Potential Games with Continuous Player Sets

William H. Sandholm*
Department of Economics
University of Wisconsin
1180 Observatory Drive
Madison, WI 53706
whs@ssc.wisc.edu
www.ssc.wisc.edu/~whs

First Version: April 30, 1997

This Version: May 20, 2000

* This paper is based on Chapter 4 of my doctoral dissertation (Sandholm [29]). I thank an anonymous referee and Associate Editor, Josef Hofbauer, and seminar audiences at Caltech, Chicago, Columbia, Harvard (Kennedy), Michigan, Northwestern, Rochester, Washington University, and Wisconsin for their comments. I especially thank Eddie Dekel and Jeroen Swinkels for their advice and encouragement. Financial support from a State Farm Companies Foundation Dissertation Fellowship is gratefully acknowledged.

Abstract

We study potential games with continuous player sets, a class of games characterized by an externality symmetry condition. Examples of these games include random matching games with common payoffs and congestion games. We offer a simple description of equilibria which are locally stable under a broad class of evolutionary dynamics, and prove that behavior converges to Nash equilibrium from all initial conditions. We consider a subclass of potential games in which evolution leads to efficient play. Finally, we show that the games studied here are the limits of convergent sequences of the finite player potential games studied by Monderer and Shapley [22].

JEL Classification Numbers: C72, C73, D62, R41.

1. Introduction

Nash equilibrium is the cornerstone of non-cooperative game theory, providing a necessary condition for stable behavior among rational agents. Still, to justify the prediction of Nash equilibrium play, one must explain how players arrive at a Nash equilibrium; if equilibrium is not reached, the fact that it is self-sustaining becomes moot. This question has launched a search for procedures by which players can learn to play Nash equilibria, and for games in which such procedures are effective.

In this paper, we study *potential games with continuous player sets*, a class of games in which a wide range of evolutionary processes converge to Nash equilibrium. We offer simple characterizations of all equilibria of these games and of the equilibria which are locally stable under evolutionary dynamics. We establish the global convergence of solution trajectories to equilibria. We describe a subclass of potential games in which evolution leads to efficient play. Finally, we characterize the games studied here as the limiting case of the finite player potential games of Monderer and Shapley [22].

Monderer and Shapley's [22] finite player potential games are games which admit a potential function: a real-valued function defined on the space of pure strategy profiles such that the change in any player's payoffs from a unilateral deviation is exactly matched by the change in potential. It follows immediately that Nash equilibria are the local maximizers of potential. Moreover, since profitable deviations increase potential, better reply adjustment processes lead to equilibrium play.

There are many reasons why introducing a notion of potential games for infinite populations is worthwhile. The basic convergence results for potential games concern myopic adjustment processes, which are most natural to study when the number of players is large; since games with large, finite populations can be cumbersome to analyze, infinite population models offer a convenient alternative. Moreover, since most work in evolutionary game theory concerns such models, our results on infinite player potential games allows us to connect this notion to a large segment of the literature. Most importantly, the infinite player model allows us to base our analysis on calculus. This allows us to find a simple, economically meaningful condition which characterizes infinite player potential games. It also enables us to derive conditions under which evolution leads to efficient play; we do not know of a finite player analogue of these efficiency results.

Formally, a game with continuous player sets is a *potential game* if it admits a *potential function*: in this context, a real valued function on the space of strategy distributions whose gradient is the vector of payoff functions. If payoffs are smooth, an equivalent definition of potential games is that payoffs satisfy *externality symmetry*: that for any pair of strategies i and j, the effect of adding players choosing strategy i on the payoffs of those choosing strategy j is equal to the effect of adding players choosing strategy j on the payoffs of those choosing strategy i. Examples of games satisfying this property are random matching games in which all players in a match receive the same payoff, and congestion games, a class of games used to model congestion in networks.[1]

Externality symmetry guarantees that reasonable behavior adjustment processes converge to Nash equilibria. The class of processes we consider is defined by two natural conditions. The first, *positive correlation*, requires that strategies' growth rates be positively correlated with their payoffs.[2] Positive correlation is the weakest monotonicity condition used in the evolutionary literature.[3] We show that all dynamics which satisfy positive correlation ascend the potential function. This observation is the key to our convergence results.

While we are able to prove a number of results using positive correlation alone, for others we must also be sure that players eventually take advantage of opportunities to improve their payoffs. Our second condition, noncomplacency, formalizes this idea by requiring all rest points of the dynamics to be Nash equilibria.

In finite player potential games, all equilibria are local maximizers of potential; since better reply adjustment processes increase potential, all equilibria are locally stable. In infinite player settings, these statements are false. We characterize Nash equilibria as the states which satisfy the Kuhn-Tucker first order conditions for a maximizer of potential. Thus, while all maximizers are equilibria, not all equilibria are maximizers. However, since dynamics satisfying positive correlation must

---

1 A related model of congestion is considered by Beckmann, McGuire, and Winsten [1], who use a potential function to characterize equilibrium and to establish conditions under which equilibrium is unique. Rosenthal [27] defines finite player congestion games and uses a potential function argument to establish the existence of a pure strategy equilibrium.

2 The condition we use is actually somewhat weaker than this – see Section 4.

3 Positive correlation is satisfied by both the replicator dynamics (Taylor and Jonker [33]) and the best response dynamics (Gilboa and Matsui [13], Matsui [20], Hofbauer [14]). Nachbar [25], Friedman [11], Samuelson and Zhang [28], Swinkels [32], Ritzberger and Weibull [25], and Hofbauer and Weibull [17] study classes of dynamics whose members satisfy some basic evolutionary desiderata. Of these, the conditions considered by Friedman [11] and Swinkels [32] are both the weakest and the closest to the condition considered here.

ascend the potential function, we are able to show that the states which locally maximize potential are precisely those which are locally stable. Thus, by restricting attention to locally stable equilibria, we recover the link with the finite player analysis.

A full evolutionary justification of Nash equilibrium requires a global stability result: solution trajectories from all initial conditions must converge to a Nash equilibrium. Despite the presence of unstable equilibria, we are able to extend Monderer and Shapley's [22] global convergence result to the infinite player setting: all trajectories of dynamics satisfying positive correlation and noncomplacency lead to Nash equilibria.

Our efficiency results concern potential games which are *homogenous*: that is, in which each strategy's payoff function is a homogenous function of the same degree. All random matching games with common payoffs are homogenous potential games, as are congestion games in which all facilities (e.g., streets) are equally sensitive to congestion.

We measure efficiency in terms of the aggregate payoffs earned by all players in the game. Together, homogeneity and externality symmetry imply that a player's payoff to choosing a strategy is always proportional to the marginal impact of his choice on aggregate payoffs. Using this observation, we show that every homogenous potential game has a homogenous potential function which is proportional to aggregate payoffs. We can therefore establish that evolution increases aggregate payoffs, that locally stable equilibria are precisely those which are locally efficient, and that unique equilibria are not only globally stable, but also globally efficient.

While the results on behavior adjustment obtained by Monderer and Shapley [22] are quite similar to those obtained here, the formal connections between the finite and infinite player models are not obvious. To draw comparisons, we restrict attention to finite player potential games in which players are anonymous and identical. We find a simple representation for these games in terms of an extended potential function, and use this representation to define a notion of convergence for sequences of games whose populations grow without bound. We then prove that the limits of such sequences are the infinite player potential games studied here. The existence of this fundamental link between the two models renders the choice between them a matter of analytical convenience.[4]

Building on work of Fisher [10] and Kimura [18], Hofbauer and Sigmund [15, p. 240-241] consider single population potential games in a population genetics setting. They show that the replicator dynamics must ascend potential,[5] and observe that homogenous potential functions are proportional to average payoffs. We establish these results for multipopulation settings, and show that positive correlation alone ensures that evolution increases potential. Moreover, by introducing non-complacency, a plausible requirement for economic models, we are able to establish a number of stronger results, including global convergence to Nash equilibrium.

The existence of a potential function is a rather strong requirement; this may lead one to question the practical relevance of our results. However, in mechanism design settings, one often assumes the presence of a social planner who is uncertain about players' preferences, but is able to alter their payoffs using transfer payments. Such a planner can use transfers to create a game which admits a potential function, ensuring convergence to equilibrium play. In Sandholm [30], we use this observation as the basis for an evolutionary approach to implementation theory.

Section 2 defines potential games and introduces three classes of examples. Section 3 characterizes equilibria. Section 4 studies evolutionary dynamics. Section 5 introduces homogenous potential games and investigates their efficiency properties. Finally, Section 6 establishes the connections between the finite player potential games of Monderer and Shapley [22] and the infinite player potential games studied here. Readers who are especially interested in these last results may prefer to read Section 6 immediately following Section 2.

---

4 Monderer and Shapley [22] propose the global maximizer of potential as an equilibrium selection device. A number of papers provide formal justifications for this idea. Blume [3, 4, 5] shows that if players are randomly matched to play a finite player potential game, then the global maximizer of potential is the unique stochastically stable outcome when mutation probabilities are determined via the log-linear choice rule. He also proves related results for local interaction models. Hofbauer and Sorger [16] consider populations of players who are randomly matched to play a game with common payoffs. They show that in perfect foresight equilibria (Matsui and Matsuyama [21]) of these games, behavior converges to the global maximizer of the potential function of the random matching game.

5 In fact, they use differential geometry techniques of Shahshahani [31] to prove that if the state space is stretched appropriately, the replicator dynamics climb the potential function at a maximal rate.

2. Potential Games

2.1 Population Games, Potential Games, and Externality Symmetry

A *population game* with r continuous populations of players is defined by a mass and a strategy set for each population and a payoff function for each strategy. The set of populations is denoted $P = \{1, \dots, r\}$, where $r \geq 1$; population p has mass m^p. The set of strategies for population p is denoted $S^p = \{1, \dots, n^p\}$, and $n = \sum_p n^p$ equals the total number of pure strategies.

The set of strategy distributions within population $p \in P$ is denoted $X^p = \{x \in \mathbf{R}^{n^p}_+ \colon \sum_i x_i^p = m^p\}$, while $X = \{x = (x^1, \dots, x^r) \in \mathbf{R}^n_+ \colon x^p \in X^p\}$ is the set of overall strategy distributions. While behavior is always described by a point in X, it will be useful to define payoffs on the set $\bar{X} = \{x \in \mathbf{R}^n_+ \colon m^p - \varepsilon \leq \sum_i x_i^p \leq m^p + \varepsilon \;\; \forall\, p \in P\}$, where ε is a positive constant. The set $\bar{X}$ contains the strategy distributions which can arise if each population's mass stays within ε of m^p. Defining payoffs on this set is useful because it enables us to speak of the marginal impact of a newcomer, but is otherwise innocuous: versions of all of our results hold if payoffs are only defined on X.

The payoff function for strategy $i \in S^p$ is denoted $F_i^p \colon \bar{X} \to \mathbf{R}$, and is assumed to be continuous. Note that the payoffs to a strategy in population p can depend on the strategy distribution within population p itself. We let $F^p \colon \bar{X} \to \mathbf{R}^{n^p}$ refer to the vector of payoff functions for strategies belonging to population p and let $F \colon \bar{X} \to \mathbf{R}^n$ denote the vector of all payoff functions. Similar notational conventions are used throughout the paper. However, when considering single population games, we omit superscripts and assume that the population mass equals one.

We call F a *potential game* if condition (P) holds:

(P) There exists a C^1 function $f \colon \bar{X} \to \mathbf{R}$ such that $\dfrac{\partial f}{\partial x_i^p}(x) = F_i^p(x)$ for all $x \in \bar{X}$, $i \in S^p$, and $p \in P$.

Condition (P) says that there is a continuously differentiable function f whose gradient, ∇f, equals the payoff vector F. The function f, which is unique up to an additive constant, is called the *potential function* of the game.

For intuition, consider a state $x \in X$ at which $F_i^p(x) > F_j^p(x)$. At such a state, a player choosing strategy j would prefer to switch to strategy i. But since $\frac{\partial f}{\partial (x_i^p - x_j^p)}(x) \equiv$

$\frac{\partial f}{\partial x_i^p}(x) - \frac{\partial f}{\partial x_j^p}(x) = F_i^p(x) - F_j^p(x) > 0$, this profitable strategy change leads to a marginal increase in potential. More generally, we show in Section 4 that the uphill directions of the potential function include all those in which reasonable adjustment processes might lead. This property lies at the heart of our analysis.

We do not require payoff functions to be differentiable, but if they are we can characterize potential games in a more intuitive fashion. In particular, if payoffs are C^1 (continuously differentiable), then condition (P) is equivalent to *externality symmetry* (ES):

$$\text{(ES)} \qquad \frac{\partial F_i^p}{\partial x_j^q} \equiv \frac{\partial F_j^q}{\partial x_i^p} \text{ for all } i \in S^p, j \in S^q, \text{ and } p, q \in P.$$

Externality symmetry requires that the marginal effect of adding a player choosing strategy i on the payoffs of players choosing strategy j is the same as the marginal effect of adding a player choosing strategy j on the payoffs of players choosing strategy i. This symmetry property has striking implications for the evolution of aggregate behavior.

2.2 Examples

2.2.2 Random Matching Games with Common Payoffs

Most work in evolutionary game theory focuses on populations of players who are randomly matched to play normal form games. In this setting, externality symmetry requires that the players in any random match all obtain the same payoffs.

An r player normal form game is defined by a payoff function $U^p: S^1 \times \ldots \times S^r \to \mathbf{R}$ for each player p; $U^p(i^1, \ldots, i^r)$ is the payoff player p receives if pure strategy profile $(i^1, \ldots, i^r)$ is followed. The game exhibits *common payoffs* if $U^p = U$ for all p: that is, if each pure strategy profile yields the same payoff for every player.[6]

[6] These normal form games are potential games in the sense of Monderer and Shapley [22]. In fact, as we observe in Section 6, Monderer and Shapley's games are precisely those whose payoffs take the form $U^p = U + A^p$, where $A^p = A^p(i^1, \ldots, i^{p-1}, i^{p+1}, \ldots, i^r)$ is independent of i^p. Since the A^p terms do not alter players' incentives, our results on evolution continue to hold when players are randomly matched to play these games. However, our efficiency results, which are defined in terms of aggregate payoffs, do not generalize to these games.

Most of the literature considers a single population which is randomly matched to play a *symmetric* two player game: one in which $S^1 = S^2$ and $U^1(i, j) = U^2(j, i)$ for all i and j. The latter condition implies that a player's payoff only depends on his strategy and that of his opponent, not on whether he is called player 1 or player 2. If a continuum of players is randomly matched to play such a game, the (expected) payoffs from a match are given by

$$F_i(x) = \sum_{j \in S^2} U^1(i, j)\, x_j .$$

If a symmetric two player game exhibits common payoffs, its payoff matrix U is symmetric: $U(i, j) = U^1(i, j) = U^2(j, i) = U(j, i)$. Hence, payoffs in the random matching game satisfy externality symmetry: $\frac{\partial F_i}{\partial x_j} \equiv U(i, j) = U(j, i) \equiv \frac{\partial F_j}{\partial x_i}$. The potential function for this matching game is

$$f(x) = \tfrac{1}{2} \sum_{(i,j) \in S^1 \times S^2} U(i, j)\, x_i\, x_j .$$

The discussion following condition (P) shows that profitable changes in behavior must ascend the function f. Since $f(x)$ is equal to one-half of average payoffs at state x, evolution leads to locally efficient states.

Analogous results hold in the multipopulation case, in which a separate population of mass one is assigned to each role in the game. If there are r populations, the payoffs to strategy $i^p \in S^p$ in the random matching game are

$$F^p_{i^p}(x) = \sum_{\substack{(i^1, \ldots, i^{p-1}, i^{p+1}, \ldots, i^r) \in \\ S^1 \times \ldots \times S^{p-1} \times S^{p+1} \times \ldots \times S^r}} \Big(U^p(i^1, \ldots, i^r) \prod_{q \neq p} x^q_{i^q}\Big).$$

If $\{U^p\}_{p \in P}$ exhibits common payoffs, then the matching game F admits the potential function

$$f(x) = \sum_{(i^1, \ldots, i^r) \in S^1 \times \ldots \times S^r} \Big(U(i^1, \ldots, i^r) \prod_{p \in P} x^p_{i^p}\Big),$$

which equals aggregate payoffs divided by r.[7]

While the games we have just described have linear and multilinear payoffs,

---

[7] Versions of our stability and efficiency results still hold if the payoffs to different players in the underlying game are multiples of one another: that is, if for all $p \in P$, $U^p = k^p U$ for some $k^p > 0$.

population games which are not based on random matching generally have non-linear payoffs. In such cases, externality symmetry does not reduce to commonality of payoffs; when externality symmetry does hold, efficiency does not immediately follow. The next class of examples illustrates these points.

2.2.2 Congestion Games

Consider a collection of towns connected by a network of streets. We associate each of r pairs of home and work locations with a group of commuters who must travel between them. Each player chooses a route (i.e., a subset of the streets) connecting home to work; his driving time depends upon the traffic on the streets he has chosen.

A *congestion model* is a collection $\{P, \{m^p\}_{p\in P}, \{S^p\}_{p\in P}, \{\Phi_i^p\}_{i\in S^p, p\in P}, \{c_\phi\}_{\phi\in\Phi}\}$. P is a set of one or more populations, one for each home/work location pair. The finite set $\Phi = \bigcup_{i\in S^p}\bigcup_{p\in P}\Phi_i^p$ contains all available streets. We associate each strategy $i \in S^p$ with a complete route Φ_i^p connecting the home/work pair of population p.[8]

Let $\rho^p(\phi) = \{i \in S^p\colon \phi \in \Phi_i^p\}$ denote the set of population p strategies which require street ϕ. The *utilization* of street $\phi \in \Phi$ is the total mass of the players whose strategies use that street:

$$u_\phi(x) = \sum_{p\in S^p}\sum_{i\in\rho^p(\phi)} x_i^p.$$

When a player selects a route, he experiences the delays on each street in the route. The *street costs*, $c_\phi\colon \mathbf{R}_+ \to \mathbf{R}$, are continuous functions which report the delays on a street as a function of its utilization. The *congestion game* derived from a congestion model is defined by its payoff functions:

$$F_i^p(x) = -\sum_{\phi\in\Phi_i^p} c_\phi(u_\phi(x)).$$

In models of traffic flow, the cost functions c_ϕ are increasing; we can study settings with positive externalities by using c_ϕ which are decreasing.

Since payoffs to the strategies in a congestion game are sums of street costs, the payoffs to any pair of strategies are bound together by the streets used in common.

---

8 We do not assume any graph theoretic structure on the set Φ. Hence, the congestion model is applicable in settings in which the set of facilities used by the players does not possess such a structure.

Increasing the proportion of players from population p using route i affects the players taking route $j \in S^q$ through increased traffic on streets in $\Phi_i^p \cap \Phi_j^q$. If street costs are differentiable, the marginal effect of this increase can be expressed as $\frac{\partial F_j^q}{\partial x_i^p} \equiv -\sum_{\phi \in \Phi_i^p \cap \Phi_j^q} c'_\phi$. An increase in the use of route j has an the same marginal effect on route i drivers $\frac{\partial F_j^q}{\partial x_i^p} \equiv -\sum_{\phi \in \Phi_i^p \cap \Phi_j^q} c'_\phi$. Hence, externality symmetry holds: $\frac{\partial F_i^p}{\partial x_j^q} \equiv \frac{\partial F_j^q}{\partial x_i^p}$.

Even if street costs are not differentiable, we can verify directly that F is a potential game by observing that

$$f(x) = -\sum_{\phi \in \Phi} \int_0^{u_\phi(x)} c_\phi(z)\,dz$$

is a potential function for F. The discussion after condition (P) shows that profitable deviations ascend this function. However, the function f is typically not proportional to aggregate payoffs, so evolution need not lead to efficient play. In Section 5, we show that if all streets are equally sensitive to congestion, efficient behavior is ensured.[9]

2.2.3 Two Strategy Games

If a game played by a single population has only two strategies, all strategy distributions lie on a line. Since continuous dynamics on a line are easy to analyze, two strategy games are quite common in the evolutionary literature. It is therefore worth noting that these games are all potential games. Given any continuous payoff functions $F_1, F_2: X \to \mathbf{R}$, a potential function satisfying condition (P) is given by[10]

$$f(x_1, x_2) = \int_0^{x_1} F_1(z, 1-z)\,dz + \int_0^{x_2} F_2(1-z, z)\,dz.$$

---

[9] Monderer and Shapley [22] establish an equivalence between finite player potential games and Rosenthal's [27] finite player congestion games; Voorneveld *et. al.* [34] provide a simpler proof. It is easy to generate examples which show that this result does not extend to the infinite player setting. The equivalence proofs rely on constructions in which the number of facilities grows exponentially in the population size; this growth persists under the symmetry conditions we impose in Section 6. Thus, when there are a continuum of players, these constructions cannot be used.

[10] To fully satisfy condition (P), we must extend the payoff functions from X to $\overline{X}$ by letting $F_1(x_1, x_2) = F_1(x_1, 1 - x_1)$ and $F_2(x_1, x_2) = F_2(1 - x_2, x_2)$ for all $(x_1, x_2) \in \overline{X} - X$. While in the previous examples there are natural interpretations of payoffs at points outside X, in the current example such payoffs have no obvious interpretation.

3. Equilibrium

We begin our analysis of potential games by characterizing their Nash equilibria. Let the best response correspondence, BR: $X \to X$, map each state $x \in X$ to the set of states whose supports consist entirely of best responses to x. Letting $C^p(x^p) = \{i \in S_p\colon x_i > 0\}$ denote the support of x^p, we define BR^p and BR by

$$BR^p(x) = \{z^p \in X^p\colon\ C^p(z^p) \subset \operatorname*{arg\,max}_{j \in S^p} F_j^p(x)\}, \text{ and}$$
$$BR(x) = \{z \in X\colon\ z^p \in BR^p(x)\ \forall\, p \in P\}.$$

A *Nash equilibrium* is a state whose support consists solely of best responses to itself: $x \in BR(x)$.

We noted earlier that all profitable strategy revisions lead to increases in potential. This suggests that the Nash equilibria of the game are related to the local maximizers of potential. The Lagrangian for this maximization problem is

$$L(x, \mu, \lambda) = f(x) + \sum_{p \in P} \mu^p (m^p - \sum_{i \in S^p} x_i^p) + \sum_{p \in P} \sum_{i \in S^p} \lambda_i^p x_i^p,$$

so the Kuhn-Tucker first-order necessary conditions are

(KT1) $\quad \frac{\partial f}{\partial x_i^p}(x) = \mu^p - \lambda_i^p,$

(KT2) $\quad \lambda_i^p x_i^p = 0,$ and

(KT3) $\quad \lambda_i^p \geq 0$

for all $i \in S^p$ and $p \in P$. The Kuhn-Tucker conditions completely characterize the set of Nash equilibria.

Proposition 3.1: *The state $x \in X$ is a Nash equilibrium of the potential game F if and only if (x, μ, λ) satisfies* (KT1), (KT2), *and* (KT3) *for some $\lambda \in \mathbf{R}^n$ and $\mu \in \mathbf{R}^r$.*

Proof: If x is a Nash equilibrium of F, then since $F(x) = \nabla f(x)$, the Kuhn-Tucker conditions are satisfied by x, $\mu^p = \max_j F_j^p(x)$, and $\lambda_i^p = \mu^p - F_i^p(x)$.

Conversely, if (x, μ, λ) satisfies the Kuhn-Tucker conditions, then for every $p \in P$, (KT1) and (KT2) imply that $F_i^p(x) = \frac{\partial f}{\partial x_i^p}(x) = \mu^p$ for all $i \in C^p(x^p)$. Furthermore, (KT1) and (KT3) imply that $F_j^p(x) = \mu^p - \lambda_j^p \leq \mu^p$ for all $j \in S^p$. Hence, $C^p(x^p) \subset \operatorname{arg\,max}_j F_j^p(x)$, and so x is a Nash equilibrium of F. ■

Observe that the multiplier μ^p equals the equilibrium payoffs in population p.

Since the set X satisfies constraint qualification, satisfaction of the Kuhn-Tucker conditions is necessary but not sufficient for local maximization of potential. Therefore, while all local maximizers of potential are equilibria, not all equilibria locally maximize potential. In contrast, in Monderer and Shapley's [22] finite player games, the only equilibria are the local maximizers of potential.

For intuition, consider a single population of players who are randomly matched to play the coordination game in Figure 1. Whether the population is finite or infinite, (expected) payoffs in this game are given by $F_1(x_1, x_2) = x_1$ and $F_2(x_1, x_2) = x_2$.[11]

1, 1	0, 0
0, 0	1, 1

Figure 1

If N ($< \infty$) players are matched to play this game, there are two equilibria in which all players choose the same pure strategy, but there is no pure strategy equilibrium corresponding to mixed equilibrium of the normal form game. If exactly $\frac{N}{2}$ players chose each strategy, then the payoffs to each are $\frac{1}{2}$. But if a player switches strategies, he alters the distribution of strategies in the population by $\frac{1}{N}$ in favor of the strategy he switches to; since the underlying game is a coordination game, this deviation is profitable. In contrast, a player in an infinite population cannot change the distribution of strategies in the population; hence, an even split of the population between the two strategies constitutes an equilibrium.

Both the finite and infinite versions of this matching game are potential games. In the infinite player case, the potential function is $f(x_1, x_2) = \frac{1}{2}((x_1)^2 + (x_2)^2)$. The equilibria (1, 0) and (0, 1) are the maximizers of potential on the set $X = \{(x_1, x_2) \in \mathbf{R}^2_+ : x_1 + x_2 = 1\}$; the equilibrium $(\frac{1}{2}, \frac{1}{2})$ minimizes potential on this set, but still satisfies

[11] In the finite population case, these are the payoffs which arise if players can be matched against themselves; an analogous analysis would hold without self-matching.

the Kuhn-Tucker first order conditions for a maximum.[12]

Of course, if even slightly more than half of the players choose one strategy, that strategy offers higher payoffs, so we would expect a population to move away from the $(\frac{1}{2}, \frac{1}{2})$ equilibrium. We show in the next section that under a broad class of evolutionary dynamics, the locally stable equilibria are precisely those which maximize potential. Thus, restricting attention to stable equilibria reestablishes the connections between our results and those of Monderer and Shapley [22].

4. Evolutionary Dynamics

Throughout this section we use terminology which is standard in dynamical systems and in evolutionary game theory. Formal definitions omitted from the text can be found in the Appendix.

4.1 Positive Correlation and Noncomplacency

Many papers in the evolutionary literature study behavior adjustment under some fixed equation of motion, most often the replicator dynamics (Taylor and Jonker [33]) or the best response dynamics (Gilboa and Matsui [13], Matsui [20], Hofbauer [14]). Rather than restrict attention to one particular specification of the dynamics, we instead establish results which hold for any dynamics within a broadly defined class.

Evolutionary dynamics are described by a vector field V: $X \to \mathbf{R}^n$ which implicitly defines an equation of motion $\dot{x} = V(x)$. We require V to satisfy Lipschitz continuity (LC) and forward invariance (FI):

(LC) V is Lipschitz continuous.

(FI) $V_i^p(x) \geq 0$ whenever $x_i^p = 0$, and $\sum_{i \in S^p} V_i^p(x) = 0$ for all $p \in P$.

These conditions guarantee the existence of unique solution trajectories which do

---

12 In the finite player matching game, the potential function is $P^N(k_1/N,\ k_2/N) = \sum_{a=0}^{k_1}(a/N) + \sum_{b=0}^{k_2}(b/N)$. This is a rescaled, discrete approximation of f; the two pure equilibria maximize this function. For a formal treatment of the finite player model and its connections with the infinite player model, see Section 6.

not leave X.[13]

Our main condition on the dynamics is called *positive correlation*.

(PC) $$V(x) \cdot F(x) = \sum_{p \in P} \sum_{i \in S^p} V_i^p(x) F_i^p(x) > 0 \text{ whenever } V(x) \neq 0.$$

To see why this condition is so named, observe that by condition (FI),

$$\begin{aligned}\sum_{p \in P} \sum_{i \in S^p} V_i^p(x) F_i^p(x) &= \sum_{p \in P} \left(\sum_{i \in S^p} (V_i^p(x) - 0)(F_i^p(x) - \tfrac{1}{n^p} \sum_{j \in S^p} F_j^p(x)) \right) \\ &= \sum_{p \in P} n^p \, Cov(V^p, F^p).\end{aligned}$$

$Cov(V^p, F^p)$ denotes the covariance between strategy growth rates and payoffs in population p. Hence, condition (PC) holds if there is a positive correlation between growth rates and payoffs in each population. However, it only requires the weighted sum of the covariances in each population to be positive. The conditions closest to positive correlation in the evolutionary game theory literature are those of Friedman [11] and Swinkels [32], who impose restrictions similar to positive correlation on each population.[14]

Since $F(x) = \nabla f(x)$, $F(x)$ is the direction of steepest ascent of the potential function. Hence, geometrically, positive correlation requires that whenever the population is moving, it is moving uphill. This observation underlies our main technical lemma. We call a C^1 function f: $X \to \mathbf{R}$ a *global Lyapunov function* for the dynamical system $\dot{x} = V(x)$ if for every solution trajectory $\{x_t\}_{t \geq 0}$, (*i*) $\frac{d}{dt} f(x_t) \geq 0$ for all t, and (*ii*) $\frac{d}{dt} f(x_t) = 0$ implies that $V(x_t) = 0$. Condition (*i*) requires that the function f is weakly increasing along all solution trajectories, while condition (*ii*) demands that f is strictly increasing except at rest points of V. Lemma 4.1 establishes that the potential function is a global Lyapunov function under any dynamics satisfying positive correlation, providing a powerful tool for characterizing evolution.

---

13 For a proof, see Ely and Sandholm [9].

14 Friedman [11] considers *weak compatibility*, which combines positive correlation within each population with *extinction*: $x_i^p = 0$ implies that $V_i^p(x) = 0$. Swinkels [32] studies *myopic adjustment dynamics*, which satisfy positive correlation within each population, but with a weak inequality replacing the strict one and with the additional requirement that all Nash equilibria are rest points (although this latter requirement is omitted in some of his results). For other conditions on evolutionary game dynamics which are stronger than positive correlation, see Nachbar [24], Samuelson and Zhang [28], Ritzberger and Weibull [25], and Hofbauer and Weibull [17]. See Weibull [35] for a survey.

Lemma 4.1: *If F is a potential game and V satisfies* (PC), *then the potential function of F is a global Lyapunov function for* $\dot{x} = V(x)$.

Proof: Positive correlation implies that $\frac{d}{dt} f(x_t) = \nabla f(x_t) \cdot \dot{x}_t = F(x_t) \cdot V(x_t) \geq 0$ and that $V(x_t) = 0$ whenever $\frac{d}{dt} f(x_t) = 0$. ■

Positive correlation restricts the dynamics at all states besides rest points. Our next condition, *noncomplacency* (NC), specifies which states can be rest points.

(NC) $V(x) = 0$ implies that x is a Nash equilibrium of F.

If a profitable deviation is available, one should expect some players to eventually take advantage of this opportunity. Noncomplacency formalizes this notion.

An example of dynamics satisfying all four conditions are the *Brown-von Neumann-Nash* (*BNN*) *dynamics*, introduced for symmetric zero-sum games by Brown and von Neumann [6] and studied more recently by Berger and Hofbauer [2]. Let

$$k_i^p = \max\left\{ F_i^p(x) - \tfrac{1}{m^p} \sum_{j \in S^p} x_j^p \, F_j^p(x),\ 0 \right\}$$

denote the excess payoff to strategy i relative to the average payoff in its population. Then the BNN dynamics are defined by[15]

$$\text{(BNN)} \qquad \dot{x}_i^p = m^p k_i^p - x_i^p \sum_{j \in S^p} k_j^p .$$

Proposition 4.2: *The BNN dynamics satisfy* (LC), (FI), (PC), *and* (NC).

Proof: In the Appendix (or see Berger and Hofbauer [2]).

It is worth pointing out that the replicator dynamics do not satisfy

---

15 An interpretation of the BNN dynamics is as follows: During any short time interval, all players in a population are equally likely to switch strategies, and do so at a rate proportional to the sum of the excess payoffs in the population. Those who switch choose strategies with above average payoffs, choosing each with probability proportional to the strategy's excess payoff.

noncomplacency: these dynamics do not allow extinct strategies to resurface, and so exhibit non-Nash rest points on the boundaries of the state space. We feel that this property of the replicator dynamics is unreasonable in most economic contexts. However, as noncomplacency is not needed for all of our results, we will note explicitly when it is required.[16]

4.2 Evolutionary Stability

Our first stability result compares the rest points of the dynamics V to the Nash equilibria of the game F. At a Nash equilibrium, no agent can unilaterally improve his payoffs. Hence, escape from a Nash equilibrium violates positive correlation. All Nash equilibria are therefore rest points. Under the additional assumption of noncomplacency, only Nash equilibria can be rest points under V.

Proposition 4.3: (*i*) *If V satisfies* (PC), *all Nash equilibria of F are rest points of* $\dot{x} = V(x)$.

(*ii*) *If V also satisfies* (NC), *Nash equilibria of F and rest points of* $\dot{x} = V(x)$ *coincide*.

As the proof of this proposition does not depend on the existence of a potential function, the result is valid for any population game.

Proof: To prove part (*i*), let x be a Nash equilibrium of F, and let V be dynamics satisfying positive correlation. Suppose $D^p(x)$ is the set of strategies in S^p that are in decline at x: $D^p(x) = \{i \in S^p\colon\ V_i^p(x) < 0\}$. Then forward invariance and the definition of equilibrium imply that $D^p(x) \subset C^p(x^p) \subset \arg\max_j F_j^p(x)$. But forward invariance also implies that $\sum_{i \in S^p} V_i^p(x) = 0$; therefore, the inclusion implies that $\sum_{i \in S^p} V_i^p(x) F_i^p(x) \le 0$. Summing over p, we see that $\sum_{p \in P} \sum_{i \in S^p} V_i^p(x) F_i^p(x) \le 0$. Positive correlation then implies that $V(x) = 0$.

The proof of part (*ii*) follows immediately from part (*i*) and the definition of noncomplacency. ■

---

[16] Since all violations of (NC) under the replicator dynamics occur on the boundary X, versions of our results requiring both conditions hold under the replicator dynamics if attention is restricted to solution trajectories which avoid the boundary of X. The best response dynamics also fail to satisfy all four of our conditions: they fail the Lipschitz continuity condition (LC).

Since dynamics satisfying positive correlation ascend the potential function, it seems plausible that connections between its local maximizers and the game's locally stable equilibria exist. These connections are established in Theorem 4.4.

There are two main conditions used in evolutionary game theory to capture local stability. Roughly speaking, an equilibrium is *Lyapunov stable* if no small change in behavior can lead the population away from the equilibrium. The stronger criterion of *asymptotic stability* requires that in addition, the population eventually returns to equilibrium. Formal definitions of these criteria can be found in the Appendix.

The statement of Theorem 4.4 requires three additional definitions. A set $A \subset X$ is a *local maximizer set* of the potential function f if (*i*) A is connected; (*ii*) f is constant on A; and (*iii*) there exists a neighborhood B of A such that $f(y) < f(x)$ for all $y \in B - A$ and $x \in A$. Since f is continuous, all local maximizer sets are closed. Moreover, Proposition 3.1 implies that all local maximizer sets consist entirely of Nash equilibria. We call a closed set *isolated* if there is a neighborhood of the set containing no Nash equilibria outside the set. Finally, set $A \subset X$ is *smoothly connected* if for any points x and y in A there exists a continuous, piecewise differentiable curve γ contained in A whose endpoints are x and y.[17]

We now state our local stability result.

Theorem 4.4: *Let F be a potential game with potential function f. Then*:

(*i*) *If V satisfies* (PC), *then any local maximizer set is Lyapunov stable.*

(*ii*) *If V also satisfies* (NC), *then*

(*a*) *Any isolated local maximizer set is a minimal asymptotically stable set*;

(*b*) *Any smoothly connected, minimal asymptotically stable set is an isolated local maximizer set.*

Proof: In the Appendix.

Part (*i*) of the theorem tells us that under dynamics satisfying positive correlation, all local maximizers of potential are Lyapunov stable: small perturbations in behavior are not enough to move the population away from these sets. Without noncomplacency we cannot say more: since the population can become stuck at non-Nash states, local maximizer sets need not be asymptotically

---

[17] Smooth connectedness is a slightly stronger property than connectedness. For examples of sets which are connected but not smoothly connected, see Munkres [23, p. 156-158].

stable, and sets larger than local maximizers can be locally stable. However, part (*ii*) of the theorem shows that if we assume both positive correlation and noncomplacency, the local maximizer sets and the asymptotically stable sets coincide.

Local stability results are most important once a population reaches equilibrium, as they establish whether we should expect the equilibrium to persist. However, they do not guarantee that equilibrium will ever be reached. In general, evolutionary game dynamics can exhibit closed orbits and chaotic behavior, with solution trajectories perpetually avoiding neighborhoods of rest points.[18] When this occurs, equilibrium prediction is obviously inappropriate, and characterizations of local stability are of less interest.

Fortunately, we are able to establish that in potential games, convergence to equilibrium is assured. Let $\{x_t\}_{t\geq 0}$ be the solution trajectory with initial condition x_0. The *limit set* of x_0, $\omega(x_0)$, is the set of limit points of this solution trajectory: $\omega(x_0) = \{z \in X\text{: } \lim_{k\to\infty} x_{t_k} = z \text{ for some } t_k \to \infty\}$.

Theorem 4.5: *Let F be a potential game. Then*:

(*i*) *If V satisfies* (PC), *each limit set* $\omega(x)$ *is a closed*, *connected set of rest points of V*.

(*ii*) *If V also satisfies* (NC), *these limit sets only contain Nash equilibria.*

Proof: Follows from Lemma 4.1, Proposition 4.3, and Lemma A.1 in the Appendix. ■

The first claim of Theorem 4.5 establishes that under positive correlation, solution trajectories starting from each initial condition must converge to rest points: closed orbits and chaotic behavior cannot occur. If noncomplacency holds as well, all of these limit points must be Nash equilibria. This result fully justifies Nash equilibrium prediction.

---

[18] Examples of limit cycles under the replicator dynamics and under the best response dynamics can be found in Weibull [35] and Gaunersdorfer and Hofbauer [12], respectively. Cowan [7] analyzes an example in which fictitious play of a 3 x 3 game leads to chaotic behavior.

5. Efficiency

Nash equilibria often fail to be social optima: because players do not consider how their actions affect opponents' payoffs, equilibrium behavior is often inefficient. In homogenous potential games, individual and social payoffs are perfectly aligned; for this reason, self-interested choices lead to efficient play.

5.1 Homogenous Potential Games

A potential game F is *homogenous of degree* k if each of its payoff functions F_i^p: $\bar{X} \to \mathbf{R}$ is C^1 and homogenous of degree k; we assume throughout that $k \neq -1$. We will explain why this condition ensures efficient behavior after presenting Lemma 5.2 below; before doing so, we offer some examples.

5.1.1 Random Matching Games with Common Payoffs

We saw earlier that all random matching games with common payoffs are potential games; in fact, all are homogenous potential games. In the single population case, the payoffs to each strategy are linear in the population state x: $F_i(x) = \sum_{j \in S^2} U(i,j) x_j$. Therefore, such games are homogenous of degree 1. In the multiple population case, the payoffs to population p's strategies are multilinear in $(x^1, \ldots, x^{p-1}, x^{p+1}, \ldots, x^r)$; such games are homogenous of degree $r - 1$.

5.1.2 Isoelastic Congestion Games

Recall that the payoffs of congestion games take the form $F_i^p(x) = -\sum_{\phi \in \Phi_i^p} c_\phi(u_\phi(x))$, where the functions c_ϕ represent the costs of using each street. Let

$$\eta_\phi(u) = \frac{u c_\phi'(u)}{c_\phi(u)}$$

denote the cost elasticity of street ϕ, which is well defined whenever $c_\phi(u) \neq 0$. We call a congestion game *isoelastic with elasticity* η if $\eta_\phi \equiv \eta$ for all $\phi \in \Phi$: that is, if all streets are equally sensitive to congestion at all levels of use. This condition implies homogeneity.

Proposition 5.1: *Any isoelastic congestion game with elasticity η is a homogenous potential game of degree η.*

Proof: Since the street costs are isoelastic functions with elasticity η, they must take the form $c_\phi(u) = \alpha_\phi u^\eta$, where the α_ϕ are constants. (Observe that η cannot be negative, as this would force street costs to become infinite at $u = 0$.) Since each u_ϕ is linear in x, each payoff function F_i^p is a sum of functions which are homogenous of degree η in x, and so is itself homogenous of degree η. ■

5.2 Evolution and Efficiency

Lemma 5.2 provides the basis for our efficiency results. We measure efficiency in terms of aggregate payoffs $\bar{F}\colon \bar{X} \to \mathbf{R}$, defined by

$$\bar{F}(x) = \sum_{p \in P} \sum_{i \in S^p} x_i^p F_i^p(x).$$

Lemma 5.2: *If a potential game F is homogenous of degree k, the function $\frac{1}{k+1}\bar{F}(x)$ is a potential function for F.*

Proof: Externality symmetry and Euler's law imply that

$$\begin{aligned}
\frac{\partial}{\partial x_i^p}\left(\tfrac{1}{k+1}\bar{F}(x)\right) &= \tfrac{1}{k+1}\left(\sum_{q \in P} \sum_{j \in S^p} x_j^q \frac{\partial F_j^q}{\partial x_i^p}(x) + F_i^p(x)\right) \\
&= \tfrac{1}{k+1}\left(\sum_{q \in P} \sum_{j \in S^p} x_j^q \frac{\partial F_i^p}{\partial x_j^q}(x) + F_i^p(x)\right) \\
&= \tfrac{1}{k+1}\left(k F_i^p(x) + F_i^p(x)\right) = F_i^p(x). \ ■
\end{aligned}$$

To see why homogeneity leads to efficient play, consider the expression $\frac{\partial}{\partial x_i^p}\bar{F}(x)$, which represents the impact on aggregate payoffs of introducing a new player choosing strategy i to the game. We can split this impact into two terms: the first, $\sum_q \sum_j x_j^q \frac{\partial F_j^q}{\partial x_i^p}(x)$, represents the effect that the new player has on the incumbent population; the second, $F_i^p(x)$, represents the new player's own payoffs. In homogenous potential games, these two effects are precisely balanced: the payoff a player receives from choosing a strategy is directly proportional to the social impact of his choice. For this reason, self-interested behavior leads to desirable social outcomes.[19]

We saw in Section 4 that evolution always increases potential. But in homogenous potential games, potential measures aggregate payoffs. Hence, aggregate payoffs must increase over all evolutionary paths.[20]

Theorem 5.3: *If the potential game F is homogenous of degree* $k > -1$ *and the dynamics V satisfy* (PC), *then all solutions of* $\dot{x} = V(x)$ *satisfy* $\frac{d}{dt}\bar{F}(x_t) \geq 0$, *with equality only at rest points of V. If* $k < -1$, *then solutions satisfy* $\frac{d}{dt}\bar{F}(x_t) \leq 0$.

Proof: Follows from Lemmas 4.1 and 5.2. ■

Theorem 4.4 established connections between local maximizers of potential and locally stable equilibria. By introducing Lemma 5.2, we can link these states with the *locally efficient* states: those which locally maximize aggregate payoffs.

Theorem 5.4: *Let F be a potential game which is homogenous of degree* $k > -1$.

(*i*) *If V satisfies* (PC), *then all locally efficient states are locally stable.*

(*ii*) *If V also satisfies* (NC), *then all locally stable states are locally efficient.*

Proof: Follows from Theorem 4.4 and Lemma 5.2. ■

In general, homogeneity only ensures local stability: for example, coordination games with common payoffs admit multiple stable equilibria, but only those with the highest payoffs are globally efficient. However, if a game admits a unique equilibrium, this situation cannot arise. Theorem 5.5 shows that in homogenous potential games, the existence of a unique equilibrium ensures both global stability and global efficiency.

Theorem 5.5: *Let F be a potential game which is homogenous of degree* $k > -1$. *If F admits a unique Nash equilibrium, this equilibrium is globally efficient and is*

---

[19] We note that homogeneity is not a complementarity condition. For example, in congestion games homogeneity is consistent both with positive externalities (i.e., decreasing facility costs), which lead to multiple equilibria, and with negative externalities (increasing facility costs), which generate unique equilibria. For an analysis of the latter case, see Corollary 5.6.

[20] Fisher's [10] Fundamental Theorem of Natural Selection, a basic result from population genetics, is Theorem 5.3 specialized to single populations, the replicator dynamics, and linear payoffs.

Theorem 5.3 also shows that in homogenous potential games with $k < -1$, evolution *decreases* social efficiency. In these games, individual and social incentives are perfectly misaligned. However, note that in random matching games with common payoffs, $k \geq 1$, while in isoelastic congestion games, $k \geq 0$.

globally stable under all dynamics satisfying (PC) *and* (NC).

Proof: Since the potential function f is continuous and the set X is compact, the set $\operatorname{argmax}_{x \in X} f(x)$ is nonempty. Proposition 3.1 implies that the unique equilibrium of F must lie in this set. Global efficiency then follows from Lemma 5.2, and global stability from Theorem 4.5. ■

We conclude this section by applying our results from the last two sections to congestion games. To model traffic flows or other settings with negative externalities using congestion games, one assumes that the cost functions $c_\phi(u_\phi)$ are increasing in the utilization levels u_ϕ. This ensures uniqueness and global stability of equilibrium. If costs are also isoelastic, global efficiency is also guaranteed.[21]

Corollary 5.6: *Let F be a congestion game whose costs satisfy $c'_\phi > 0$, and suppose that V satisfies* (PC) *and* (NC). *Then F has a unique, globally stable equilibrium. If F is also isoleastic, this equilibrium is globally efficient.*

Proof: Recall that the potential function of the congestion game F is $f(x) = -\sum_{\phi \in \Phi} \int_0^{u_\phi(x)} c_\phi(z)\,dz$. Since street costs are strictly increasing, it is easily verified that f is strictly concave. Thus, uniqueness of equilibrium follows from Proposition 3.1, global stability from Theorem 4.5, and global efficiency from Theorem 5.5. ■

6. Potential Games as Limits of Finite Player Games

In this final section, we establish connections between infinite population potential games and the finite player potential games (*FPP games*) of Monderer and Shapley [22]. We prove that infinite population potential games are precisely the limits of convergent sequences of FPP games in which the number of players approaches infinity. For notational convenience, we restrict attention to the case of single population games; our results are easily extended to the multipopulation case.

---

[21] Beckmann, McGuire, and Winsten [1] prove a uniqueness result for their congestion model using a potential function argument. Dafermos and Sparrow [8] use different techniques to prove a global efficiency result for congestion games; their conditions on costs are somewhat stronger than those used in Corollary 5.6. Neither of these works proves evolutionary stability results.

An N player normal form game is defined by a strategy set S^α and a utility function U^α: $\prod_\alpha S^\alpha \to \mathbf{R}$ for each player $\alpha \in \{1, \dots, N\}$. We let $S^{-\alpha} = \prod_{\beta \neq \alpha} S^\beta$ denote the set of strategy profiles for α's opponents. Following Monderer and Shapley [22], we call a game a *finite player potential game* if there exists a potential function U: $\prod_\alpha S^\alpha \to \mathbf{R}$ such that

$$U^\alpha(\hat{i}^\alpha, i^{-\alpha}) - U^\alpha(i^\alpha, i^{-\alpha}) = U(\hat{i}^\alpha, i^{-\alpha}) - U(i^\alpha, i^{-\alpha})$$

for all $\hat{i}^\alpha$, $i^\alpha \in S^\alpha$, $i^{-\alpha} \in S^{-\alpha}$, and $\alpha \in \{1, \dots, N\}$. That is, any unilateral deviation has the same effect on both the deviator's payoffs and potential. Thus, the potential function serves as proxy for each player's payoff function when the strategies of his opponents are held fixed. It is easily verified that FPP games can be characterized as the class of games which admit the representation

$$U^\alpha(i^\alpha, i^{-\alpha}) = U(i^\alpha, i^{-\alpha}) + A^\alpha(i^{-\alpha}),$$

where for each player α, A^α is a function from opponents' strategy profiles $S^{-\alpha}$ to the real line.

To make sense of the notion of a convergent sequence of FPP games, we restrict attention to games in which players are identical and anonymous. In particular, we assume that all players share the same strategy set S and payoff functions $\{u_i\}_{i \in S}$, and that the payoff and potential functions only condition on the population's aggregate behavior. We call FPP games which satisfy these requirements *anonymous finite player potential games* (*AFP games*).

Payoff functions of AFP games must take the form

$$u_i(x) = P(x) + a(x - \tfrac{e_i}{N})$$

where P denotes the potential function, x the current strategy distribution, and e_i a basis vector in $\mathbf{R}^n$. The domains of the functions P, u_i, and a are X^N, X_i^N, and X_d^N, respectively, where

$$X^N = \{x \in \mathbf{R}^n_+ : \textstyle\sum_j x_j = 1, \text{ and } Nx_j \in \mathbf{Z} \text{ for all } j \in S\},$$
$$X_i^N = \{x \in X^N : x_i \neq 0\}, \text{ and}$$
$$X_d^N = \{x \in \mathbf{R}^n_+ : \textstyle\sum_j x_j = \tfrac{N-1}{N}, \text{ and } Nx_j \in \mathbf{Z} \text{ for all } j \in S\}.$$

The set X^N contains all possible strategy distributions when the population size is N, while X_i^N is the set of strategy distributions in which at least one player chooses strategy i. We call X_d^N the set of *diminished strategy distributions*; points in this set represent distributions of strategies in a subpopulation with one absent player.

The connection between AFP games and infinite player potential games becomes clearer if we represent the former in a slightly different way. Without loss of generality, we can extend the domain of the potential function P from X^N to $X^N \cup X_d^N$ by defining $P(y) = -a(y)$ for all $y \in X_d^N$. We can then express payoffs as

$$u_i(x) = P(x) - P(x - \tfrac{e_i}{N}).$$

In this representation, the extended potential function summarizes all information about payoffs. The payoff to a player choosing strategy i when the strategy distribution is x is the difference between the values of the potential function at two points: the current distribution x, and the diminished distribution $x - \frac{e_i}{N}$ which would arise if the player left the population.

This suggests that in the infinite population limit, the payoffs to strategy i should be related to the partial derivative of P with respect to i. The first step in formalizing this intuition is to define a notion of convergence. We say that a sequence of AFP games $\left\{\{u_i^N\}_{i\in S}, P^N\right\}_{N=N_0}^{\infty}$ *converges* if there exists a C^1 function P: $\overline{X} \to \mathbf{R}$ and a vanishing sequence of real numbers $\{K^N\}_{N=N_0}^{\infty}$ such that

$$\text{(C)} \qquad \left|(\tfrac{1}{N}P^N(x) - P(x)) - (\tfrac{1}{N}P^N(y) - P(y))\right| \le K^N \left\|x - y\right\|$$

for all x and y in $X^N \cup X_d^N$ and all N. In words: for each N, we require the difference between $\frac{1}{N}P^N$ and P to be a Lipschitz continuous function; the Lipschitz constant K^N must vanish as N grows large.

To understand this condition, first observe that in the finite player games, the potential functions have magnitudes of order N; for this reason, we consider the rescaled potential functions $\frac{1}{N}P^N$. Second, notice that potential functions are only unique up to an additive constant. Since $\overline{X}$ is bounded, condition (C) implies that the functions $\frac{1}{N}P^N(x) - c^N \equiv \frac{1}{N}P^N(x) - (\frac{1}{N}P^N(e_1) - P(e_1))$ converge uniformly to $P(x)$. Finally, condition (C) requires that for each fixed N, the difference $\frac{1}{N}P^N(x) - P(x)$ varies little when x is changed by a small amount; the larger is N, the less the difference may vary.[22]

If the potential functions of a sequence of AFP games converge, their payoff functions also converge. Furthermore, the collection $\{\{u_i\}_{i\in S}, P\}$ defined by the various limits is an infinite player potential game. This is the content of our convergence theorem.

Theorem 6.1: *Let* $\left\{\{u_i^N\}_{i\in S}, P^N\right\}_{N=N_0}^{\infty}$ *be a convergent sequence of AFP games with limit potential function P. Then*

(*i*) *The sequences of payoff functions are uniformly convergent: for each i* $\in S$ *there exists a* C^0 *function* u_i: $\overline{X} \to \mathbf{R}$ *such that*

$$\lim_{N\to\infty} \sup_{x\in X_i^N} \left|u_i^N(x) - u_i(x)\right| = 0.$$

(*ii*) *The limit payoff functions and potential function define an infinite player potential game. That is,*

$$u_i(x) = \tfrac{\partial P}{\partial x_i}(x) \qquad \textit{for all } x \in \overline{X} \textit{ and } i \in S.$$

Proof: Fix $i \in S$ and $x \in X_i^N$. By the Mean Value Theorem,

$$P(x) - P\left(x - \tfrac{e_i}{N}\right) = \tfrac{1}{N}\tfrac{\partial P}{\partial x_i}(z_i^N(x))$$

for some $z_i^N(x)$ on the linear segment connecting x and $x - \frac{e_i}{N}$. Therefore, equation (C) implies that

$$\begin{aligned}
\left|u_i^N(x) - \tfrac{\partial P}{\partial x_i}(x)\right| &= \left|(P^N(x) - P^N(x - \tfrac{e_i}{N})) - \tfrac{\partial P}{\partial x_i}(x)\right| \\
&= \left|(P^N(x) - P^N(x - \tfrac{e_i}{N})) - N(P(x) - P(x - \tfrac{e_i}{N}))\right| + \left|\tfrac{\partial P}{\partial x_i}(z_i^N(x)) - \tfrac{\partial P}{\partial x_i}(x)\right| \\
&= N\left|(\tfrac{1}{N}P^N(x) - P(x)) - (\tfrac{1}{N}P^N(x - \tfrac{e_i}{N}) - P(x - \tfrac{e_i}{N}))\right| + \left|\tfrac{\partial P}{\partial x_i}(z_i^N(x)) - \tfrac{\partial P}{\partial x_i}(x)\right| \\
&\le K^N + \left|\tfrac{\partial P}{\partial x_i}(z_i^N(x)) - \tfrac{\partial P}{\partial x_i}(x)\right|.
\end{aligned}$$

Since K^N vanishes, and since P is C^1 on the compact set $\overline{X}$, we conclude that

$$\lim_{N\to\infty} \sup_{x\in X_i^N} \left|u_i^N(x) - \tfrac{\partial P}{\partial x_i}(x)\right| = 0. \blacksquare$$

22 For an example of a convergent sequence of AFP games, see the end of Section 3.

By restricting attention to games in which players are anonymous and by choosing an appropriate representation for these games, we are able to establish a basic link between potential games with finite and infinite populations. Theorem 6.1 shows that the condition which defines infinite population potential games is the limiting version of the conditions defining AFP games. Together with our stability results, the convergence theorem shows that in settings in which players are anonymous, the choice between the finite and infinite player models is a matter of analytical convenience.

Appendix

A.1 Evolutionary Dynamics: Definitions and Auxiliary Results

Most of the definitions we require are stated in Section 4; some definitions which were omitted are provided here. A *neighborhood* of a closed set $A \subset X$ is a set which is open relative to X and contains A. A closed set $A \subset X$ is *Lyapunov stable* if every neighborhood B of A contains a neighborhood B' of A such that every solution trajectory starting in B' never leaves B. In other words, solutions starting at all points sufficiently close to A always remain close to A.

Let $\{x_t\}_{t\geq 0}$ denote the solution trajectory of $\dot{x} = V(x)$ with initial condition x_0. The *limit set* of x_0, $\omega(x_0)$, is the set of accumulation points of this solution trajectory: $\omega(x_0) = \{z \in X\text{: } \lim_{k\to\infty} x_{t_k} = z \text{ for some } t_k \to \infty\}$. A closed set of $A \subset X$ is *asymptotically stable* if it is Lyapunov stable and there exists a neighborhood B of A such that $\omega(x) \subset A$ for all $x \in B$: in other words, solutions starting from all points close enough to A remain nearby and eventually converge to A. The existence of a global Lyapunov function allows a strong characterization of the limit behavior of a dynamical system, as the following lemma shows. The lemma combines results found in Losert and Akin [19, Proposition 1] and Robinson [26, Theorem 5.4.1].

Lemma A.1: *If f: $X \to \mathbf{R}$ is a global Lyapunov function for $\dot{x} = V(x)$, then each limit set $\omega(x)$ is a non-empty, compact, and connected set consisting entirely of rest points of V and upon which f is constant.*

Global Lyapunov functions can also be used to establish Lyapunov stability. The following lemma follows from Theorem 6.4 of Weibull [35].

Lemma A.2: *If f: $X \to \mathbf{R}$ is a global Lyapunov function for $\dot{x} = V(x)$ and A is a local maximizer set of f, then A is Lyapunov stable.*

To prove asymptotic stability results, we need slightly stronger conditions. We call a C^1 function f: $X \to \mathbf{R}$ a *strict local Lyapunov function* for the set A under $\dot{x} = V(x)$ if (*i*) A is a local maximizer set of f, and (*ii*) there exists a neighborhood B of A such that $\frac{d}{dt} f(x_t) > 0$ whenever $x \in B - A$. The existence of a strict local Lyapunov function for the set A implies its asymptotic stability. This result also follows from Theorem 6.4 of Weibull [35].

Lemma A.3: *If f: $X \to \mathbf{R}$ is a strict local Lyapunov function for A, then A is asymptotically stable.*

A.2 Proofs Omitted from the Text

Proof of Proposition 4.2:

We present the proof of the case in which there is a single population of mass one; the proof of the general case is a straightforward extension. It is easily verified that (BNN) satisfies (LC) and (FI). To check (PC), suppose that x is not a rest point of (BNN): $\dot{x} = V(x) \neq 0$. It follows that $k_i > 0$ for some strategy $i \in S^1$. Hence, letting $\hat{S} = \{ i \in S^1 \colon k_i > 0\}$, we find that

$$\begin{aligned}
\sum_{i \in S^1} V_i(x) F_i(x) &= \sum_{i \in S^1} k_i F_i(x) - \sum_{i \in S^1} x_i F_i(x) \Big(\sum_{j \in S^1} k_j\Big) \\
&= \sum_{i \in \hat{S}} (F_i(x) - \bar{F}(x)) F_i(x) - \bar{F}(x) \sum_{j \in \hat{S}} (F_j(x) - \bar{F}(x)) \\
&= \sum_{i \in \hat{S}} (F_i(x) - \bar{F}(x))^2 \\
&= \sum_{i \in \hat{S}} (k_i)^2 > 0.
\end{aligned}$$

Thus, property (PC) holds.

To establish (NC), suppose that x is not a Nash equilibrium. Then there is a strategy $i \in S^1$ such that $x_i > 0$ and $F_i(x) < \max_{j \in S^1} F_j(x)$. Let strategy $h \in S^1$ satisfy $F_h(x) = \max_{j \in S^1} F_j(x)$. Then it is clear that $F_h(x) > \bar{F}(x)$, and so $k_h > 0$. Thus, if $x_h = 0$, then $\dot{x}_h = k_h > 0$, so x is not a rest point. On the other hand, if $x_h > 0$, then since $F_h(x) > \bar{F}(x)$, there must be a strategy $l \in S^1$ such that $x_l > 0$ and $F_l(x) < \bar{F}(x)$. But then $k_l = 0$, and so $\dot{x}_l < 0$; hence, x is not a rest point. This establishes property (NC), completing the proof. ■

Proof of Theorem 4.3:

Part (i) of the theorem follows immediately from Lemma 4.1 and Lemma A.2. To prove part (ii)(a), let A be an isolated local maximizer set. Since A is isolated, there is a neighborhood B of A such that all Nash equilibria in B are in A. Hence, noncomplacency implies that $V(x) \neq 0$ for all $x \in B - A$, and so positive correlation implies that $\frac{d}{dt} f(x_t) = F(x_t) \cdot V(x_t) > 0$ whenever $x_t \in B - A$. Thus, f is a strict local Lyapunov function for A. Lemma A.3 then implies that A is asymptotically stable. Moreover, since A consists entirely of Nash equilibria, noncomplacency implies that no strict subset of A can be asymptotically stable; therefore, A is a minimal asymptotically stable set.

To prove part (ii)(b), suppose that A is a minimal asymptotically stable set which is smoothly connected. Let A' be the set of points in A which are Nash equilibria. Because A is asymptotically stable, there is a neighborhood B of A such that all trajectories starting in B converge to A. By Lemma A.1 and Proposition 4.3 (ii), the limit sets of these trajectories must be contained in A'. Therefore, A' is non-empty and asymptotically stable; it is closed by definition. But since A is a minimal asymptotically stable set, we conclude that $A = A'$. Thus, the set A consists entirely of Nash equilibria.

We continue with a lemma:

Lemma A.4: *The potential function f is constant on any smoothly connected set of Nash equilibria of F.*

Proof: Let A be a smoothly connected set of Nash equilibria of F, and let y and z be elements of A. Then there exists a piecewise smooth function γ: $[0, 1] \to A \subset X$ with $\gamma(0) = y$ and $\gamma(1) = z$. Since

$$f(z) - f(y) = \int_\gamma \nabla f \cdot d\check{x} = \int_0^1 \big(F(\gamma(t)) \cdot \gamma'(t)\big)\, dt = \sum_p \int_0^1 \big(F^p(\gamma(t)) \cdot (\gamma^p)'(t)\big)\, dt,$$

it is sufficient to show that for each $p \in P$, the integrand $F^p(\gamma(t)) \cdot (\gamma^p)'(t)$ equals zero at all points at which γ is differentiable.

Fix $p \in P$, and let t be a point of differentiability of γ. Observe that if $\gamma_i^p(t) = 0$, then $(\gamma_i^p)'(t) = 0$; otherwise, the path γ would leave X at time t. On the other hand, if $\gamma_i^p(t) \neq 0$, then since $\gamma(t)$ is a Nash equilibrium, $F_i^p(\gamma(t)) = \mu^p(\gamma(t)) \equiv \max_{j \in S^p} F_j^p(\gamma(t))$. Hence,

$$
\begin{aligned}
F^p(\gamma(t))\cdot(\gamma^p)'(t) &= \sum_{i\in C^p(\gamma^p(t))}\Big(F_i^p(\gamma(t))\cdot(\gamma_i^p)'(t)\Big)\\
&= \sum_{i\in C^p(\gamma^p(t))}\Big(\mu^p(\gamma(t))\cdot(\gamma_i^p)'(t)\Big)\\
&= \mu^p(\gamma(t))(\check{1}\cdot(\gamma^p)'(t)),
\end{aligned}
$$

where $\check{1} = (1, \ldots, 1)$. But since γ stays in X, $\check{1}\cdot\gamma^p(t) \equiv m^p$, so differentiating with respect to t yields $\check{1}\cdot(\gamma^p)'(t) = 0$. Therefore, $F^p(\gamma(t))\cdot(\gamma^p)'(t) = 0$. ❑

Now let B be the basin of attraction for A described above. By definition, $B - A$ contains no rest points, and hence (by Proposition 4.3 (*i*)) no Nash equilibria. Hence, A is isolated. By Lemma A.4, f takes a unique value on A. Call this value c. Let x_0 be an arbitrary point in $B - A$, and let $\{x_t\}_{t\geq 0}$ be the solution trajectory starting from x_0. By the definition of B, $\omega(x_0) \subset A$, so $\lim_{t\to\infty} f(x_t) = c$. Because f is a strict local Lyapunov function for the set A, it follows that $f(x_0) < c$. Since A is connected and f is constant on A, we conclude that A is a local maximizer set. ■

References

1. M. Beckmann, C. B. McGuire, and C. B. Winsten, "Studies in the Economics of Transportation," Yale University Press, New Haven, 1956.
2. U. Berger and J. Hofbauer, "The Nash Dynamics," mimeo, Universität Wien, 2000.
3. L. E. Blume, The statistical mechanics of strategic interaction, *Games Econ. Behav.* **5** (1993), 387-424.
4. L. E. Blume, Population games, *in* "The Economy as an Evolving Complex System II," (W. B. Arthur, S. N. Durlauf, and D. A. Lane, Eds.), Addison-Wesley, Reading, MA, 1997.
5. L. E. Blume, "How Noise Matters," mimeo, Cornell University, 1999.
6. G. W. Brown and J. von Neumann, Solutions of games by differential equations, *in* "Contributions to the Theory of Games I," (H. W. Kuhn and A. W. Tucker, Eds.), Annals of Mathematics Studies 24, Princeton University Press Princeton, 1950.
7. S. Cowan, "Dynamical Systems Arising from Game Theory," Ph.D. dissertation, University of California, Berkeley, 1992.
8. S. Dafermos and F. T. Sparrow, The traffic assignment problem for a general

network, *J. Res. Nat. Bur. Standards B* **73** (1969), 91-118.

9. J. C. Ely and W. H. Sandholm, "Evolution with Diverse Preferences," SSRI Working Paper #2005, University of Wisconsin, 2000.
10. R. A. Fisher, "The Genetical Theory of Natural Selection," Clarendon Press, Oxford, 1930.
11. D. Friedman, Evolutionary games in economics, *Econometrica* **59** (1991), 637-666.
12. A. Gaunersdorfer and J. Hofbauer, Fictitious play, Shapley polygons, and the replicator equation, *Games Econ. Behav.* **11** (1995), 279-303.
13. I. Gilboa and A. Matsui, Social stability and equilibrium, *Econometrica* **59** (1991), 859-867.
14. J. Hofbauer, "Stability for the Best Response Dynamics," mimeo, Universität Wien, 1995.
15. J. Hofbauer and K. Sigmund, "Theory of Evolution and Dynamical Systems," Cambridge University Press, Cambridge, 1988.
16. J. Hofbauer and G. Sorger, Perfect foresight and equilibrium selection in symmetric potential games, *J. Econ. Theory* **85** (1999), 1-23.
17. J. Hofbauer and J. W. Weibull, Evolutionary selection against dominated strategies, *J. Econ. Theory* **71** (1996), 558-573.
18. M. Kimura, On the change of population fitness by natural selection. *Heredity* **12** (1958), 145-167.
19. V. Losert and E. Akin, Dynamics of games and genes: Discrete versus continuous time, *J. Math. Biology* **17** (1983), 241-251.
20. A. Matsui, Best response dynamics and socially stable strategies, *J. Econ. Theory* **57** (1992), 343-362.
21. A. Matsui and K. Matsuyama, An approach to equilibrium selection, *J. Econ. Theory* **65** (1995), 415-434.
22. D. Monderer and L. S. Shapley, Potential games, *Games Econ. Behav.* **14** (1996), 124-143.
23. J. R. Munkres, "Topology: A First Course," Prentice-Hall, Englewood Cliffs, NJ, 1975.
24. J. H. Nachbar, 'Evolutionary' selection dynamics in games: Convergence and limit properties, *Int. J. Game Theory* **19** (1990), 59-89.
25. K. Ritzberger and J. W. Weibull, Evolutionary selection in normal form games, *Econometrica* **63** (1995), 1371-1399.
26. C. Robinson, "Dynamical Systems," CRC Press, Boca Raton, FL, 1995.

27. R. W. Rosenthal, A class of games possessing pure strategy Nash equilibria, *Int. J. Game Theory* **2** (1973), 65-67.
28. L. Samuelson and J. Zhang, Evolutionary stability in asymmetric games, *J. Econ. Theory* **57** (1992), 363-391.
29. W. H. Sandholm, "Evolutionary Justification of Nash Equilibrium," Ph.D. dissertation, Northwestern University, 1998.
30. W. H. Sandholm, "Evolutionary Implementation and Congestion Pricing," SSRI Working Paper #9938, University of Wisconsin, 1999.
31. S. Shahshahani, A new mathematical framework for the study of linkage and selection, *Memoirs AMS* **211** (1979).
32. J. M. Swinkels, Adjustment dynamics and rational play in games, *Games Econ. Behav*. **5** (1993), 455-484.
33. P. Taylor and L. Jonker, Evolutionary stable strategies and game dynamics, *Math. Biosci.* **40** (1978), 145-156.
34. M. Voorneveld, P. Borm, F. van Megen, S. Tijs, and G. Facchini, Congestion games and potentials reconsidered, *Int. Game Theory Rev*. **1** (1999), 283-299.
35. J. Weibull, "Evolutionary Game Theory," MIT Press, Cambridge, 1995.