues that despite a basic difference between the two dynamics—that one is based on imitation of opponents, and the other on "revision driven by insecurity" and direct selection of new strategies—the replicator dynamic and the projection dynamic exhibit surprisingly similar behavior.

2. Definitions

To keep the presentation self-contained, we briefly review some definitions and results from Lahkar and Sandholm (2008).

[2]Hofbauer and Sandholm (2006), building on the work of Berger and Hofbauer (2006), construct a game that possesses a strictly dominated strategy, but that causes a large class of evolutionary dynamics to admit an interior attractor. The analysis in that paper concerns dynamics that are continuous in the population state, and therefore does not apply to the projection dynamic.

[3]In contrast, other standard dynamics, including the best response, logit, BNN, and Smith dynamics, converge to equilibrium in null stable games: see Hofbauer and Sandholm (2008, 2007).

2.1 Preliminaries

To simplify our notation, we focus on games played by a single unit-mass population of agents who choose pure strategies from the set $S = \{1, \ldots, n\}$. The set of *population states* (or *strategy distributions*) is thus the simplex $X = \{x \in \mathbf{R}^n_+ : \sum_{i \in S} x_i = 1\}$, where the scalar $x_i \in \mathbf{R}_+$ represents the mass of players choosing strategy $i \in S$.

We take the strategy set S as given and identify a population game with its payoff function $F : X \to \mathbf{R}^n$, a Lipschitz continuous map that assigns each population state x a vector of payoffs $F(x)$. The component function $F_i : X \to \mathbf{R}$ denotes the payoffs to strategy $i \in S$. State $x \in X$ is a *Nash equilibrium* of F, denoted $x \in NE(F)$, if $x_i > 0$ implies that $i \in \operatorname{argmax}_{j \in S} F_j(x)$.

The *tangent space* of X, denoted $TX = \{z \in \mathbf{R}^n : \sum_{i \in S} z_i = 0\}$, contains those vectors describing motions between points in X. The orthogonal projection onto the subspace $TX \subset \mathbf{R}^n$ is represented by the matrix $\Phi = I - \frac{1}{n}\mathbf{1}\mathbf{1}' \in \mathbf{R}^{n \times n}$, where $\mathbf{1} = (1, ..., 1)'$ is the vector of ones. Since $\Phi v = v - \mathbf{1} \cdot \frac{1}{n} \sum_{k \in S} v_k$, component $(\Phi v)_i$ is the difference between the v_i and the unweighted average payoff of the components of v. Thus, if v is a payoff vector, Φv discards information about the absolute level of payoffs while preserving information about relative payoffs.

The *tangent cone* of X at state $x \in X$ is the set of directions of motion from x that initially remain in X:

$$\begin{aligned} TX(x) &= \{z \in \mathbf{R}^n : z = \alpha(y - x) \text{ for some } y \in X \text{ and some } \alpha \geq 0\} \\ &= \{z \in TX : z_i \geq 0 \text{ whenever } x_i = 0\}. \end{aligned}$$

The *closest point projection* onto $TX(x)$ is given by

$$\Pi_{TX(x)}(v) = \operatorname*{argmin}_{z \in TX(x)} \left| z - v \right|.$$

It is easy to verify that if $x \in \operatorname{int}(X)$, then $TX(x) = TX$, so that $\Pi_{TX(x)}(v) = \Phi v$. More generally, Lahkar and Sandholm (2008) show that

$$(\Pi_{TX(x)}(v))_i = \begin{cases} v_i - \frac{1}{\#\mathcal{S}(v,x)} \sum_{j \in \mathcal{S}(v,x)} v_j & \text{if } i \in \mathcal{S}(v, x), \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

where the set $\mathcal{S}(v, x) \subseteq S$ contains all strategies in support(x), along with any subset of $S -$ support(x) that maximizes the average $\frac{1}{\#\mathcal{S}(v,x)} \sum_{j \in \mathcal{S}(v,x)} v_j$.

2.2 The Replicator Dynamic and the Projection Dynamic

An *evolutionary dynamic* is a map that assigns each population game F a differential equation $\dot{x} = V^F(x)$ on the state space X. The best-known evolutionary dynamic is the *replicator dynamic* (Taylor and Jonker (1978)), defined by

$$\text{(R)} \qquad \dot{x}_i = x_i \left(F_i(x) - \sum_{k \in S} x_k F_k(x) \right).$$

In words, equation (R) says that the *percentage* growth rate of strategy i equals the difference between the payoff to strategy i and the *weighted* average payoff under F at x (that is, the average payoff obtained by members of the population).

The *projection dynamic* (Nagurney and Zhang (1997)) assigns each population game F the differential equation

$$\text{(P)} \qquad \dot{x} = \Pi_{TX(x)}(F(x)).$$

Under the projection dynamic, the direction of motion is always given by the closest approximation of the payoff vector $F(x)$ by a feasible direction of motion. When $x \in \text{int}(X)$, the tangent cone $TX(x)$ is just the subspace TX, so the explicit formula for (P) is simply $\dot{x} = \Phi F(x)$; otherwise, the formula is obtained from equation (1).

To begin to draw connections between the two dynamics, note that at interior population states, the projection dynamic can be written as[4]

$$\text{(2)} \qquad \dot{x}_i = (\Phi F(x))_i = F_i(x) - \frac{1}{n} \sum_{i \in S} F_k(S).$$

In words, the projection dynamic requires the *absolute* growth rate of strategy i to be the difference between strategy i's payoff and the *unweighted* average payoff of all strategies.[5] Comparing equations (R) and (2), we see that at interior population states, the replicator and projection dynamics convert payoff vector fields in differential equations in similar fashions, the key difference being that the replicator dynamic uses relative definitions, while the projection dynamic employs the corresponding absolute definitions. The remainder of this paper explores game-theoretic ramifications of this link.

When F is generated by random matching to play the normal form game A (i.e., when

[4]At interior population states (but not boundary states), the projection dynamic is identical to the *linear dynamic* of Friedman (1991); see Lahkar and Sandholm (2008) for further discussion.

[5]At boundary states, some poorly performing unused strategies (namely, those not in $\mathcal{S}(F(x), x)$) are ignored, while the absolute growth rates of the remaining strategies are defined as before.

F takes the linear form $F(x) = Ax$), the dynamic (P) is especially simple. On $\mathrm{int}(X)$, the dynamic is described by the linear equation $\dot{x} = \Phi Ax$; more generally, it is given by

$$(3) \qquad \dot{x}_i = (\Pi_{TX(x)}(Ax))_i = \begin{cases} (Ax)_i - \frac{1}{\#\mathcal{S}(Ax,x)} \sum_{j \in \mathcal{S}(Ax,x)} (Ax)_j & \text{if } i \in \mathcal{S}(Ax, x), \\ 0 & \text{otherwise.} \end{cases}$$

Notice that once the set of strategies $\mathcal{S}(Ax, x)$ is fixed, the right hand side of (3) is a linear function of x. Thus, under single population random matching, the projection dynamic is piecewise linear. In Section 4, this observation plays a key role in our proof that strictly dominated strategies can survive under (P).

3. Microfoundations

We derive evolutionary dynamics from a description of individual behavior by introducing the notion of a *revision protocol* $\rho : \mathbf{R}^n \times X \to \mathbf{R}^{n\times n}_+$. Suppose that as time passes, agents are randomly offered opportunities to switch strategies. The *conditional switch rate* $\rho_{ij}(F(x), x) \in \mathbf{R}_+$ is proportional to the probability with which an i player who receives an opportunity switches to strategy j.

Given this specification of individual decision making, aggregate behavior in the game F is described by the *mean dynamic*

$$(\mathrm{M}) \qquad \dot{x}_i = \sum_{j \in S} x_j \rho_{ji}(F(x), x) - x_i \sum_{j \in S} \rho_{ij}(F(x), x).$$

Here the first term describes the inflow into strategy i from other strategies, while the second term describes the outflow from i to other strategies.

Consider these three examples of revision protocols:

$$(4a) \qquad \rho_{ij} = x_j [F_j(x) - F_i(x)]_+,$$

$$(4b) \qquad \rho_{ij} = x_j (K - F_i(x)),$$

$$(4c) \qquad \rho_{ij} = x_j (F_j(x) + K).$$

The x_j term in these three protocols reveals that they are models of imitation. For instance, to implement protocol (4a), an agent who receives a revision opportunity picks an opponent from at random; he then imitates this opponent only if the opponents' payoff is higher than his own, doing so with probability proportional to the payoff difference.[6]

[6] Protocol (4a) is the *pairwise proportional imitation* protocol of Schlag (1998); protocol (4b), called *pure*

It is well known that the replicator dynamic can be viewed as the aggregate result of evolution by imitation. In fact, all three protocols above generate the replicator dynamic as their mean dynamics. For protocol (4a), one computes that

$$\begin{aligned}\dot{x}_i &= \sum_{j\in S} x_j\rho_{ji} - x_i \sum\nolimits_{j\in S} \rho_{ij} \\ &= \sum_{j\in S} x_j x_i [F_i(x) - F_j(x)]_+ - x_i \sum_{j\in S} x_j [F_j(x) - F_i(x)]_+ \\ &= x_i \sum_{j\in S} x_j (F_i(x) - F_j(x)) \\ &= x_i \left(F_i(x) - \sum_{j\in S} x_j F_j(x) \right).\end{aligned}$$

The derivations for the other two protocols are similar.

To draw connections with the projection dynamic, we replace the x_j term with $\frac{1}{nx_i}$ in each of the protocols above:

$$\rho_{ij} = \frac{1}{nx_i}[F_j(x) - F_i(x)]_+, \tag{5a}$$

$$\rho_{ij} = \frac{1}{nx_i}(K - F_i(x)), \tag{5b}$$

$$\rho_{ij} = \frac{1}{nx_i}(F_j(x) + K). \tag{5c}$$

While protocols (4a)–(4c) captured imitation, protocols (5a)–(5c) instead capture *revision driven by insecurity*: agents are quick to abandon strategies that are used by few of their fellows. For instance, under protocol (5a), an agent who receives a revision opportunity first considers whether to actively reconsider his choice of strategy, opting to do so with probability inversely proportional to the mass of agents currently choosing his strategy. If he does consider revising, he chooses a *strategy* at random, and then switches to this strategy with probability proportional to the the difference between its payoff and his current payoff.

Protocols (5a)–(5c) are only well-defined on $\text{int}(X)$. But on that set, each of the protocols generates the projection dynamic. For protocol (5a), this is verified as follows:

$$\dot{x}_i = \sum_{j\in S} x_j\rho_{ji} - x_i \sum_{j\in S} \rho_{ij}$$

imitation driven by dissatisfaction, is due to Björnerstedt and Weibull (1996), protocol (4c), which we call *imitation of success*, can be found in Hofbauer (1995).

$$
\begin{aligned}
&= \sum_{j\in S} x_j \frac{[F_i(x) - F_j(x)]_+}{nx_j} - x_i \sum_{j\in S} \frac{[F_j(x) - F_i(x)]_+}{nx_i} \\
&= \frac{1}{n} \sum_{j\in S} (F_i(x) - F_j(x)) \\
&= F_i(x) - \frac{1}{n} \sum_{j\in S} F_j(x).
\end{aligned}
$$

Again, the derivations for the other protocols are similar.

On the boundary of the simplex X, protocols (5a)–(5c) no longer make sense. Still, it is possible to construct a matched pair of revision protocols that generate dynamics (R) and (P) throughout the simplex—see Lahkar and Sandholm (2008) for details.

4. Inflow-Outflow Symmetry and Dominated Strategies

It is natural to expect evolutionary dynamics to eliminate dominated strategies. The first positive result on this question was proved by Akin (1980), who showed that the replicator dynamic eliminates strictly dominated strategies so long as the initial state is interior. Akin's (1980) result was subsequently extended to broader classes of imitative dynamics by Samuelson and Zhang (1992) and Hofbauer and Weibull (1996). But while these results seem encouraging, they are actually quite special: Hofbauer and Sandholm (2006) show that continuous evolutionary dynamics that are not based exclusively on imitation do not eliminate strictly dominated strategies in all games.

In this section, we show that the projection dynamic shares with the replicator dynamic a property called inflow-outflow symmetry, and we explain why this property leads to selection against dominated strategies on the interior of X under both of these dynamics. Despite this shared property of the two dynamics, the long run prospects for dominated strategies under these dynamics are quite different. Using the fact that solutions to the projection dynamic can enter and exit the boundary of X, we prove that inflow-ouflow symmetry is not enough to ensure that dominated strategies are eliminated.

In the general expression for the mean dynamic (M),

$$
\text{(M)} \qquad \dot{x}_i = \sum_{j\in S} x_j \rho_{ji}(F(x), x) - x_i \sum_{j\in S} \rho_{ij}(F(x), x),
$$

the term x_i, representing the mass of players choosing strategy i, appears in an asymmetric fashion. Since in order for an agent to switch away from strategy i, he must first be selected at random for a revision opportunity, x_i appears in the (negative) outflow term. But since agents switching to strategy i were previously playing other strategies, x_i does not appear in the inflow term.

We say that an evolutionary dynamic satisfies *inflow-outflow symmetry* if this asymmetry in equation (M) is eliminated by the dependence of the revision protocol ρ on the population state x. Under the replicator dynamic and other imitative dynamics, ρ_{ji} is proportional to x_i, making both the inflow and outflow terms in (M) proportional to x_i; thus, these dynamics exhibit inflow-outflow symmetry. Similarly, under the projection dynamic, which is based on abandonment, ρ_{ij} is inversely proportional to x_i whenever x_i is positive. As a result, neither the inflow nor the outflow term in equation (M) depends directly on x_i, yielding inflow-ouflow symmetry on int(X).

Importantly, inflow-outflow symmetry implies that a strictly dominated strategy i will always lose ground to the strategy j that dominates it. In the case of the replicator dynamic, the ratio x_i/x_j falls over time throughout int(X):

$$
\begin{aligned}
(6) \qquad \frac{d}{dt}\left(\frac{x_i}{x_j}\right) &= \frac{\dot{x}_i x_j - \dot{x}_j x_i}{x_j^2} \\
&= \frac{x_i\left(F_i(x) - \sum_{k\in S} x_k F_k(x)\right)\cdot x_j - x_j\left(F_j(x) - \sum_{k\in S} x_k F_k(x)\right)\cdot x_i}{x_j^2} \\
&= \frac{x_i}{x_j}(F_i(x) - F_j(x)) \\
&< 0.
\end{aligned}
$$

Under the the projection dynamic, it is the difference $x_i - x_j$ that falls on int(X):

$$
\begin{aligned}
(7) \qquad \frac{d}{dt}\left(x_i - x_j\right) &= \left(F_i(x) - \tfrac{1}{n}\sum_{k\in S} F_k(x)\right) - \left(F_j(x) - \tfrac{1}{n}\sum_{k\in S} F_k(x)\right) \\
&= F_i(x) - F_j(x) \\
&< 0.
\end{aligned}
$$

By combining equation (6) with the fact that int(X) is invariant under (R), it is easy to prove that the replicator dynamic eliminates strictly dominated strategies along solutions in int(X); this is Akin's (1980) result. But because solutions of the projection dynamic can enter and leave int(X), the analogous argument based on equation (7) does not go through: while i will lose ground to j in the interior of X, it might gain ground back on the boundary of X, leaving open the possibility of survival.

To pursue this idea, we consider the following game, introduced by Berger and Hofbauer (2006) in their analysis of survival of dominated strategies under the BNN dynamic:

$$F(x) = Ax = \begin{pmatrix} 0 & -3 & 2 & 2 \\ 2 & 0 & -3 & -3 \\ -3 & 2 & 0 & 0 \\ -3-c & 2-c & -c & -c \end{pmatrix} \begin{pmatrix} x_R \\ x_P \\ x_S \\ x_T \end{pmatrix}. \tag{8}$$

The game defined by the first three strategies is bad Rock-Paper-Scissors with winning benefit $w = 2$ and losing cost $l = 3$. In this three-strategy game, solutions of (P) other than the one at the Nash equilibrium $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ enter a closed orbit that enters and exits the three edges of the simplex (see Figure 7(iii) of Lahkar and Sandholm (2008)).

The fourth strategy of game (8), Twin, is a duplicate of Scissors, except that its payoff is always $c \geq 0$ lower than that of Scissors. When $c = 0$, the set of Nash equilibria of game (8) is the line segment L between $x^* = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$ and $(\frac{1}{3}, \frac{1}{3}, 0, \frac{1}{3})$. If $c > 0$, then Twin is strictly dominated, and the game's unique Nash equilibrium is x^*.

Figure 1 presents the solution to (P) from initial condition $(\frac{97}{300}, \frac{103}{300}, \frac{1}{100}, \frac{97}{300})$ for payoff parameter $c = \frac{1}{10}$. At first, the trajectory spirals down line segment L, as agents switch from Rock to Paper to Scissors/Twin to Rock, with Scissors replacing Twin as time passes (since $\dot{x}_T - \dot{x}_S = -\frac{1}{10}$ on $\text{int}(X)$). When Twin is eliminated, both it and Scissors earn less than the average of the payoffs to Rock, Paper, and Scissors; therefore, x_S falls while x_T stays fixed at 0, and so $x_T - x_S$ *rises*. Soon the solution enters $\text{int}(X)$, and so $\dot{x}_T - \dot{x}_S = -\frac{1}{10}$ once again. When the solution reenters face *RPS*, it does so at a state further away from the Nash equilibrium x^* than the initial point of contact. Eventually, the trajectory appears to enter a closed orbit on which the mass on Twin varies between 0 and roughly .36.

The existence and stability of this closed orbit is established rigorously in Theorem 4.1, whose proof can be found in the appendix.

Theorem 4.1. *In game* (8) *with* $c = \frac{1}{10}$, *the projection dynamic* (P) *has an asymptotically stable closed orbit* γ *that absorbs all solutions from nearby states in finite time. This orbit, pictured in Figure 2, combines eight segments of solutions to linear differential equations as described in equation* (3)*; the approximate endpoints and exit times of these segments are presented in Table 1. Along the orbit, the value of* x_T *varies between* 0 *and approximately* .359116.

Recall that in random matching games, the projection dynamic is piecewise linear: on the interior of X, the dynamic is described by $\dot{x} = \Phi A x$; on the boundary of X, it is described by equation (3). When the only unused strategy is strategy i, equation (3) provides only two possibilities for $\dot{x}$: if $F_i(x)$ does not exceed the average payoff of the other three strategies, then $\dot{x}_i = 0$, so the solution travels along the face of X where strategy

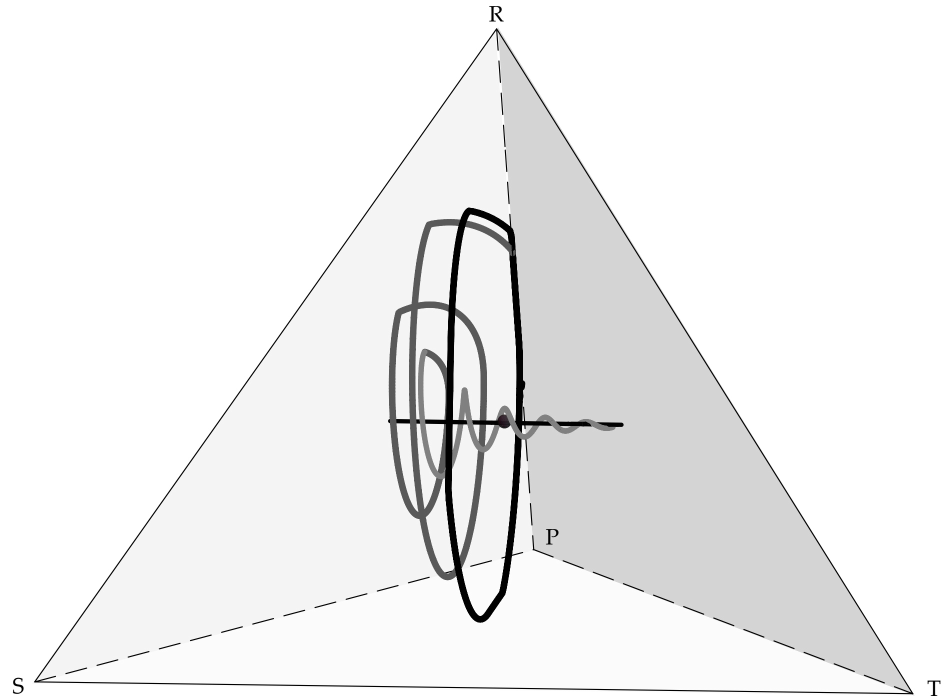


Figure 1: A solution to (P) in Bad Rock-Paper-Scissors-Twin.

i is absent; if instead $F_i(x)$ is greater than this average payoff, then the solution from x immediately enters int(X). We illustrate these regions in Figure 2, where we shade the portions of the faces of X to which solutions "stick".

Similar considerations determine the behavior of (P) on the edges of the simplex. For instance, solutions starting at vertex R travel along edge RP until reaching state $\xi = (\frac{7}{15}, \frac{8}{15}, 0, 0)$, at which point they enter face RPS.[7]

The proof of Theorem 4.1 takes advantage of the piecewise linearity of the dynamic, the Lipschitz continuity of its solutions in their initial conditions, and the fact that solutions to the dynamic can merge in finite time. Because of piecewise linearity, we can obtain analytic solutions to (P) within each region where (P) is linear. The point where a solution leaves one of these regions generally cannot be expressed analytically, but it can be approximated numerically to an arbitrary degree of precision. This approximation introduces a small error; however, the Lipschitz continuity of solutions places a tight bound on how quickly this error can propagate. Ultimately, our approximate solution starting from state ξ returns to edge RP. Since solutions cycle outward, edge RP is reached between ξ and vertex R. While the point of contact we compute is only approximate, solutions from all states between vertex R and state ξ pass through state ξ. Therefore, since our total

[7] State ξ lies between the vertices on edge RP of the "sticky" regions in faces RPT and RPS. These vertices lie at states $(\frac{71}{150}, \frac{79}{150}, 0, 0)$ and $(\frac{67}{150}, \frac{83}{150}, 0, 0)$, respectively.

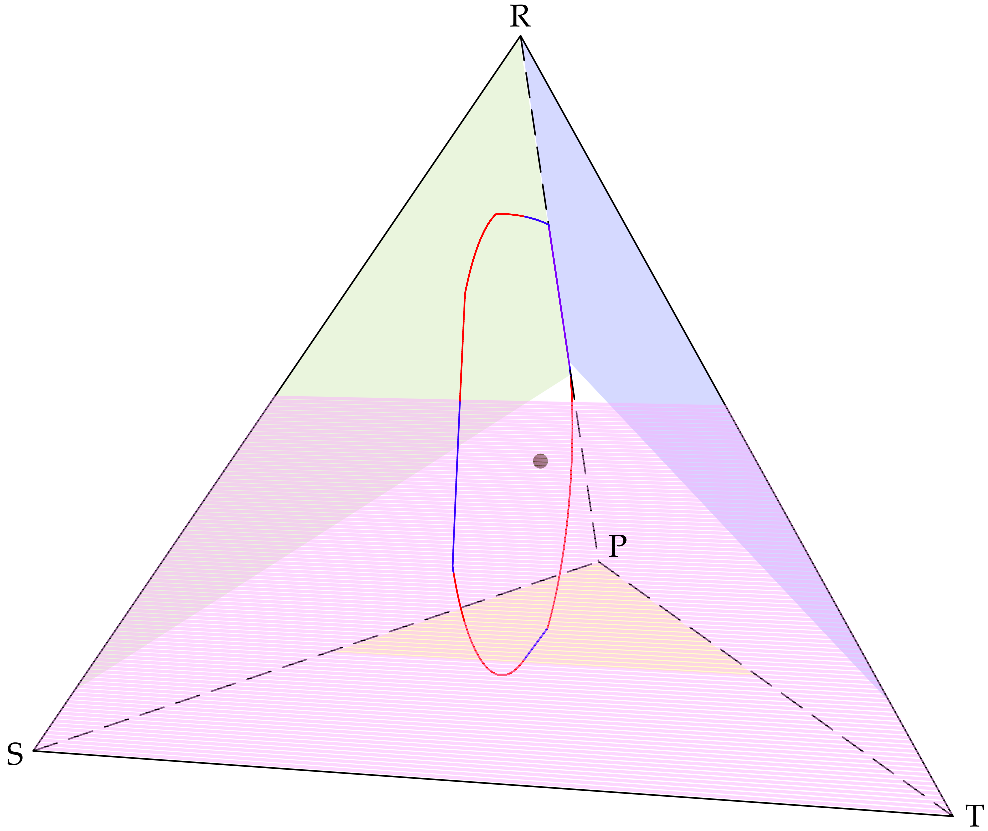


Figure 2: The closed orbit of (P) in Bad Rock-Paper-Scissors-Twin.

Segment	Support = $\mathcal{S}(Ax, x)$	Exit point	Exit time
(initial state)	RP	(.466667, .533333, 0, 0)	0
1	RPS	(.446354, .552864, .000782, 0)	.015678
2	$RPST$	(0, .564668, .227024, .208308)	.324883
3	PST	(0, .413636, .307144, .279219)	.416973
4	$RPST$	(.256155 ,0, .395751, .348094)	.656509
5	RST	(.473913, 0, .288747, .237340)	.793914
6	$RPST$	(.709788, .244655, .045576, 0)	1.028310
7	RPS	(.693072, .306928, 0, 0)	1.065574
8	RP	(.466667, .533333, 0, 0)	1.252812

Table 1: Approximate transition points and transition times of the closed orbit γ.

approximation error is very small, our calculations prove that the true solution must return to state ξ.

Hofbauer and Sandholm (2006) offer general conditions on evolutionary dynamics that are sufficient to ensure the survival of strictly dominated strategies in some games. Their conditions, though mild, include the requirement that the dynamic be continuous in the population state; this requirement this condition is used in the essential way in the proof of their result. By contrast, the projection dynamic is discontinuous at the boundaries of the simplex, and as we have seen, this discontinuity is used in an essential way in our proof of Theorem 4.1.

5. Global Behavior

In this final section of the paper, we illustrate connections between the global behaviors of the projection and replicator dynamics in stable games and in potential games.

5.1 Convergence and Cycling in Stable Games

Population game F is a *stable game* (Hofbauer and Sandholm (2008)) if

$$(y - x)'(F(y) - F(x)) \leq 0 \text{ for all } x, y \in X. \tag{9}$$

If inequality (9) is strict whenever $y \neq x$, then F is a *strictly stable game*; if (9) is always satisfied with equality, then F is a *null stable game*.

Let x^* be a Nash equilibrium of F, and let

$$E_{x^*}(x) = \left| x - x^* \right|^2,$$

denote the squared Euclidean distance from x^*. Nagurney and Zhang (1997) and Lahkar and Sandholm (2008) show that the value of E_{x^*} is nonincreasing along solutions of the projection dynamic if F is a stable game. This value is decreasing if F is strictly stable, and it is constant along interior portions of solution trajectories if $x^* \in \text{int}(X)$ and F is null stable.

One can establish precisely analogous statements for the replicator dynamic by replacing the distance function E_{x^*} with the "distance-like function"

$$\mathcal{E}_{x^*}(x) = \sum_{i:\, x_i^* > 0} x_i^* \log \frac{x_i^*}{x_i};$$

see Hofbauer et al. (1979), Zeeman (1980), and Akin (1990).[8] Thus, by taking advantage of these Lyapunov functions, one can show that the replicator dynamic and the projection dynamic converge to equilibrium from all initial conditions in strictly stable games (actually, all interior initial conditions in the case of the replicator dynamic), and that both admit constants of motion in null stable games.

We illustrate this point in Figure 3, where we present phase diagrams for the projection and replicator dynamics atop contour plots of E_{x^*} and $\mathcal{E}_{x^*}$ in the (standard) Rock-Paper-Scissors game

$$F(x) = \begin{pmatrix} F_R(x) \\ F_P(x) \\ F_S(x) \end{pmatrix} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_R \\ x_P \\ x_S \end{pmatrix}.$$

Since F is null stable with unique Nash equilibrium $x^* = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, interior solutions of the projection dynamic are closed orbits that lie on the level sets of E_{x^*}, while interior solutions of the replicator dynamic are closed orbits that lie on the level sets of $\mathcal{E}_{x^*}$. This behavior stands in contrast with that of many other evolutionary dynamics, including the best response, logit, BNN, and Smith dynamics, all of which converge to equiilbrium in null stable games (Hofbauer and Sandholm (2007, 2008)).

5.2 Gradient Systems for Potential Games

The population game $F : X \to \mathbf{R}^n$ a *potential game* (Monderer and Shapley (1996), Sandholm (2001, 2008)) if it admits a *potential function* $f : X \to \mathbf{R}$ satisfying[9]

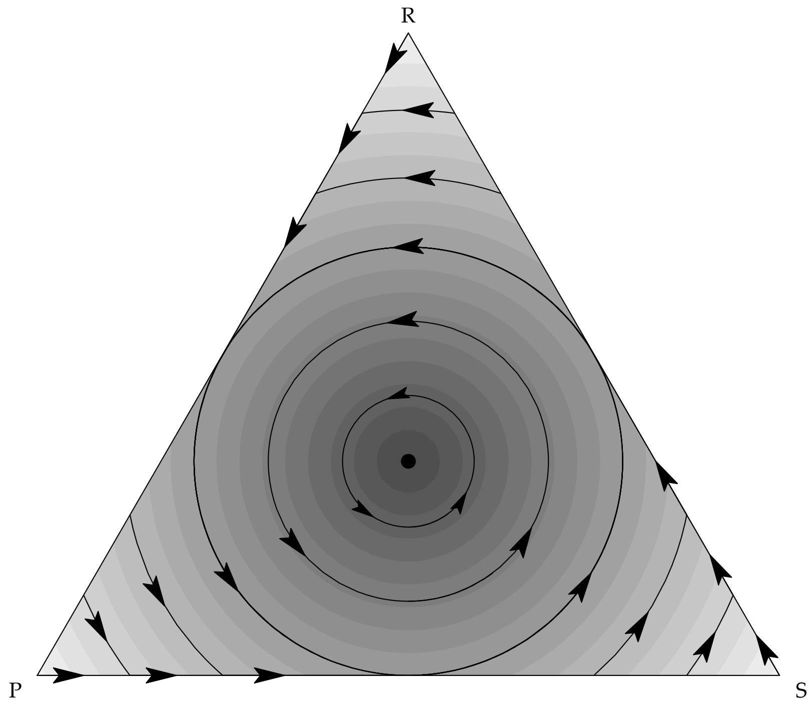
$$\text{(10)} \qquad \nabla f(x) = \Phi F(x) \text{ for all } x \in X.$$

Lahkar and Sandholm (2008) note that if F is a potential game, the potential function f serves as a strict Lyapunov function for the projection dynamic: its value increases along solutions to (P), strictly so except at Nash equilibria. In this respect the projection dynamic is similar to most other evolutionary dynamics considered in the literature; see Sandholm (2001) and Hofbauer and Sandholm (2007).

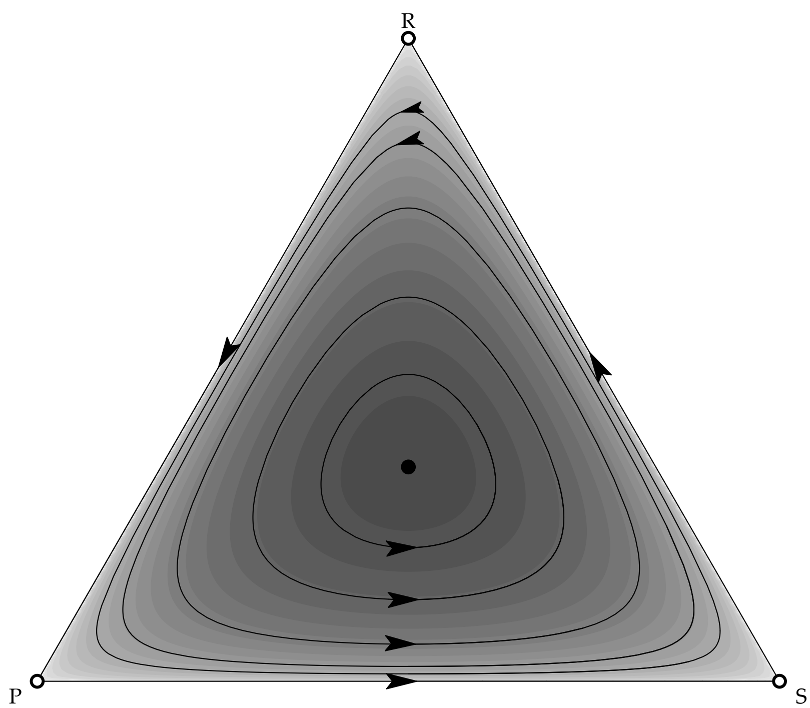
One can obtain a much stronger conclusion by restricting attention to the interior of

[8] $\mathcal{E}_{x^*}(x)$ is the *relative entropy* of x^* with respect to x. While $\mathcal{E}_{(\cdot)}(\cdot)$ is not a true distance, $\mathcal{E}_{x^*}(\cdot)$ is strictly concave, nonegative, and equal to 0 only when its argument x equals x^*.

[9] Since the domain of f is X, the gradient vector $\nabla f(x)$ is the unique vector in the tangent space TX that represents the derivative of f at X, in the sense that $f(y) = f(x) + \nabla f(x)'(y - x) + o(|y - x|)$ for all $y \in X$.



(i) The projection dynamic, plotted over E_{x^*}.



(ii) The replicator dynamic, plotted over $\mathcal{E}_{x^*}$.

Figure 3: Phase diagrams of the projection and replicator dynamics in standard RPS. Grayscale represents the values of the Lyapunov functions E_{x^*} and $\mathcal{E}_{x^*}$: lighter shades indicate higher values.

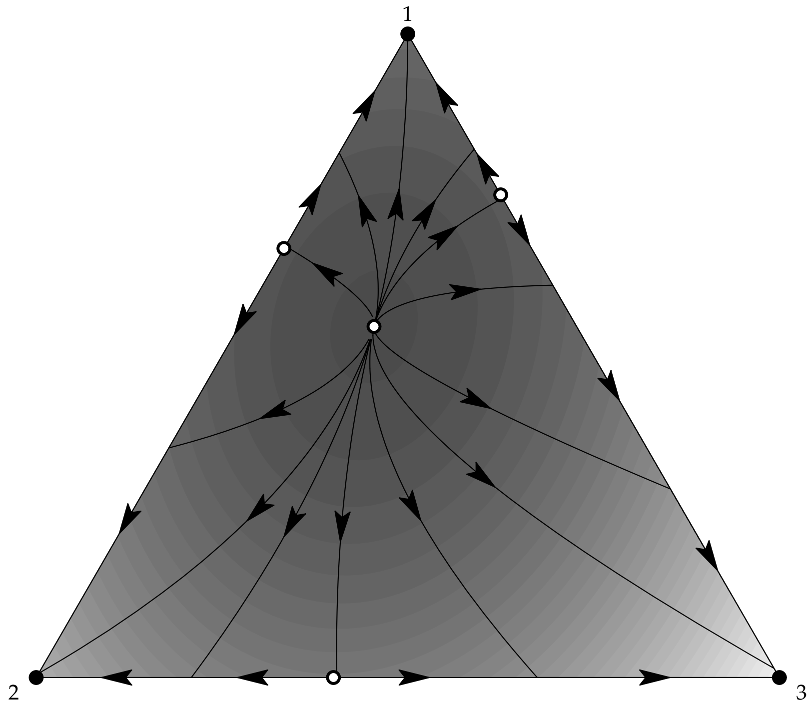


Figure 4: Phase diagram of (P) for coordination game (12). Grayscale represents the value of potential: lighter colors indicate higher values.

the simplex. There the projection dynamic is actually the *gradient system* for f:

(11) $\qquad \dot{x} = \nabla f(x) \;\text{ on }\; \mathrm{int}(X),$

In geometric terms, (11) says that interior solutions to (P) cross the level sets of the f orthogonally. We illustrate this point in Figure 4, where we present the phase diagram of the projection dynamic in the pure coordination game

(12) $$F(x) = \begin{pmatrix} F_1(x) \\ F_2(x) \\ F_3(x) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

The contour plot in this figure shows the level sets of the game's potential function,

$$f(x) = \frac{1}{2}\left((x_1)^2 + 2(x_2)^2 + 3(x_3)^2\right).$$

Evidently, solutions trajectories of (P) in the interior of the simplex cross the level sets of f at right angles.

Remarkably enough, it is also possible to view the replicator dynamic for F as a gradient system for the potential function f. Shahshahani (1979), building on the early work of Kimura (1958), showed that the replicator dynamic for a potential game is a gradient dynamic after a "change in geometry"—in particular, after the introduction of an appropriate Riemannian metric on $\text{int}(X)$. Subsequently, Akin (1979) (see also Akin (1990)) established that Shahshahani's (1979) Riemannian manifold is isometric to the set $\mathcal{X} = \{\chi \in \mathbf{R}^n_+ : \sum_{i \in S} \chi_i^2 = 4\}$, the portion of the raidus 2 sphere lying in the positive orthant, endowed with the usual Euclidean metric. It follows that if we use Akin's (1979) isometry to transport the replicator dynamic for the potential game F to the set $\mathcal{X}$, this transported dynamic is a gradient system in the usual Euclidean sense. To conclude the paper, we provide a direct proof of this striking fact, a proof that does not require intermediate steps through differential geometry.

Akin's (1979) transformation, which we denote by $H : \text{int}(\mathbf{R}^n_+) \to \text{int}(\mathbf{R}^n_+)$, is defined by $H_i(x) = 2\sqrt{x_i}$. As we noted earlier, H is a diffeomorphism that maps the simplex X onto the set $\mathcal{X}$. We wish to prove

Theorem 5.1. *Let $F : X \to \mathbf{R}^n$ be a potential game with potential function $f : X \to \mathbf{R}$. Suppose we transport the replicator dynamic for F on* $\text{int}(X)$ *to the set* $\text{int}(\mathcal{X})$ *using the transformation H. Then the resulting dynamic is the (Euclidean) gradient dynamic for the transported potential function $\phi = f \circ H^{-1}$.*

Since $H_i(x) = 2\sqrt{x_i}$, the transformation H makes changes in component x_i look large when x_i itself is small. Therefore, Theorem 5.1 tells us that the replicator dynamic is a gradient dynamic on $\text{int}(X)$ after a change of variable that makes changes in the use of rare strategies look important relative to changes in the use of common ones. Intuitively, this reweighting accounts for the fact that under imitative dynamics, both increases and decreases in the use of rare strategies are necessarily slow.

Proof. We prove Theorem 5.1 in two steps: first, we derive the transported version of the replicator dynamic; then we derive the gradient system for the transported version of the potential function, and show that it is the same dynamic on $\mathcal{X}$. The following notation will simplify our calculations: when $y \in \mathbf{R}^n_+$ and $a \in \mathbf{R}$, we let $[y^a] \in \mathbf{R}^n$ be the vector whose ith component is $(y_i)^a$.

We can express the replicator dynamic on X as

$$\dot{x} = R(x) = \text{diag}(x)\,(F(x) - \mathbf{1}x'F(x)) = (\text{diag}\,(x) - xx')\,F(x).$$

The transported version of this dynamic can be computed as

$$\dot{\chi} = \mathcal{R}(\chi) = DH(H^{-1}(\chi))R(H^{-1}(\chi)).$$

In words: given a state $\chi \in \mathcal{X}$, we first find the corresponding state $x = H^{-1}(\chi) \in X$ and direction of motion $R(x)$. Since $R(x)$ represents a displacement from state x, we transport it to $\mathcal{X}$ by premultiplying it by $DH(x)$, the derivative of H evaluated at x.

Since $\chi = H(x) = 2\,[x^{1/2}]$, the derivative of H at x is given by $DH(x) = \mathrm{diag}([x^{-1/2}])$ Using this fact, we derive a primitive expression for $\mathcal{R}(\chi)$ in terms of $x = H^{-1}(\chi) = \frac{1}{4}[\chi^2]$:

$$\begin{aligned}
\dot{\chi} &= \mathcal{R}(\chi) \\
&= DH(x)R(x) \\
&= \mathrm{diag}([x^{-1/2}])(\mathrm{diag}(x) - xx')F(x) \\
&= \left(\mathrm{diag}([x^{1/2}]) - [x^{1/2}]x'\right)F(x).
\end{aligned} \tag{13}$$

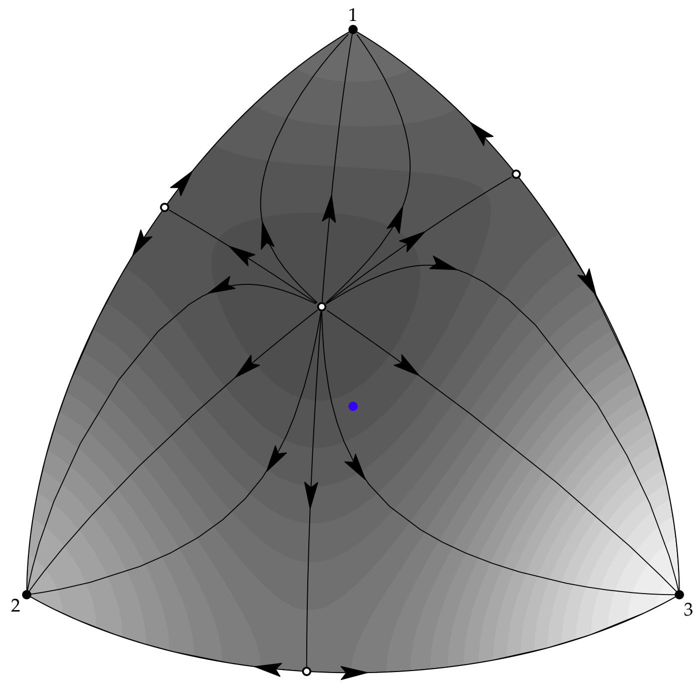
Now, we derive the gradient system on $\mathcal{X}$ generated by $\phi = f \circ H^{-1}$. To compute $\nabla\phi(\chi)$, we need to define an extension of ϕ to all of $\mathbf{R}^n_+$, compute its gradient, and then project the result onto the tangent space of $\mathcal{X}$ at χ. The easiest way to proceed is to let $\tilde{f} : \mathrm{int}(\mathbf{R}^n_+) \to \mathbf{R}$ be an arbitrary C^1 extension of f, and to define the extension $\tilde{\phi} : \mathrm{int}(\mathbf{R}^n_+) \to \mathbf{R}$ by $\tilde{\phi} = \tilde{f} \circ H^{-1}$.

Since $\mathcal{X}$ is a portion of a sphere centered at the origin, the tangent space of $\mathcal{X}$ at χ is the subspace $T\mathcal{X}(\chi) = \{z \in \mathbf{R}^n : \chi' z = 0\}$. The orthogonal projection onto this set is represented by the $n \times n$ matrix

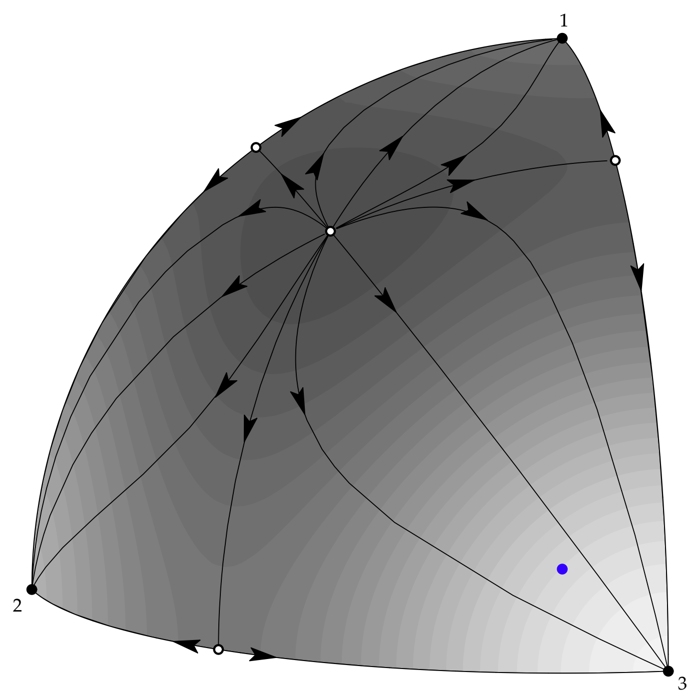
$$P_{T\mathcal{X}(\chi)} = I - \frac{1}{\chi'\chi}\chi\chi' = I - \frac{1}{4}\chi\chi' = I - [x^{1/2}][x^{1/2}]'.$$

Also, since $\Phi\nabla\tilde{f}(x) = \nabla f(x) = \Phi F(x)$ by construction, it follows that $\nabla\tilde{f}(x) = F(x) + c(x)\mathbf{1}$ for some scalar-valued function $c : X \to \mathbf{R}$. Therefore, the gradient system on $\mathcal{X}$ generated by ϕ is

$$\begin{aligned}
\dot{\chi} &= \nabla\phi(\chi) \\
&= P_{T\mathcal{X}(\chi)}\nabla\tilde{\phi}(\chi) \\
&= P_{T\mathcal{X}(\chi)}DH^{-1}(\chi)'\nabla\tilde{f}(x) \\
&= P_{T\mathcal{X}(\chi)}\,(DH(x)^{-1})'\,(F(x) + c(x)\mathbf{1}) \\
&= \left(I - [x^{1/2}][x^{1/2}]'\right)\mathrm{diag}([x^{1/2}])\,(F(x) + c(x)\mathbf{1}) \\
&= \left(\mathrm{diag}([x^{1/2}]) - [x^{1/2}]x'\right)(F(x) + c(x)\mathbf{1})
\end{aligned}$$



(i) origin of projection = $H(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$



(ii) origin of projection = $H(\frac{1}{7}, \frac{1}{7}, \frac{5}{7})$

Figure 5: The phase diagram of the transported replicator dynamic $\dot{\chi} = \mathcal{R}(\chi)$ for a coordination game. Grayscale represents the value of the transported potential function.

$$= \left(\mathrm{diag}([x^{1/2}]) - [x^{1/2}]x'\right) F(x).$$

This agrees with equation (13), completing the proof of the theorem. ■

In Figure 5, we illustrate Theorem 5.1 with phase diagrams of the transported replicator dynamic $\dot{\chi} = \mathcal{R}(\chi)$ for the three-strategy coordination game from equation (12). These phase diagrams on $\mathcal{X}$ are drawn atop contour plots of the transported potential function $\phi(\chi) = (f \circ H^{-1})(\chi) = \frac{1}{32}((\chi_1)^4 + 2(\chi_2)^4 + 3(\chi_3)^4)$. According to Theorem 5.1, the solution trajectories of $\mathcal{R}$ should cross the level sets of ϕ orthogonally.

Looking at Figure 5(i), we find that the crossings look orthogonal at the center of the figure, but not by the boundaries. This is an artifact of our drawing a portion of the sphere in $\mathbf{R}^3$ by projecting it orthogonally onto a sheet of paper.[10] To check whether the crossings near a given state $\chi \in \mathcal{X}$ are truly orthogonal, we can minimize the distortion of angles near χ by making χ the origin of the projection.[11] We mark the projection origins in Figures 5(i) and Figures 5(ii) with dots; evidently, the crossings are orthogonal near these points.

A. Appendix

The Proof of Theorem 4.1

The method used to construct the approximate closed orbit of (P) is described in the text after the statement of the theorem. Here, we verify that this approximation implies the existence of an exact closed orbit of (P). A minor modification of our argument shows that this orbit absorbs all nearby trajectories in finite time.

Let us review the construction of the approximate closed orbit. We begin by choosing the initial state $\xi^0 = \xi = (\frac{7}{15}, \frac{8}{15}, 0, 0)$. The (exact) solution to (P) from ξ initially travels through face *RPS* in a fashion described by the linear differential equation (3), and so be computed analytically. The solution exits face *RPS* into the interior of X when it hits the line on which the payoff to T equals the average of the payoffs to R, P, and S. The exit point cannot be determined analytically, but it can be approximated to any desired degree of accuracy. We call this approximate exit point, which we compute to 16 decimal places, $\xi^1 \approx (.446354, .552864, .000782, 0)$, and we call the time that the solution to (P) reaches this point $t^1 \approx .015678$.

[10]For the same reason, latitude and longitude lines in an orthographic projection of the Earth only appear to cross at right angles in the center of the projection, not on the left and right sides.

[11]The origin of the projection, $o \in \mathcal{X}$, is the point where the sphere touches the sheet of paper. If we view the projection from any point on the ray that exits the sheet of paper orthogonally from o, then the center of the sphere is directly behind o.

Next, we consider the (exact) solution to (P) from starting from state ξ^1. This solution travels through int(X) until it reaches face PST. We again compute an approximate exit point ξ^2, and we let t^2 be the total time expended during the first two solution segments. Continuing in this fashion, we compute the approximate exit points $\xi^3, \dots, \xi^7$, and the transition times $t^3, \dots, t^7$.

Now for each state ξ^k, let $\{x_t^k\}_{t \geq t^k}$ be the solution to (P) that starts from state ξ^k at time t^k. Because solutions to (P) are Lipschitz continuous in their initial conditions (see Lahkar and Sandholm (2008)), we can bound the distance between state $x_{t^7}^0$, which is the location of the solution to (P) from state $\xi^0 = \xi$ at time t^7, and state $x_{t^7}^7 = \xi^7$, as follows:

$$\left|x_{t^7}^0 - x_{t^7}^7\right| \leq \sum_{k=1}^{7} \left|x_{t^7}^{k-1} - x_{t^7}^k\right| \leq \sum_{k=1}^{7} \mathrm{e}^{K(t_7 - t_k)} \varepsilon.$$

Here, K is the Lipschitz coefficient for the payoff vector field F, and ε is an upper bound on the roundoff error introduced when we compute the approximate exit point ξ^k for the solution to (P) from state ξ^{k-1}.

Since $F(x) = Ax$ is linear, its Lipschitz coefficient is the spectral norm of the payoff matrix A: that is, the square root of the largest eigenvalue of $A'A$ (see Horn and Johnson (1985)). A computation reveals that the spectral norm of A is approximately 5.718145. Since we compute our approximate exit points to 16 decimal places, our roundoff errors are no greater than 5×10^{-17}. Thus, since $t^7 - t^1 = 1.049900$, we obtain the following bound on the distance between $x_{t^7}^0$ and $x_{t^7}^7$:

$$\left|x_{t^7}^0 - x_{t^7}^7\right| \leq 7\left(\mathrm{e}^{K(t_7 - t_k)} \varepsilon\right) \approx 7\left(\mathrm{e}^{(5.718145)(1.049900)} \times (5 \times 10^{-17})\right) \approx 1.416920 \times 10^{-13}.$$

It is easy to verify that any solution to (P) that starts within this distance of state $\xi^7 \approx (.693072, .306928, 0, 0)$ will hit edge RP between vertex R and state ξ, and so continue on to ξ. We therefore conclude that $\{x_t^0\}_{t \geq 0}$, the exact solution to (P) starting from state ξ, must return to state ξ. This completes the proof of the theorem. ■

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