Evolution, Population Growth, and History Dependence[*]

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Abstract

We consider an evolutionary model with mutations which incorporates stochastic population growth. We provide a complete characterization of the effects of population growth on the evolution of play. In particular, we show that if the rate of population growth is at least logarithmic, the stochastic process describing play converges: only one equilibrium will be played from a certain point forward. If in addition the rate of mutation is taken to zero, the probability that the equilibrium selected is the first equilibrium played approaches one. Thus, population growth generates history dependence: the contingency of equilibrium selection on historical conditions.

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1. Introduction

Kandori, Mailath, and Rob (1993) introduce their model of stochastic evolution by describing two major difficulties of non-cooperative game theory. First, Nash equilibrium requires more stringent rationality and knowledge assumptions than are typically natural to assume, raising questions about why we should expect Nash equilibria to be played. Second, many games exhibit multiple equilibria. In such cases, unless a compelling justification of why players coordinate on a specific equilibrium exists, the predictive content of the equilibrium concept is cast into doubt. The program of equilibrium refinements has not overcome these difficulties. In most cases, refinements do not generate unique predictions. Moreover, as Samuelson (1993) notes, different refinements yield different predictions, transforming the problem of multiple equilibria into a problem of multiple refinements.

Recent work in evolutionary game theory has engendered hope of addressing these issues. Evolutionary models need not assume that players are particularly rational, and are able to study whether boundedly rational players can learn to play a Nash equilibrium as a group. Moreover, multiplicity of equilibria has a natural interpretation as a multiplicity of predictions: uniqueness is not required to justify coordination, which occurs as a consequence of the evolutionary process itself.

Using only weak rationality and knowledge assumptions, Kandori, Mailath, and Rob (1993, henceforth KMR) show that populations of players can learn to play Nash equilibria, and give unique predictions about the equilibrium which will be played.[1] A fixed population of players is repeatedly randomly matched to play a 2 x 2 symmetric game. A deterministic dynamic describes the movement of the population towards the selection of strategies that have performed well in the past. This dynamic is perturbed by introducing small, independent probabilities of mutation by each player. The evolution of play is thus represented by a Markov chain on the state space of distributions of strategy choices of the members of the population. For any positive rate of mutation, this Markov chain has a unique stationary distribution which is also the ergodic distribution of the process: the long run proportion of time spent in each state. KMR consider the *long run equilibria* of the game, which they define to be the states that receive positive weight in the stationary distribution when the rate of mutation is taken to zero. Their main result is that in coordination games, all long run equilibria entail coordination on the risk dominant strategy.

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[1] Other seminal models in this stream of the literature include Foster and Young (1990) and Young (1993).

In this paper, we extend the KMR model by introducing stochastic population growth. While most work in evolutionary economics has focused on the behavior of a fixed group of agents, it seems quite natural to consider the effects of alterations in the membership of the population itself. Such alterations play a central role in biological evolutionary game theory. The replicator dynamic (and, implicitly, the notion of evolutionary stability) is driven entirely by population adjustment: the payoffs of the underlying game represent the reproductive fitnesses of the strategies, and the birth and death rates of individuals programmed to play each strategy determine the course of evolution. Evolutionary economic models have focused on the effects of myopic strategy adjustments in fixed populations. However, in analogue with the biological models, it is reasonable to allow economic evolution to be influenced by entry and exit, which takes place in accordance with the benefits of membership in the population.

As an example, consider consumer technology choice, a fit candidate for an evolutionary economic analysis.[2] When a new technology becomes available, consumers only slowly avail themselves of its potential; it may be a matter of years after its introduction before the technology becomes established to its fullest degree. Consequently, it is natural to model technology choice using a small initial population which grows over time, and in particular to ask whether the behavior of the original agents has a disproportionate influence on the ultimate course of events.

In the sequel, we provide a complete characterization of the effects of population growth on the evolutionary process. Under slow enough rates of population growth, the equilibrium selection results of KMR are strengthened: the limiting distribution of the evolutionary process puts all weight on the risk dominant equilibrium, even when the rate of mutation is positive. For a small intermediate range of growth rates, this equilibrium selection result is strengthened further: with probability one, the population settles upon the risk dominant equilibrium, playing it exclusively from a certain time forward. In contrast, under faster rates of population growth, the evolutionary process fails to select the risk dominant equilibrium. The population eventually settles upon an equilibrium, and each pure strategy equilibrium is this limit with positive probability. More importantly, as the rate of mutation approaches zero, the probability that the population coordinates in all periods on the equilibrium in

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[2] Models of consumer technology choice involve large populations of agents who must select among a variety of options for satisfying a technological need. Each option exhibits network externalities: consumers prefer options that are used by more of their fellows. In this setting, it is sensible to model consumers as myopic decision makers who base their choices on the population shares using each available option. See Kandori and Rob (1993) for an evolutionary model of technology choice. For examples and discussion of network externalities, see Katz and Shapiro (1985, 1986).

whose basin of attraction play began approaches one. Hence, the long run behavior of the evolutionary process depends entirely on its initial conditions.

Surprisingly, the critical rates of population growth at which the character of evolution changes are logarithmic. Logarithmic growth is extremely slow. As this slow rate of growth is a sufficient condition for history dependence to occur, we conclude that in evolutionary situations involving population growth, dependence of equilibrium selection on historical conditions should be expected.

KMR's model of evolution suggests that in the long run, initial conditions do not matter. In contrast, our model indicates that only initial conditions matter. Thus, rather than depending on underlying strategic considerations, the evolution of a convention may often be a consequence of a precedent set by a small but fundamental advance guard. A convention need not be optimal, nor even secure against risk; it often simply needs to be established first.

The intuition behind our results can be explained as follows. In both KMR's model and our model, deterministic dynamics divide a discrete analogue of mixed strategy space into two basins of attraction, one for each pure strategy equilibrium. Coordination on an equilibrium breaks down if enough mutations occur to cause the population to jump between basins of attraction. In KMR, since the number of players is fixed, the probability of simultaneous mutations by a fraction of the population large enough to break coordination, while quite small, is fixed. Thus, such events will occur an infinite number of times and generate the ergodic behavior which drives KMR's results. However, when the population is growing, the probability of a shock great enough to disturb coordination decreases over time. If this probability falls fast enough, we are assured that from some point onward, no switches occur. Logarithmic growth is both necessary and sufficient for this conclusion. If in addition the rate of mutation is taken to zero, the probability that simultaneous mutations will disrupt coordination in any finite time span vanishes. Consequently, we are able to show that logarithmic growth together with arbitrarily small mutation rates yield complete dependence on initial conditions.

In an effort to keep the model as simple as possible we study 2 x 2 symmetric coordination games; however, our results are actually quite general. In particular, our main result can be extended to any strict Nash equilibrium of any $n \times n$ symmetric game.

KMR has been criticized on the grounds that its predictions, which are based on asymptotic properties of the evolutionary process, require inordinately long times to gain economic relevance. When the underlying contest is a coordination game, the waiting time to jump between equilibria is roughly ε^{-N}, where N is the population size and ε the rate of mutation. Since evolutionary models are most naturally applied to large populations, and since in the analysis the rate of mutation is taken to zero, the predictive power of the limiting stationary distribution is called into question.[3] While our results may appear to be a restatement of this critique, they differ from it in two fundamental ways. While the waiting time critique strains the interpretation of the equilibrium selection results of KMR, the results themselves remain correct. In our model, the equilibrium selection results fail to hold. On the other hand, we show that KMR's equilibrium selection results still obtain, and are in fact strengthened, under sublogarithmic growth. It is therefore clear that the unboundedness of the population size alone does not drive convergence in our model: the rate of population growth itself plays a crucial role.

The formal results of stochastic evolutionary models describe a population's behavior in the infinite time horizon. In economic applications of these models, the relevant time span is bounded; the limiting results should be interpreted as an approximate characterization of behavior after some finite span. In our growing population model, both the time span and the population size are unbounded; again, the results should be interpreted as an approximate characterization of a bounded model. We investigate the implications of our results in bounded settings in Section 4.2. Fixing a positive rate of mutation, we show that for any finite time span, the population size required to virtually guarantee history dependence over that span is logarithmic in the length of the span. Unless the relevant time horizon is exceptionally long, this requirement on the population size is quite weak. While this finding is worthy of mention on its own, it also implies a stringent necessary condition for KMR's fixed population equilibrium selection results to be applicable over finite periods of time: unless the time span of interest is more than exponentially greater than the size of population, history dependence should be expected.

Binmore, Samuelson, and Vaughan (1995) study the effects of varying the order in which limits are taken in stochastic evolutionary models. They find that if the limit as time tends to infinity is taken before the population limit, equilibrium selection results like those of KMR are generated, while if the population limit is taken first, then the

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[3] KMR, noting these difficulties in interpretation, suggest that their model is best applied to small populations. Nevertheless, an evolutionary economic analysis seems most relevant when the number of players is large, as this appears necessary to justify the implicit anonymity assumption and the myopic decision criteria used by the players. See Ellison (1993, 1995), Binmore, Samuelson, and Vaughan (1995), and Binmore and Samuelson (1997) for further discussion of waiting times in stochastic evolutionary models.

evolution of play closely approximates a deterministic dynamic (in their model, the replicator dynamic) for arbitrarily long finite periods of time. The authors conclude that because of the waiting times necessary for the equilibrium selection results to become meaningful, modelers should expect that in most instances, the deterministic dynamic yields the relevant prediction. The model presented here can be viewed as an attempt to refine the analysis of Binmore, Samuelson, and Vaughan (1995) by taking the time and population limits simultaneously. Our results reinforce the assessment of these authors that in most evolutionary settings, deterministic dynamics provide more realistic predictions than analyses of stochastic stability.

In independently conceived work, Robles (1995) obtains similar results to those presented here by allowing declining rates of mutation. In the original KMR model, stationary distributions are determined for each fixed rate of mutation, and the equilibrium concept used concerns the limit of the stationary distributions as the rate of mutation tends to zero. In contrast, Robles (1995) allows the rate of mutation to decrease as time tends to infinity. Necessary and sufficient conditions are given for the existence of a limiting distribution of the evolutionary process which is independent of the initial state. The analysis is similar to our own, as, roughly speaking, both sets of results are also driven by decreasing the ratio of probability of mutation to population size. In Section 4.1, we show how our results can be extended to the case of declining mutation rates, and provide a tighter characterization of the limit behavior in this setting than does Robles (1995)[4,5]

We view our model as an effort towards generating more realistic predictions of the behavior of large populations. KMR and much of the literature which it has spawned have focused on evolutionary models yielding unique predictions. While uniqueness of

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[4] For complementary work focusing on mutation in evolutionary models, see Bergin and Lipman (1996) and Blume (1994).

[5] Ellison (1993) models economic evolution under the assumption that players are only matched against opponents who are nearby with respect to some neighborhood structure. He shows that in coordination games, the risk dominant equilibria remain the long run equilibria, but that the waiting times to reach these equilibria are dramatically smaller than in the KMR model. If one adds population growth but fixes the neighborhood size, it seems likely that the risk dominant equilibrium would continue to be selected, as the critical number of mutations needed to disturb coordination on the risk dominated equilibrium stays fixed. The predictions of KMR would thus appear to be more robust to population growth in specific cases in which a local interaction structure exists. For related work on local interaction models, see Blume (1993, 1995) and Ely (1995).

In the stochastic evolutionary model of Young (1993), as in that of KMR, the risk dominant equilibrium is selected in 2 x 2 coordination games. However, the two models are in other respects quite different. In Young's (1993) model, a representative from each of n populations is chosen in each period to play an n player normal form game. These representatives select best responses to incomplete memories of plays by past representatives. The sizes of the populations are irrelevant in this model; hence, population growth would have no effect.

equilibrium is obviously desirable in the setting of one-shot simultaneous move games, we feel that such a conclusion is not always sensible when we consider actual evolutionary settings. It is natural to expect that the evolution of the behavior of large groups of players will often depend on historical conditions. This necessarily implies non-unique predictions, since different initial conditions can generate different outcomes. Therefore, allowing for the possibility of multiple equilibria, although less satisfactory from the perspective of "traditional" economic analysis, can often provide a more realistic analysis of the behavior of large populations.

2. The Model

Players are repeatedly matched to play the symmetric 2 x 2 game in Figure 1.

	s^1	s^2
s^1	a, a	b, c
s^2	c, b	d, d

Figure 1

We identify 2 x 2 symmetric games with vectors $G = (a, b, c, d) \in \mathbf{R}^4$. We restrict our attention to coordination games; these are games satisfying $a > c$ and $d > b$. Let x^* be the proportion of players selecting strategy s^1 in the symmetric mixed strategy equilibrium of G:

$$x^* = \frac{d-b}{(a-c)+(d-b)}.$$

We assume without loss of generality that $x^* \geq \frac{1}{2}$, so that strategy s^2 is risk dominant.

Following KMR, we first consider a fixed population of n players who are repeatedly randomly matched to play the game G. The distribution of strategies in the population is an element of $Z_n = \{0, 1, 2, \dots, n\}$, representing the number of players who are currently using strategy s^1. States 0 and n in Z_n will be called the *unanimous* states; these are the states towards which the evolutionary process gravitates.

Since players are never randomly matched against themselves, expected payoffs are given by

$$\pi^1(z,n) = \frac{(z-1)}{(n-1)}a + \frac{(n-z)}{(n-1)}b \qquad \text{for } z \in \{1, 2, \ldots, n\},$$
$$\pi^2(z,n) = \frac{z}{(n-1)}c + \frac{(n-z-1)}{(n-1)}d \qquad \text{for } z \in \{0, 1, \ldots, n-1\}.$$

We assume *clever myopic adjustment*: players evaluate their strategies by comparing their current expected payoffs to their expected payoffs if they were to switch strategies, given that the remaining players do not switch strategies. That is, players selecting s^1 want to switch at state z if and only if $\pi^1(z, n) < \pi^2(z-1, n)$, while s^2 players want to switch precisely when $\pi^2(z, n) < \pi^1(z+1, n)$.[6] We consider two dynamic adjustment processes based upon these comparisons. In both cases, the dynamics report the net number of switches, with positive numbers representing an increase in the number of players choosing s^1.

Under the *best response dynamics*, players constantly monitor their strategies, always updating to play a best response whenever they are not doing so. Formally, the best response dynamics are defined as follows:

$$D^{BR}(z, n) = \begin{cases} -z & \text{if } z \in [0, nx^*], \\ n-z & \text{if } z \in (nx^*, n]. \end{cases}$$

The best response dynamics are the fastest dynamics consistent with clever myopic adjustment.

As an alternative, it may be desirable in an evolutionary economic model to assume that players do not constantly monitor their strategies. Rather, it seems more consistent with myopia to assume that players only occasionally consider updating their choice of action. This observation motivates the *Bernoulli dynamics*.[7] Under these dynamics, in each period, each player independently with probability $\theta > 0$ receives a *learning draw*: the opportunity to reevaluate his strategy choice. If he is currently playing the myopic best response, he stands pat; otherwise, he switches. We formally define the Bernoulli dynamics as follows. Let $(\Omega, \mathscr{F}, P)$ be a probability space. Fixing θ, we define a collection of i. i. d. random variables $U_{t,i}$, $t \in \mathbf{N}_0 = \{0, 1, 2, \ldots\}$, $i \in \mathbf{N}$, with $P(U_{t,i} = 1) = \theta$

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[6] In contrast, KMR assume *simple myopic adjustment*: players evaluate their strategies by comparing their current expected payoffs to those of an opponent currently playing the other strategy. For small population sizes, simple myopic adjustment can lead to counterintuitive results: for example, coordination on strictly dominated strategies. Nevertheless, versions of the results in this paper can still be proved under the assumption of simple myopic adjustment. See Sandholm (1996) for a discussion of the differences between the payoff evaluation methods.

[7] Such dynamics were first introduced by Samuelson (1994).

and $P(U_{t,i} = 0) = 1 - \theta$. The Bernoulli dynamics, denoted D^θ, are given by:

$$D^\theta(z, n) = -1_{\{\pi^1(z,n)<\pi^2(z-1,n)\}}(z,n)\sum_{i=1}^{z} U_{t,i} + 1_{\{\pi^2(z,n)<\pi^1(z+1,n)\}}(z,n)\sum_{j=z+1}^{n} U_{t,j},$$

As usual, the ones represent indicator functions.

Under clever myopic adjustment, for generic payoffs there is a single state at which players using either strategy want to switch. Our best response dynamics implicitly assume that only players of one of the two strategies switch at this state. This greatly simplifies the analysis of the resulting evolutionary process. The Bernoulli dynamics do not incorporate this simplifying assumption.

Mutation is modeled using a collection of i. i. d. random variables $X_{t,i}$, $t \in \mathbf{N}_0$, $i \in \mathbf{N}$, with $P(X_{t,i} = 1) = \varepsilon$ and $P(X_{t,i} = 0) = 1 - \varepsilon$. For each fixed t, these random variables are also assumed to be independent of $U_{s,i}$ for all $s \le t$ and all i. The change in the number of s^1 players during period t due to mutations is given by

$$M_t(z,n) = -\sum_{i=1}^{z} X_{t,i} + \sum_{j=z+1}^{n} X_{t,j}.$$

The evolution of play is described by a nonhomogenous Markov chain $\{(N_t, \zeta_t)\}_{t=0}^{\infty}$ on $\mathbf{N} \times \mathbf{N}_0$. At each time $t \ge 0$, $N_t \ge 2$ denotes the population size, while $\zeta_t \le N_t$ represents the number of players selecting strategy s^1. The initial state, (N_0, ζ_0), with $\zeta_0 \le N_0$, is given; the evolution of the states will be explained below. We assume that for all i and t, $U_{t,i}$ and $X_{t,i}$ are independent of N_s and ζ_s for all $s \le t$. Let $\{\mathscr{F}_t\}_{t=0}^{\infty}$ be the filtration generated by $\{(N_t, \zeta_t)\}_{t=0}^{\infty}$: that is, $\mathscr{F}_t = \sigma((N_s, \zeta_s)\colon s \le t)$. Observe that each $\mathscr{F}_t$ is countably generated, and that $\mathscr{F}_0 = \{\emptyset, \Omega\}$.

Population growth occurs via an entry ("birth") process $\{B_t\}_{t=0}^{\infty}$. The number of births which occur during each period is Markovian: the distribution of births can depend on the current state, but on nothing else. That is, $B_t\colon \Omega \to \mathbf{N}_0$ satisfies the following Markov condition:

$$P(B_t \in A \mid (N_s, \zeta_s), s \le t) = P(B_t \in A \mid (N_t, \zeta_t)) \text{ for all } A \subseteq \mathbf{N}_0.$$

The adjustments in the population size are thus stochastic functions of the current period's state. One of many possible interpretations is that the change in population size is a noisy function of the average payoffs received by players in the current period. Of course, other interpretations are possible depending on the exact specification of the process.

We assume that entrants, like players receiving the learning draw, play a myopic best response to the current behavior of the population. Formally, the entrants' behavior, stated in terms of the number of entrants who choose strategy s^1, is given by

$$e(B_t, z, n) = \begin{cases} 0 & \text{if } z \leq nx^*, \\ B_t & \text{otherwise.} \end{cases}$$

As presented thus far, our model does not admit the possibility of players exiting the population ("dying"), nor is it flexible concerning players' behavior upon entry. However, as we discuss in Section 4.3, the model can be adapted to admit these possibilities. In particular, all of the results stated under the best response dynamics are unaffected if both entry and exit are permitted.

The law of motion of $\left\{(N_{t,}\, \zeta_t)\right\}_{t=0}^{\infty}$ is generated by combining mutation and entry with the base dynamic D_t:

$$\begin{aligned} N_{t+1} &= N_t + B_t, \\ \zeta_{t+1} &= \zeta_t + M_t(\zeta_t, N_t) + D_t(\zeta_t + M_t(\zeta_t, N_t), N_t) \\ &\qquad + e(B_t, (\zeta_t + M_t(\zeta_t, N_t) + D_t(\zeta_t + M_t(\zeta_t, N_t), N_t)), N_t). \end{aligned}$$

It is clear that $\left\{(N_{t,}\, \zeta_t)\right\}_{t=0}^{\infty}$ is a Markov chain on $\mathbf{N} \times \mathbf{N}_0$.

The critical assumption needed for our results are bounds on the asymptotic growth rate of the population. We now define notation characterizing the asymptotic behavior of deterministic functions and random variables. We say that a function f: $\mathbf{N}_0 \rightarrow \mathbf{N}$ is *asymptotically at least* $k \ln t$, denoted $f(t) \geq^a k \ln t$, if there exists a $T > 0$ such that for all $t \geq T$, $f(t) \geq k \ln t$. The condition $f(t) \leq^a k \ln t$ is defined similarly. Our growth conditions for random variables require that they be uniformly bounded by deterministic functions. For example, we say that a sequence of random variables $\left\{X_t\right\}_{t=0}^{\infty}$ *uniformly approaches infinity*, denoted $X_t \rightarrow^u \infty$, if there exists a function f such that $f(t) \rightarrow \infty$ as $t \rightarrow \infty$ and $X_t \geq f(t)$ almost surely for all t. We say that $\left\{X_t\right\}_{t=0}^{\infty}$ is *asymptotically uniformly greater than* $k \ln t$, denoted $X_t >^u k \ln t$, if there exist a function f and a positive constant α such that $f(t) \geq^a (k + \alpha) \ln t$ and $X_t \geq f(t)$ almost surely for all t. Similarly $X_t <^u k \ln t$ if there exist a function f and a positive constant α such that $f(t) \leq^a (k - \alpha) \ln t$ and $X_t \leq f(t)$ almost surely for all t. We last define

$$\Omega^u(\ln t) = \left\{ \{X_t\}_{t=0}^{\infty} \middle| \exists k > 0 \text{ such that } X_t >^u k \ln t \right\},$$
$$o^u(\ln t) = \left\{ \{X_t\}_{t=0}^{\infty} \middle| X_t <^u k \ln t \; \forall \; k > 0 \right\}.$$

Intuitively, $\Omega^u(\ln t)$ is the set of sequences of random variables whose asymptotic growth rates are at least logarithmic, while $o^u(\ln t)$ is the set of sequences of random variables whose asymptotic growth rates are less than logarithmic.

It will be convenient to state our results by considering the movement of the population between the basins of attraction of the unanimous states. We track this movement by defining a new process, $\{z_t\}_{t=0}^{\infty}$, which reports which basin of attraction the population inhabits at each time t. Most of our results concern evolution under the best response dynamics, D^{BR}. Under these dynamics, we define the process $\{z_t\}_{t=0}^{\infty}$ as follows:

$$z_t(\omega) = \begin{cases} 1 & \text{if } \zeta_t(\omega) > N_t(\omega)x^*, \\ 2 & \text{if } \zeta_t(\omega) \leq N_t(\omega)x^*. \end{cases}$$

Observe that under the best response dynamics, after period zero the population must be at a unanimous state; hence, z_t and N_t are together enough to determine ζ_t.

Under the Bernoulli dynamics D^θ, defining the process $\{z_t\}_{t=0}^{\infty}$ is a bit more complicated, because for generic payoffs there is a single state from which it is possible to arrive at either unanimous state without a mutation. At this exceptional state, we let z_t take the value zero. This extra possibility will not prove relevant to our analysis. Still, we define

$$z_t(\omega) = \begin{cases} 0 & \text{if } \pi^1(\zeta_t, N_t) < \pi^2(\zeta_t - 1, N_t) \text{ and } \pi^2(\zeta_t, N_t) < \pi^1(\zeta_t + 1, N_t), \\ 1 & \text{if } \pi^1(\zeta_t, N_t) \geq \pi^2(\zeta_t - 1, N_t) \text{ and } \pi^2(\zeta_t, N_t) < \pi^1(\zeta_t + 1, N_t), \\ 2 & \text{if } \pi^1(\zeta_t, N_t) < \pi^2(\zeta_t - 1, N_t) \text{ and } \pi^2(\zeta_t, N_t) \geq \pi^1(\zeta_t + 1, N_t). \end{cases}$$

Our analysis focuses on the effect of population growth on the behavior of $\{z_t\}_{t=0}^{\infty}$. We find that even logarithmic rates of population growth are fast enough to make this process converge: from some point forward, the population coordinates on a single equilibrium. If in addition the rate of mutation small enough, the process is constant and equal to its initial value z_0 with arbitrarily high probability. In other words, long run behavior is completely determined by initial conditions.

3. Results

We begin by reviewing the results of KMR in the current context. To do so, we simply need assume that the population process is constant: $N_t \equiv N$ for all t. We assume that $x^* > \frac{1}{2}$, so that s^2 is the unique risk dominant strategy, and consider evolution under the best response dynamics.

Theorem 1 (Kandori, Mailath, and Rob 1993): *Assume that $N_t \equiv N$ for all t and that $x^* \geq \frac{1}{2}\left(\frac{N+1}{N}\right) > \frac{1}{2}$. Then under the best response dynamics D^{BR}, for any $\varepsilon < 1 - x^*$ there exists a probability vector $\mu_\varepsilon = (\mu_\varepsilon^1, \mu_\varepsilon^2)$ such that*

(*a*) (*Wandering*) $P(\lim_{t\to\infty} z_t \text{ exists}) = 0.$

(*b*) (*Convergence in distribution*) $\lim_{t\to\infty} P(z_t = i) = \mu_\varepsilon^i$ *for i = 1, 2.*

(*Ergodicity*) $\lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} 1_{\{z_t = i\}} = \mu_\varepsilon^i$ *almost surely for i = 1, 2.*

Moreover, $(\mu_\varepsilon^1, \mu_\varepsilon^2) \to (0, 1)$ as $\varepsilon \to 0$.

KMR open their analysis by studying the limit behavior of their constant population size model for a fixed positive rate of mutation. Under the best response dynamics, $\{z_t\}_{t=1}^{\infty}$ is a stationary Markov chain with strictly positive transition probabilities. Hence, a unique stationary distribution exists, and by standard results in Markov chain theory, it is the limiting distribution of the process as well as the ergodic distribution of the process: with probability one, the time averaged behavior of the process approaches this distribution. KMR then show that as the rate of mutation approaches zero, all of the mass in the stationary distribution is placed on the state corresponding to the risk dominant equilibrium. Hence, when mutations are rare, in the long run we should expect to see the population coordinating on the risk dominant equilibrium.

We have noted in part (*a*) that the process $\{z_t\}_{t=0}^{\infty}$ *wanders*: $P(\lim_{t\to\infty} z_t$ exists) = 0. Recall that the process $\{z_t\}_{t=0}^{\infty}$ records the movements of the population between the two basins of attraction. If the limit of this process fails to exist for a particular realization $\{z_t(\omega)\}_{t=0}^{\infty}$, then in this realization the population jumps between the two basins of attraction forever. Part (*a*) states that with probability one, this limit does not exist.

Consequently, the equilibrium selection results of KMR described above are stochastic in nature: even when the risk dominant equilibrium is reached, it will almost surely be abandoned at some future date.

Adding population growth to the KMR model produces two effects which influence the course of evolution. The first effect is absolute: population growth makes jumps between the equilibria become less likely over time. Thus, for fast enough rates of growth, these jumps might cease, preventing the process $\{z_t\}_{t=0}^{\infty}$ from wandering between states forever. The second effect is relative: as the population size grows arbitrarily large, the difficulty in leaving the risk dominant equilibrium via mutations becomes arbitrarily greater than the difficulty in leaving the risk dominated equilibrium. As we shall see, the latter effect manifests itself whenever population growth is unbounded, while the former effect requires the more stringent condition of logarithmic growth.

Of course, population growth renders the evolutionary process nonstationary, so a stationary distribution cannot exist for positive rates of mutation. Nevertheless, we can still completely characterize the long run behavior of the process. In order to state our equilibrium selection result under slow population growth, we need conditions concerning the relative sizes of the probabilities of leaving each equilibrium. Define

$$R_t = \frac{P(z_{t+1} = 1 | z_t = 2)}{P(z_{t+1} = 1 | z_t = 2) + P(z_{t+1} = 2 | z_t = 1)}.$$

Our sufficient conditions for equilibrium selection are

(J1) $\lim_{t \to \infty} R_t = 0$;

(J2) $\sum_{t=0}^{\infty} |R_t - R_{t+1}| < \infty$.

Condition (J1) states that jumps from the risk dominant equilibrium become arbitrarily more difficult than jumps from the risk dominated equilibrium. Condition (J2) states that the sequence of jump probability ratios is of bounded variation. These conditions hold if, for example, the population process $\{N_t\}_{t=0}^{\infty}$ is deterministic, increasing, and unbounded.[8]

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[8] We prove this claim after the proof of Theorem 2(*i*). When the population process is stochastic, conditions (J1) and (J2) are difficult to check because the jump probabilities depend on the population size, which in turn can depend in a complicated way on the history of the process. Intuitively, these conditions will hold if the population does not tend to be larger at the risk dominated equilibrium than at

We also define three functions which prove critical to our characterization:

$$\rho(\varepsilon, a) = \left(\tfrac{\varepsilon}{a}\right)^{a}\left(\tfrac{1-\varepsilon}{1-a}\right)^{1-a},$$
$$l(\varepsilon, a) = -(\ln \rho(\varepsilon, a))^{-1},$$
$$u(\varepsilon, a) = -(\ln \rho(\varepsilon, 1-a))^{-1}.$$

We now state our first result, which characterizes evolution under the best response dynamics.

Theorem 2: *Suppose* $x^* > \frac{1}{2}$ *and* $\varepsilon \in (0, 1 - x^*)$, *and set* $l = l(\varepsilon, x^*)$ *and* $u = u(\varepsilon, x^*)$. *Then under the best response dynamics* D^{BR},

(*i*) *If* $N_t \rightarrow^{u} \infty$ *and* $N_t <^{u} l \ln t$, *then*
  (*a*) (*Wandering*) $P(\lim_{t\to\infty} z_t \text{ exists}) = 0$.
  (*b*) (*Selection*) *Under* (J1) *and* (J2), $\lim_{t\to\infty} P(z_t = 2) = 1$.

(*ii*) *If* $N_t >^{u} l \ln t$ *and* $N_t <^{u} u \ln t$, *then*
  (*a*) (*Convergence*) $P(\lim_{t\to\infty} z_t \text{ exists}) = 1$.
  (*b*) (*Selection*) $P(\lim_{t\to\infty} z_t = 2) = 1$.

(*iii*) *If* $N_t >^{u} u \ln t$, *then*
  (*a*) (*Convergence*) $P(\lim_{t\to\infty} z_t \text{ exists}) = 1$.
  (*b*) (*Non-selection*) $P(\lim_{t\to\infty} z_t = i) > 0$ *for* $i = 1, 2$.

When the population grows without bound but sublogarithmically (or slow enough and logarithmically), only the relative effect is relevant. Thus, in case (*i*), we see that the population wanders forever. Moreover, under conditions (J1) and (J2), the limiting distribution places *all* weight on the risk dominant strategy.[9] Hence, the equilibrium selection results of KMR are strengthened by very slow population growth.

For a small range of logarithmic growth rates, the absolute effect of population

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the risk dominant equilibrium.

[9] That the population can wander between states forever and yet in the limit place all probability mass on the risk dominant strategy may appear inconsistent. To see that these claims are compatible, consider an infinite sequence of coins such that the probability that the result of toss t is tails slowly approaches zero (in particular, at a rate no faster than $(1/t)$). Clearly, the probability that the t^{th} toss comes up heads approaches one. Nevertheless, the second Borel-Cantelli Lemma implies that almost every realization of the sequence of tosses contains an infinite number of both heads and tails.

growth becomes relevant, but only affects the risk dominant equilibrium. That is, it becomes possible for the population to become stuck at the risk dominant equilibrium, but not at the risk dominated equilibrium. Consequently, in case (*ii*), the population must converge to the risk dominant equilibrium with probability one. For this limited range of growth rates, equilibrium selection occurs in an especially stark fashion.

We emphasize that cases (*i*) and (*ii*) of Theorem 2 do not simply broaden the conditions under which the equilibrium selection results of KMR obtain. Rather, slow population growth enhances the *quality* of the mode of selection. In KMR, when the rate of mutation vanishes, all of the limiting probability mass is placed on the risk dominant equilibrium; since the population wanders, equilibrium selection is stochastic. In case (*i*) above, the same conclusion is reached, but under positive rates of mutation. Moreover, in case (*ii*), the evolutionary process converges to the risk dominant equilibrium: for this narrow band of growth rates, equilibrium selection is deterministic.

For superlogarithmic growth rates (and fast enough logarithmic growth rates), the absolute effect of population growth predominates. In case (*iii*), the growth rate is sufficient to make becoming stuck at either equilibrium possible. Indeed, convergence occurs almost surely, and convergence to either equilibrium is possible. Thus, the equilibrium selection results of KMR are disrupted.

We observe once again that logarithmic growth rates are extremely slow. If population growth occurs, it is natural to expect it to occur at a superlogarithmic rate. Therefore, we believe that case (*iii*) is the best characterization of behavior under population growth.

We briefly sketch the idea behind the proof of Theorem 2; the proof itself can be found in Section 6. Fix a mixed strategy equilibrium $x^* > \frac{1}{2}$ and a mutation rate $\varepsilon < 1 - x^*$, and set $\rho = \rho(\varepsilon, x^*)$ and $l = l(\varepsilon, x^*) = -(\ln \rho(\varepsilon, x^*))^{-1}$. Suppose that at the beginning of period t, all players are playing the risk dominant strategy, s^2. Since mutations by each player are independent events with probability $\varepsilon < x^*$, the law of large deviations (Lemma 1(*ii*)) tells us that as the population size grows large, the probability that enough deviations occur during period t to cause the population to jump out of the basin of attraction of the risk dominant equilibrium is exponential in the population size. A calculation shows that the base of this exponent is ρ. Thus, $P(z_{t+1} = 1 \mid z_t = 2)$ is asymptotically equal to ρ^{N_t}. A conditional version of the Borel-Cantelli Lemmas (Lemma 2(*ii*)) can be used to show that such jumps can occur infinitely often if and only if these conditional probabilities are summable. If we let $N_t = l \ln t$, then $\rho^{N_t} = \rho^{-(\ln \rho)^{-1}(\ln t)} = \rho^{-(\ln \rho)^{-1}(\ln \rho)(\log_\rho t)} = t^{-1}$. Since the sequence $\left\{t^a\right\}_{t=0}^{\infty}$ is summable if and only if a

< -1, $N_t = l \ln t$ is precisely the critical growth rate at which jumps from the risk dominant equilibrium cease to occur infinitely often. Similar reasoning can used to assess the likelihood of jumps from the risk dominated equilibrium; we simply need to replace $\rho(\varepsilon, x^*)$ with $\rho(\varepsilon, 1 - x^*)$ and $l(\varepsilon, x^*)$ with $u(\varepsilon, x^*)$. Together, these arguments form the basis for the characterization.

Figure 2 illustrates Theorem 2, sketching the regions corresponding to its three cases. For a fixed value of x^*, it classifies the various combinations of growth and mutation rates. For each fixed x^*, the inverses of $l(\cdot, x^*)$ and $u(\cdot, x^*)$ are well defined. The figure takes into account this observation, as well as the following easily verified facts: for all $\varepsilon \in (0, 1 - x^*)$, $l(\cdot, x^*)$ and $u(\cdot, x^*)$ are increasing, and $l(\varepsilon, x^*) < u(\varepsilon, x^*)$; $\lim_{\varepsilon \downarrow 0} l(\varepsilon, x^*) = \lim_{\varepsilon \downarrow 0} u(\varepsilon, x^*) = 0$; $\lim_{\varepsilon \uparrow (1-x^*)} l(\varepsilon, x^*) = [(2x^* - 1) \ln(x^* / 1 - x^*)]^{-1}$; and $\lim_{\varepsilon \uparrow (1-x^*)} u(\varepsilon, x^*) = \infty$.

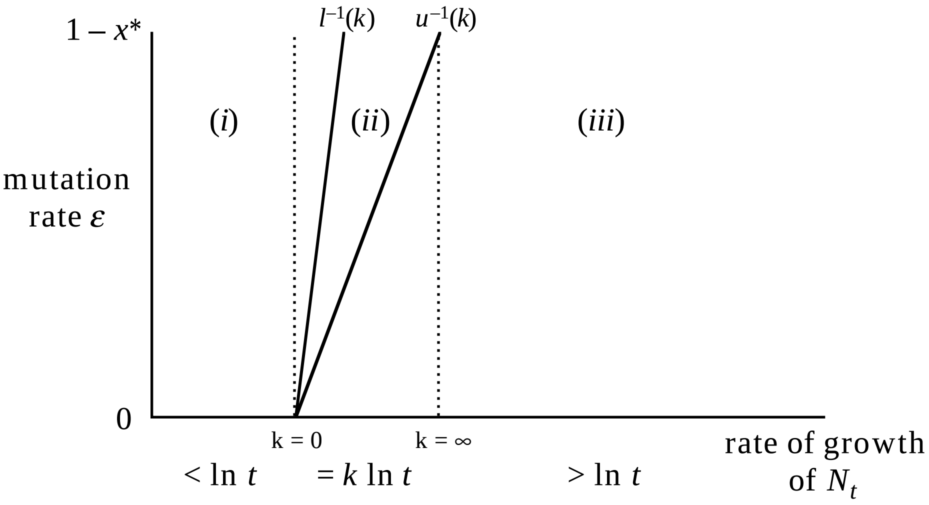

Figure 2

Following game theoretic tradition, we proceed by considering the effect on our model of taking the rate of mutation to zero. As is clear from Figure 2, lowering the mutation rate to zero pares the three cases from our first result down to two. In the limit, all that matters is whether the rate of population growth is logarithmic. If the growth rate is sublogarithmic, the equilibrium selection results of KMR are retained and even strengthened, as this immediate corollary to Theorem 2(*i*) shows.

Corollary 1: *Suppose that* $N_t \in o^u(\ln t)$. *Then under the best response dynamics* D^{BR}, *for any* $\varepsilon > 0$, $P(\lim_{t \to \infty} z_t$ *exists*$) = 0$. *If in addition conditions* (J1) *and* (J2) *hold,* $\lim_{t \to \infty} P(z_t = 2) = 1$.

On the other hand, we know from Theorem 2 that logarithmic and faster growth rates can cause the population to settle on a single equilibrium. When the rate of mutation is taken zero, we can say much more: as the mutation rate vanishes, so too does the probability of leaving the basin of attraction in which play began. This is our main result.

Theorem 3 (*History Dependence*): *Suppose that* $N_t \in \Omega^u(\ln t)$. *Then under the best response dynamics* D^{BR}, $\lim_{\varepsilon \to 0} P(z_t = z_0 \text{ for all } t) = 1$.

All of the results thus far have been stated under the best response dynamics D^{BR}. While analytically quite convenient, these dynamics possess two undesirable properties. First, they are extremely fast: all players react instantly to changes in the behavior of their fellows. Second, they are deterministic: with probability one, D^{BR} moves the population away from the boundary of the basin of attraction; there is zero probability of the population failing to adjust at all.[10]

In our final result, we show that the history dependence result persists under the Bernoulli dynamics D^{θ}. These dynamics share neither of the undesirable properties mentioned above: they are quite slow when θ is very small, and always admit the possibility that the population does not adjust at all. Nevertheless, history dependence perseveres; hence, it is not simply an artifact of the best response dynamics, but a more general consequence of population growth.

To prove this result, we need to strengthen the population growth condition by assuming a particular logarithmic growth rate. Nevertheless, this distinction does not effect the interpretation of the result. For example, growth rates satisfying this stronger condition are still less than t^a for any $a > 0$, or even $(\ln t)^b$ for any $b > 1$, both of which exhibit extremely slow growth.

We now present our final result.

Theorem 4: *Fix the Bernoulli dynamics* D^{θ} *for some* $\theta > 0$, *and suppose that* $z_0 \neq 0$. *If* $N_t >^u k \ln t$ *for some sufficiently large k, then* $\lim_{\varepsilon \to 0} P(z_t = z_0 \text{ for all } t) = 1$.

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[10] While this latter point may seem inconsequential, the distinction between probability zero events and events with very small but positive probabilities acquires paramount importance when there are an infinite number of opportunities for these events to occur.

4. Extensions

4.1 Declining Mutation Rates

We have shown that allowing population growth can prevent evolutionary models from exhibiting ergodic behavior. Intuitively, this is a consequence of the decreasing probability of jumps between the two equilibria; the results depend on the rates of decrease. Fixing the population size but allowing the rate of mutation to decline over time has a similar effect on the probability of jumps; consequently, one would expect to see similar results. By asymptotically equating the probability of jumps in the two models, we are able to show an equivalence between them. In particular, we now show how Theorem 2 can be adapted to a model with declining mutation rates, and provide a tighter characterization of this model than that of Robles (1995).

For simplicity, we consider evolution under the best response dynamics. In a model with population growth, for a given rate of mutation ε, mixed strategy equilibrium $x^* > \frac{1}{2}$, and population size N_t, we have seen that the probability of a jump from the risk dominant equilibrium is asymptotically equal to $\rho(\varepsilon, x^*)^{N_t}$ as N_t approaches infinity. Alternatively, consider an evolutionary model in which the population size is fixed at some M (with $x^* \geq \frac{1}{2}\left(\frac{M+1}{M}\right)$), but in which the rate of mutation falls over time and is equal to δ_t at time t. In such a model, the probability of a jump from the risk dominant equilibrium is equal to $\sum_{i=1}^{M-[Mx^*]} a_i(\delta_t)^{[Mx^*]+i}$, where $[\cdot]$ denotes the greatest integer function and the a_i are positive constants. This expression is asymptotically equal to $a_1(\delta_t)^{[Mx^*+1]}$ as δ_t approaches zero. If we fix ε and M, equate $\rho(\varepsilon, x^*)^{N_t}$ and $a_1\delta_t^{[Mx^*+1]}$, and solve for δ_t in terms of N_t, we see that as N_t grows large, a population size of N_t generates the same probability of jumping from the risk dominant equilibrium as a mutation rate of $\delta_t(N_t) \propto \rho(\varepsilon, x^*)^{(N_t/[Mx^*+1])}$. Recalling that the critical population growth rate for jumps from the risk dominant equilibrium is $N_t = l \ln t = -(\ln \rho(\varepsilon, x^*))^{-1} \ln t$, we conclude that the critical declining mutation rate for such jumps is[11] $\delta_t(-(\ln \rho(\varepsilon, x^*))^{-1} \ln t) \propto t^{-1/[Mx^*+1]}$.

The preceding argument is the basis for our characterization of limiting behavior under declining mutation rates. To state our characterization, we first define two functions which serve the same role as did the functions $l(\cdot, \cdot)$ and $u(\cdot, \cdot)$ in the characterization of limiting behavior under population growth.

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[11] The critical declining mutation rate can also be determined by computing the Borel-Cantelli summability condition directly.

$$L(M, x^*) = \frac{-1}{[Mx^*]+1},$$
$$U(M, x^*) = \frac{-1}{M-[Mx^*]}.$$

We also define two standard pieces of asymptotic notation.

$$\Omega(t^a) = \left\{ f\colon \mathbf{N}_0 \to \mathbf{N} \,\middle|\, \exists k, T \text{ such that } f(t) \geq kt^a \;\forall t \geq T \right\}.$$
$$O(t^a) = \left\{ f\colon \mathbf{N}_0 \to \mathbf{N} \,\middle|\, \exists k, T \text{ such that } f(t) \leq kt^a \;\forall t \geq T \right\},$$

These two notations mean "asymptotically at least t^a" and "asymptotically at most t^a", respectively.

Table I presents our characterization, stated without proof, of limiting behavior under declining mutation rates. The table makes manifest the equivalence of the population growth and declining mutation rate models.[12]

There are three main differences between our characterization of evolution under declining mutation rates and that of Robles (1995). Robles (1995) does not consider the behavior of the sample paths of the evolutionary process. Consequently, he establishes just two classes of asymptotic behavior rather than three. Additionally, in the first two cases, in which a limiting distribution exists, Robles (1995) does not calculate this distribution; hence, he does not observe the strenghtened equilibrium selection results which occur in these cases.

Behavior	Population Growth	Declining Mutation Rates
Wandering & Selection	$N_t \to^u \infty$, $N_t <^u l \ln t$.	$\delta_t \to 0$, $\delta_t \in \Omega(t^L)$.
Convergence & Selection	$N_t >^u l \ln t$, $N_t <^u u \ln t$.	$\delta_t \in O(t^{L-c})$ for some $c > 0$, $\delta_t \in \Omega(t^U)$.
Convergence & Non-selection	$N_t >^u u \ln t$.	$\delta_t \in O(t^{U-c})$ for some $c > 0$.

Table I: Asymptotic Behavior under Population Growth and Declining Mutation Rates

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[12] We should point out that it is also mathematically possible to extend Theorem 3 to the case of declining mutation rates. However, the interpretation of this extension is quite strained.

4.2 Bounded Populations and Finite Time Spans

As we discussed in the introduction, the results of the literature on stochastic evolution are highly dependent upon the order in which the time and population limits are taken. Fix a rate of mutation ε. If as in KMR we hold fixed the population size, the risk dominant equilibrium eventually will have been played in a predominant proportion of past periods. On the other hand, for any fixed time span, there is a population size large enough to virtually guarantee that the population always will play its initial equilibrium selection. Our limiting results, while suggestive, only partially arbitrate between these two equilibrium predictions. Consider, for example, a modeler who intends to use a stochastic evolutionary model to study some subject of economic interest. The relevant time span is suggested by the object of study, as is a bound on the size of the population; the small but positive rate of mutation is fixed. When should the modeler expect history dependence?

Once the time span of interest is fixed, by choosing the rate of mutation small enough, we can be virtually assured that no change from the initial state will occur. Therefore, to state an interesting history dependence result, we must fix the mutation rate *in advance*, and then present a lower bound on the population size in terms of the time span T such that history dependence will occur if the population size exceeds this bound. The bound must hold for arbitrarily long time spans given the mutation rate we have chosen.

Suppose first that the modeler believes the population to be growing. In this case, we can apply Theorem 3 to address the question posed above directly.[13] Fix a positive constant c. By Theorem 3, for any population process $\{N_t\}_{t=0}^{\infty}$ with $N_t \geq c \ln t$ and any $\alpha > 0$, there exists an $\bar{\varepsilon}$ such that $P(z_t = z_0 \text{ for all } t) > 1 - \alpha$ whenever the mutation rate is less than $\bar{\varepsilon}$. Hence, even under these positive rates of mutation, history dependence occurs over the infinite horizon with high probability. The modeler is interested only in behavior through some finite time T. Happily, it follows *a fortiori* from the previous claim that for any T, $P(z_t = z_0 \text{ for all } t \leq T) > 1 - \alpha$ under mutation rates less than $\bar{\varepsilon}$. Now recall that this conclusion requires only that $N_t \geq c \ln t$ for all t. Consequently, for all mutation rates below $\bar{\varepsilon}$ and all times T, conditional on adequate population sizes before time T, the population size that must be established by time T to virtually assure history dependence through time T is just $c \ln T$. Thus, if the modeler is only concerned with behavior through time T, he need only believe that the population will grow as

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[13] For convenience, we frame this discussion in the context of the best reply dynamics; similar claims can be made under the Bernoulli dynamics by applying Theorem 4.

large as $c \ln T$ to expect history dependence.

This reasoning immediately extends to fixed population settings. Once again, consider a growing population model in which $N_t = c \ln t$ after some initial period, rounding upward when necessary. For simplicity, assume that the initial state is unanimous. For any α we can find an $\bar{\varepsilon}(\alpha,c)$ such that $P(z_t = z_0 \text{ for all } t) > 1 - \alpha$ whenever the mutation rate is less than $\bar{\varepsilon}(\alpha,c)$. Since increasing the population size in the early periods can only make jumps during those periods less likely, we have the following result, stated under the best response dynamics.[14]

Corollary 2: *Fix $\alpha > 0$ and $c > 0$, and let the initial state be unanimous. Suppose the mutation rate is less than $\bar{\varepsilon}(\alpha,c)$. Choose a time T and a fixed population size $N_t \equiv N$. If $T \leq \exp(N/c)$, then $P(z_t = z_0$ for all $t \leq T) > 1 - \alpha$.*

This corollary can be interpreted as follows. Suppose that the modeler believes the population size in his object of study to be fixed. While the modeler will typically have some estimate of the appropriate time span and population size, he will not know these data precisely. Corollary 2 tells him that so long as the length of the relevant time span is not more than exponentially greater than the population size, history dependence should be expected. In most settings of economic interest, this condition will hold, and hence the history dependent prediction is the relevant one.

4.3 Exit, and Stochastic Behavior by Entrants

In the model presented above, players never leave the population. However, since under the best response dynamics the population always coordinates on a unanimous state, exit cannot affect the proportions of strategies in the population as a whole. Thus, Theorems 2 and 3 continue to hold if we allow players to exit, so long as the bounds on the total population size are maintained.[15] Under the Bernoulli dynamics, the population need not stay close to a unanimous state; therefore, allowing exit can change the proportion of strategies in the population and so affect our results.

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[14] The assumption that the initial state is unanimous is considerably stronger than necessary. If we suppose instead that the initial state in the growing population model is not unanimous, then for history dependence to carry over to the fixed population model, we must assume that the players added to the initial population to construct the fixed population model choose their strategies in a manner which does not increase the probability of a jump during period zero.

[15] In our model, the initial state need not be unanimous. Therefore, for Theorem 3 to continue to hold when exit is allowed, we must assume that deaths during period zero do not cause the population to jump between basins of attraction.

While we have assumed that entrants play myopic best responses, one might want to admit the possibility that entrants' behavior is more diverse. For example, one might assume that entrants select actions independently according to some probability distribution which depends on the current state. In this case, the choices of the entrants can change the proportions of strategies played in the population. Nevertheless, results can still be proved in this case. Under the best response dynamics, we can allow stochastic entry behavior if bounds are placed on the proportion of new players entering the population each period. Under stochastic updating, our results can be extended if in addition the probability with which entrants fail to play the myopic best response is made arbitrarily small.[16]

5. Conclusion

We investigate an evolutionary model with independent mutations and population growth. Our main result is that as the mutation rate is taken to zero, the probability that the population never leaves the basin of attraction of the equilibrium in which it begins approaches one. In contrast, Kandori, Mailath, and Rob (1993) predict that in the long run, a fixed population of players will act in accordance with some stationary distribution which is given independently of the initial conditions. In our model, it is the initial state which determines the equilibrium on which the system will settle.

Kandori, Mailath, and Rob (1993) has been praised for yielding unique predictions from weak rationality and knowledge assumptions. This approbation seems quite justified because of the cardinal importance of uniqueness of equilibrium in the context of solution concepts for one-shot games. When analyzing one-shot games, the goal of a unique prediction is central: multiple predictions, while suggestive, are insufficient, providing neither a clear prediction to an outside observer nor an adequate guide to play for an agent involved in the strategic interaction.

In repeated situations like those usually considered in evolutionary economics, some mechanism beyond individual introspection may serve to coordinate players on a particular action profile. We argue that in such contexts, uniqueness of prediction is not

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[16] For our results to continue to hold under stochastic updating, it is also enough to assume a deterministic upper bound (less than $1 - x^*$) on the proportion of entrants who fail to play a best response, since entry can only break coordination if a very high proportion of entrants fail to play a best response in a single period. If the behavior of each entrant is determined independently, such large scale failures, while unlikely, are always possible. For this reason, allowing independent behavior by entrants is essentially equivalent to adding a new source of mutation to the model. The requirement that the rate of failure to play a best response be taken to zero should be regarded accordingly.

only unnecessary, but in many cases is undesirable. If the evolutionary model is taken at face value, as genuinely attempting to study the development of norms in a large population, it is natural to expect that early behavior patterns of the population will often affect the equilibrium selected. The initial state, while perhaps not meaningful in an eductive analysis of the underlying normal form game, can be essential when we consider how a population learns to play.

Thus, in a model which attempts to describe the development of conventions in a large population, multiple long run predictions should be possible; which prediction is realized should depend on details of a historical character. To cite a well known example, David (1985) has studied the predominance of the long established QWERTY keyboard arrangement despite the existence of an alternative arrangement which is at least twenty percent more efficient. Our results indicate that in the absence of a centralized effort to switch, we need not expect that such well entrenched standards will change.

KMR's analysis of economic evolution suggests that in the long run, society's choices are independent from its past. In contrast, our results indicate that historical conditions can influence society's ultimate course. The existence of multiple predictions and history dependence, rather than being a cause for dissatisfaction, should be viewed as a natural consequence of economic evolution.

6. Proofs

6.1 Mathematical Preliminaries

We begin by collecting some mathematical results which will be used in the sequel. The first is a basic large deviation theory result: the probability that the sample average of n i.i.d. random variables exceeds a constant greater than its mean decreases exponentially in the number of random variables.

Lemma 1: *Let* $\{X_i\}_{i=1}^{\infty}$ *be a sequence of i. i. d. random variables with finite support, and let* $S_n = \sum_{i=1}^{n} X_i$. *Fix* $a > EX_1$. *Then there exists an* $r < 1$ *such that*

(*i*) $P(S_n \geq na) \leq r^n$ *for all* n, *and*

(*ii*) $\lim_{n\to\infty} \frac{1}{n} \ln P(S_n \geq na) = \ln r$.

Moreover,

(*iii*) *If* $P(X_1 = 1) = \varepsilon < a$ *and* $P(X_1 = 0) = 1 - \varepsilon$, *then* $r = \rho(\varepsilon, a) \equiv \left(\frac{\varepsilon}{a}\right)^a \left(\frac{1-\varepsilon}{1-a}\right)^{1-a}$.

Proof: (*i*) For all $t > 0$, $P(S_n \geq na) = P(\sum_{i=1}^n (X_i - a) \geq 0) = P(\exp(t\sum_{i=1}^n (X_i - a)) \geq 1) \leq E\exp(t\sum_{i=1}^n (X_i - a)) = M(t)^n$, where $M(t) = E\exp(t(X_1 - a))$ is the moment generating function for $(X_1 - a)$, and where the final inequality follows from Markov's inequality. As $M(0) = 1$, $M'(0) = E(X_1 - a) < 0$, and $M''(t) = E(X_1 - a)^2\ e^{tX_1} > 0$, letting $r = \min_t M(t)$ proves the result.

(*ii*) See Billingsley (1995, Theorem 9.3).

(*iii*) In this case, $M(t)$ is minimized by $t^* = \ln(\frac{a}{1-a} \cdot \frac{1-\varepsilon}{\varepsilon})$, so $r = M(t^*) = \left(\frac{\varepsilon}{a}\right)^a \left(\frac{1-\varepsilon}{1-a}\right)^{1-a}$. The asymptotic tightness of this bound follows from Theorem 9.3 of Billingsley (1995). ■

The following lemma states the first Borel-Cantelli Lemma as well as a conditional extension of both Borel-Cantelli Lemmas. The latter can be proved using martingale methods.

Lemma 2: *Let* $\{A_t\}_{t=1}^{\infty}$ *be a sequence of events, and let* $\left\{\mathscr{F}_t\right\}_{t=0}^{\infty}$ *be a filtration such that* $\mathscr{F}_0 = \{\varnothing, \Omega\}$ *and* $A_t \in \mathscr{F}_t$ *for all* $t \geq 1$. *Then*

(*i*) $\sum_{t=1}^{\infty} P(A_t) < \infty$ *implies that* $P(A_t$ infinitely often (i.o.)) = 0.

(*ii*) $\{A_t \text{ i.o.}\} = \left\{\sum_{t=1}^{\infty} P(A_t | \mathscr{F}_{t-1}) = \infty\right\}$.

Proof: (*i*) By Tonelli's Theorem, $E\left(\sum_{t=1}^{\infty} 1_{A_t}\right) = \sum_{t=1}^{\infty} P(A_t) < \infty$, so $P\left(\sum_{t=1}^{\infty} 1_{A_t} = \infty\right) = 0$.

(*ii*) See Durrett (1991, Theorem 4.3.2). ■

Since probabilities conditional on σ-fields are only defined up to an equivalence class, the equality in the latter result is interpreted as equality up to a set of probability zero.

The next lemma is a well known result from real analysis.

Lemma 3: *Let* $\left\{x_t\right\}_{t=1}^{\infty}$ *be a sequence of numbers in the interval* (0, 1). *Then* $\sum_{t=1}^{\infty} x_t < \infty$ *if and only if* $\prod_{t=1}^{\infty} (1 - x_t) > 0$.

Proof: If x_t does not converge to 0, it is clear that neither statement is true. When $\lim_{t \to \infty} x_t = 0$, the result follows from the observations that $\prod_{t=1}^{\infty} (1 - x_t) > 0$ if and only if $\sum_{t=1}^{\infty} \ln(1 - x_t) > -\infty$ and that $\lim_{x \to 0} \frac{\ln(1+x)}{x} = \frac{d}{dx} \ln(1 + x)\big|_{x=0} = 1$. ■

The next two lemmas are needed only for the proof of Theorem 2(*i*)(*b*). To state them we require one last definition: we say a sequence $\{c_n\}_{n=0}^{\infty}$ is of *bounded variation* if $\sum_{n=0}^{\infty}|c_n - c_{n+1}| < \infty$. The following lemma gives sufficient conditions for the convergence in distribution of a two state nonhomogenous Markov chain.

Lemma 4: *Let* $\{X_t\}_{t=0}^{\infty}$ *be a nonhomogenous Markov chain with state space* $S = \{1, 2\}$. *Let* M_t, *the matrix of the probabilities of transitions between periods* t *and* $t + 1$, *be given by*

$$M_t = \begin{bmatrix} 1-a_t & a_t \\ b_t & 1-b_t \end{bmatrix}.$$

for $t \geq 0$. *If* $\sum_{t=0}^{\infty}(a_t + b_t) = \infty$ *and* $\left\{\frac{b_t}{a_t+b_t}\right\}_{t=0}^{\infty}$ *is of bounded variation, then for any initial distribution,* $\lim_{t\to\infty} P(X_t = 1) = \lim_{t\to\infty}\left(\frac{b_t}{a_t+b_t}\right)$.

Proof: See Isaacson and Madsen (1976, p. 177). ■

The next lemma shows that a sequence which converges at an exponential rate is of bounded variation.

Lemma 5: *Let a sequence* $\{c_n\}_{n=0}^{\infty}$ *be given. If there exist a constant* $\beta < 1$ *and an integer* N *such that* $0 \leq c_n \leq \beta^n$ *for all* $n \geq N$, *then* $\{c_n\}_{n=0}^{\infty}$ *is of bounded variation.*

Proof: Since the first N terms of the sequence cannot affect the result, we can without loss of generality assume that $N = 0$. Let $C_n = [c_{n+1}, c_n]$ if $c_{n+1} \leq c_n$ and $[c_n, c_{n+1}]$ otherwise. For any $x \in (0, 1)$, the upper bound guarantees that c_n cannot exceed x once $n > \frac{\ln x}{\ln \beta}$. Therefore, the number of intervals C_n in which x appears is bounded above by $\frac{\ln x}{\ln \beta} + 1$. By Tonelli's Theorem,

$$\sum_{n=0}^{\infty}|c_n - c_{n+1}| = \sum_{n=0}^{\infty}\left(\int_0^1 1_{\{x\in C_n\}}dx\right) = \int_0^1\left(\sum_{n=0}^{\infty}1_{\{x\in C_n\}}\right)dx \leq \int_0^1\left(\tfrac{\ln x}{\ln \beta}+1\right)dx = 1-\tfrac{1}{\ln \beta} < \infty.$$

Therefore, $\{c_n\}_{n=0}^{\infty}$ is of bounded variation. ■

The final lemmas are key elements in the proofs of the history dependence results.

Lemma 6: *Let* $\{X_i\}_{i=1}^{\infty}$ *be a sequence of independent random variables such that* $P(X_i = 0) = 1 - \varepsilon$ *for all* $i \in \mathbf{N}$, *and let* N *be a random variable with range* $\mathbf{N}$ *which is independent of the* X_i. *Then* $\lim_{\varepsilon\to 0} P(X_i = 0\ \forall i \leq N) = 1$.

Proof: Fix $\eta > 0$; we show that for all ε small enough, $P(X_i = 0\ \forall i \le N) > 1 - \eta$. Choose m large enough that $P(N \le m) > (1 - \eta)^{(1/2)}$. Then for any $\varepsilon < 1 - (1 - \eta)^{(1/2m)}$,

$$\begin{aligned} P(X_i = 0\ \forall\ i \le N) &\ge P(X_i = 0\ \forall\ i \le N, N \le m) \\ &\ge P(X_i = 0\ \forall\ i \le m)\, P(N \le m) \\ &> (1 - \varepsilon)^m\, (1 - \eta)^{(1/2)} \\ &> (1 - \eta). \ \blacksquare \end{aligned}$$

Lemma 7: *Suppose that the function f*: $\mathbf{N} \times (0, 1) \to (0, 1)$ *satisfies* $\lim\limits_{\varepsilon \to 0} f(t, \varepsilon) \to 0$ *for all* $t \in \mathbf{N}$, *and that there exist an* $\tilde{\varepsilon} \in (0, 1)$, *an integer T, and a function g*: $\mathbf{N} \to (0, 1)$ *such that* $f(t, \varepsilon) \le g(t)$ *for all* $\varepsilon < \tilde{\varepsilon}$ *and all* $t \ge T$ *and that* $\sum_{t=1}^{\infty} g(t) < \infty$. *Then* $\lim\limits_{\varepsilon \to 0} \prod_{t=1}^{\infty}(1 - f(t, \varepsilon)) = 1$.

Proof: We show, equivalently, that $\lim\limits_{\varepsilon \to 0} \sum_{t=1}^{\infty} \ln(1 - f(t, \varepsilon)) = 0$. Since $\sum_{t=1}^{\infty} g(t) < \infty$, Lemma 3 implies that $\prod_{t=1}^{\infty}(1 - g(t)) > 0$. Therefore, $\sum_{t=1}^{\infty} \ln(1 - g(t)) > -\infty$, with each term strictly negative. Fixing some $\delta > 0$, we can find a $\hat{T} \ge T$ such that $\sum_{t=\hat{T}+1}^{\infty} \ln(1 - g(t)) > -\frac{\delta}{2}$. We can also choose $\hat{\varepsilon} < \tilde{\varepsilon}$ small enough that $\varepsilon < \hat{\varepsilon}$ implies that $f(t, \varepsilon) < 1 - \exp(\frac{-\delta}{2\hat{T}})$ for all $t \le \hat{T}$. For such ε,

$$\begin{aligned} \sum_{t=1}^{\infty} \ln(1 - f(t, \varepsilon)) &= \sum_{t=1}^{\hat{T}} \ln(1 - f(t, \varepsilon)) + \sum_{t=\hat{T}+1}^{\infty} \ln(1 - f(t, \varepsilon)) \\ &\ge \sum_{t=1}^{\hat{T}} \ln(1 - f(t, \varepsilon)) + \sum_{t=\hat{T}+1}^{\infty} \ln(1 - g(t)) \\ &> \hat{T}\, (\tfrac{-\delta}{2\hat{T}}) + (\tfrac{-\delta}{2}) = -\delta, \end{aligned}$$

completing the proof. ■

6.2 Proofs of Main Results

To conserve on notation, we will use upper bars to denote both a realization of a random variable and the event that this realization occurs. For example, $\overline{N}_t$ can refer to either a particular positive integer or to the event that the random variable N_t takes this particular value.

Theorem 2: *Suppose* $x^* > \frac{1}{2}$ *and* $\varepsilon \in (0, 1 - x^*)$, *and set* $l = l(\varepsilon, x^*)$ *and* $u = u(\varepsilon, x^*)$. *Then under the best response dynamics* D^{BR},

(*i*) *If* $N_t \to^u \infty$ *and* $N_t <^u l \ln t$, *then*

- (*a*) (*Wandering*) $P(\lim_{t\to\infty} z_t \text{ exists}) = 0$,
- (*b*) (*Selection*) *Under* (J1) *and* (J2), $\lim_{t\to\infty} P(z_t = 2) = 1$.

(*ii*) *If* $N_t >^u l \ln t$ *and* $N_t <^u u \ln t$, *then*

- (*a*) (*Convergence*) $P(\lim_{t\to\infty} z_t \text{ exists}) = 1$,
- (*b*) (*Selection*) $P(\lim_{t\to\infty} z_t = 2) = 1$.

(*iii*) *If* $N_t >^u u \ln t$, *then*

- (*a*) (*Convergence*) $P(\lim_{t\to\infty} z_t \text{ exists}) = 1$,
- (*b*) (*Non-selection*) $P(\lim_{t\to\infty} z_t = i) > 0$ *for* $i = 1, 2$.

Proof of part (*i*): We start by proving (*a*). Fix $\varepsilon > 0$. Since $N_t <^u l \ln t$, there exist a function $L(t)$, an integer S, and an $\alpha > 0$ such that $N_t \le L(t)$ a.s. for all $t \ge S$ and $L(t) \le (l(\varepsilon, x^*) - \alpha) \ln t$. Since $l(\cdot, \cdot)$ is continuous and decreasing in its second argument, there exists a $\beta > 0$ such that $l(\varepsilon, x^* + \beta) - \frac{\alpha}{2} = l(\varepsilon, x^*) - \alpha$. Let $l = l(\varepsilon, x^* + \beta)$ and $\rho = \rho(\varepsilon, x^* + \beta)$, and let $\gamma = \ln \rho + (l - \frac{\alpha}{2})^{-1}$. It is easily checked that γ is strictly positive.

Observe that for $t > 0$,

$$P(z_{t+1} \ne z_t \mid z_t = 1, \overline{N}_t) = P(M_t \le -\overline{N}_t(1 - x^*)) = P\left(\sum_{i=1}^{\overline{N}_t} X_{t,i} \ge \overline{N}_t(1 - x^*)\right),$$

and

$$P(z_{t+1} \ne z_t \mid z_t = 2, \overline{N}_t) = P(M_t > -\overline{N}_t(1 - x^*)) = P\left(\sum_{i=1}^{\overline{N}_t} X_{t,i} > \overline{N}_t x^*\right).$$

Both of these quantities are greater than $P\left(\sum_{i=1}^{\overline{N}_t} X_{t,i} \ge \overline{N}_t (x^* + \beta)\right)$.

Lemma 1 implies that there exists an m such that for $n \ge m$ and for all t, $\frac{1}{n} \ln P\left(\sum_{i=1}^{n} X_{t,i} \ge n(x^* + \beta)\right) > \ln \rho - \gamma$. Since $N_t \to^u \infty$, there exists a function $K(t)$ such that $N_t \ge K(t)$ a.s. for all t and $K(t) \to \infty$ as $t \to \infty$. Choose $T \ge S$ large enough that for all $t \ge T$, $K(t) > m$. Then for $t \ge T$, $K(t) \le N_t \le L(t)$ almost surely, so for almost all $\overline{N}_t$,

$$P\left(\sum_{i=1}^{\overline{N}_t} X_{t,i} \geq \overline{N}_t(x^* + \beta)\right) > (\rho e^{-\gamma})^{\overline{N}_t} \geq (\rho e^{-\gamma})^{L(t)} \geq (\rho e^{-\gamma})^{(l-(\alpha/2))\ln t}$$
$$= (\rho e^{-\gamma})^{(l-(\alpha/2))(\log_{(\rho \exp(-\gamma))} t)(\ln \rho e^{-\gamma})} = t^{(l-(\alpha/2))\ln(\rho e^{-\gamma})} = t^{-1}.$$

We thus conclude that for almost every $\omega \in \Omega$, $P(z_{t+1} \neq z_t \mid z_t = z_t(\omega), N_t = N_t(\omega)) > t^{-1}$.

By the Markov property, and since the $\mathscr{F}_t$ are countably generated, $P(z_{t+1} \neq z_t | \mathscr{F}_t)(\omega) = P(z_{t+1} \neq z_t | z_t = z_t(\omega), N_t = N_t(\omega))$. Therefore, for almost all $\omega \in \Omega$,

$$\begin{aligned} \sum_{t=0}^{\infty} P(z_{t+1} \neq z_t | \mathscr{F}_t)(\omega) &> \sum_{t=T}^{\infty} P(z_{t+1} \neq z_t | z_t = z_t(\omega), N_t = N_t(\omega)) \\ &> \sum_{t=T+1}^{\infty} t^{-1} = \infty. \end{aligned}$$

Thus, by Lemma 2 (*ii*), $P(z_t \neq z_{t-1}$ i.o.$) = 1$. This completes the proof of (*a*).

We continue with the proof of (*b*). Let $J_t^i = P(z_{t+1} \neq i \mid z_t = i)$ be the probability that a jump occurs at time t if the state is i, and let $j_n^i = P(z_{t+1} \neq i \mid z_t = i, N_t = n, t > 0)$ be the probability that a jump occurs from state i when the population size is n. The latter quantity is well defined since these probabilities are independent of t after period zero. Since $\{(N_t, z_t)\}_{t=0}^{\infty}$ is a Markov chain, for $t > 0$ we have that

$$\begin{aligned} &P(z_{t+1} \neq i \mid z_t = i) \\ &\quad = \sum_{\{\overline{N}_s : s \leq t\}} \sum_{\{\overline{z}_s : s < t\}} P(z_{t+1} \neq i | z_t = i; \overline{N}_s, s \leq t; \overline{z}_s, s < t)\ P(\overline{N}_s, s \leq t; \overline{z}_s, s < t | z_t = i) \\ &\quad = \sum_{\{\overline{N}_s : s \leq t\}} \sum_{\{\overline{z}_s : s < t\}} P(z_{t+1} \neq i | z_t = i; \overline{N}_t)\ P(\overline{N}_s, s \leq t; \overline{z}_s, s < t | z_t = i) \\ &\quad = \sum_{\overline{N}_t} P(z_{t+1} \neq i | z_t = i; \overline{N}_t)\ P(\overline{N}_t | z_t = i) \\ &\quad = \sum_{\overline{N}_t} j_{\overline{N}_t}^i P(\overline{N}_t | z_t = i). \end{aligned}$$

Once $t \geq T$ (where T is taken from the proof of (*a*)), this expression is simply a convex combination of elements of $\{j_n^i\colon K(t) \leq n \leq L(t)\}$, and hence is at least $j_{L(t)}^i$. Thus, the computation in the proof of part (*i*) shows that this expression is at least t^{-1}. Therefore,

$$(1) \qquad \sum_{t \geq T} J_t^1 = \sum_{t \geq T} J_t^2 = \infty.$$

Recall that $R_t = J_t^2 / (J_t^1 + J_t^2)$. We have assumed

$$(\text{J1}) \qquad \lim_{t \to \infty} R_t = 0, \text{ and}$$

(J2) $\sum_{t=0}^{\infty}|R_t - R_{t+1}| < \infty$.

If the process $\{N_t\}_{t=0}^{\infty}$ is deterministic, the process $\{z_t\}_{t=0}^{\infty}$ is by itself a Markov chain. Furthermore, the a_t and b_t from Lemma 4 correspond to J_t^1 and J_t^2, respectively. Equation (1) and assumption (J2) show that these transition probabilities satisfy the conditions of Lemma 4, and so, by (J1), $\lim_{t\to\infty} P(z_t = 1) = \lim_{t\to\infty} \frac{b_t}{a_t + b_t} = \lim_{t\to\infty} R_t = 0$, proving the result.

If $\{N_t\}_{t=0}^{\infty}$ is not deterministic, then $\{z_t\}_{t=0}^{\infty}$ by itself is not a Markov chain: its transition probabilities from period t to period $t + 1$ depend on the population size N_t, which itself can depend on values of z_s before period t. To prove the result in this case, we create an auxiliary process $\{\hat{z}_t\}_{t=0}^{\infty}$ which is a Markov chain and which has the same one-dimensional distributions as $\{z_t\}_{t=0}^{\infty}$. Using a standard technique (see, *mutatis mutandis*, Theorem 8.1 of Billingsley (1995)), we can construct a probability space $(\hat{\Omega}, \hat{\mathscr{F}}, \hat{P})$ and a Markov chain $\{\hat{z}_t\}_{t=0}^{\infty}$ defined on this space with the same initial distribution and one step transition probabilities as $\{z_t\}_{t=0}^{\infty}$:

$$\hat{P}(\hat{z}_0 = k) = P(z_0 = k);$$
$$\hat{P}(\hat{z}_{t+1} = l | \hat{z}_t = k) = P(z_{t+1} = l | z_t = k).$$

By induction, $\hat{z}_t$ and z_t have the same distribution for all $t \geq 0$. Hence, $\lim_{t\to\infty} P(z_t = 1) = \lim_{t\to\infty} \hat{P}(\hat{z}_t = 1)$, and we only need investigate the behavior of $\{\hat{z}_t\}_{t=0}^{\infty}$. However, since by definition the one step transition probabilities of $\{\hat{z}_t\}_{t=0}^{\infty}$ are the same as the one step transition probabilities of $\{z_t\}_{t=0}^{\infty}$, (1) and (J2) imply that $\{\hat{z}_t\}_{t=0}^{\infty}$ satisfies the conditions of Lemma 4. Hence, by (J1), $\lim_{t\to\infty} P(z_t = 1) = \lim_{t\to\infty} \hat{P}(\hat{z}_t = 1) = \lim_{t\to\infty} R_t = 0$. This completes the proof of (*b*).

We now show that if $\{N_t\}_{t=0}^{\infty}$ is deterministic, increasing, and unbounded, then (J1) and (J2) hold. Let $r_n = j_n^2 / (j_n^1 + j_n^2)$, and observe that since N_t is a constant, $R_t = J_t^2 / (J_t^1 + J_t^2) = j_{N_t}^2 / (j_{N_t}^1 + j_{N_t}^2) = r_{N_t}$. Since $\{N_t\}_{t=0}^{\infty}$ is increasing and $\lim_{t\to\infty} N_t = \infty$, by the triangle inequality it is sufficient to prove that the following conditions hold:

(J1') $\lim_{n\to\infty} r_n = 0$;

(J2') $\sum_{n=N_o}^{\infty} |r_n - r_{n+1}| < \infty$.

Observe that

$$\frac{j_n^2}{j_n^1} = \frac{P\left(\sum_{i=1}^{n} X_i > nx^*\right)}{P\left(\sum_{i=1}^{n} X_i \geq n(1-x^*)\right)},$$

where the X_i are i.i.d. with $P(X_i = 1) = \varepsilon$ and $P(X_i = 0) = 1 - \varepsilon$. Let $q = \rho(\varepsilon, x^*) / \rho(\varepsilon, 1 - x^*)$. Since $\rho(\varepsilon, x^*) < \rho(\varepsilon, 1 - x^*)$, we can choose $\gamma > 0$ such that $qe^{\gamma} = \frac{1+q}{2} < 1$. By Lemma 1, there exists an N such that for all $n \geq N$, $P\left(\sum_{i=1}^{n} X_i \geq n(1-x^*)\right) \geq (\rho(\varepsilon, (1 - x^*))e^{-\gamma})^n$. Therefore, by Lemma 1, for all $n \geq N$ we have that

$$r_n = \frac{j_n^2}{j_n^1 + j_n^2} \leq \frac{j_n^2}{j_n^1} \leq \left(\frac{\rho(\varepsilon, x^*)}{\rho(\varepsilon,(1-x^*))} e^{\gamma}\right)^n .$$

This implies condition (J1'). Furthermore, applying Lemma 5, we see that $\{r_n\}_{n=N_0}^{\infty}$ is of bounded variation, which is condition (J2'). ❑

Proof of part (*ii*). Observe that (*b*) implies (*a*), so we need only prove (*b*). We show that $P(z_t = 2 \text{ i.o.}) = 1$ and $P((z_t = 2 \text{ and } z_{t+1} = 1) \text{ i.o.}) = 0$, which together imply the result.

To show that $P(z_t = 2 \text{ i.o.}) = 1$, we show equivalently that for all R, $P(z_t = 1 \text{ for all } t \geq R) = 0$. Let $u = u(\varepsilon, x^*)$ and $\rho = \rho(\varepsilon, 1 - x^*)$. Since $N_t <^{u} u \ln t$, there exist a function $U(t)$ and a constant S such that $N_t \leq U(t)$ a.s. for all $t \geq S$ and $U(t) \leq (u - \alpha) \ln t$ for some $\alpha > 0$. Let $\gamma = \ln \rho + (u - \alpha)^{-1}$, and observe that $\gamma > 0$. Lemma 1 implies that there exists an m such that for $n \geq m$ and for all t, $\frac{1}{n}\ln P\left(\sum_{i=1}^{n} X_{t,i} \geq n(1-x^*)\right) > \ln \rho - \gamma$. Since $N_t >^{u} l \ln t$, we can choose $T \geq S$ large enough that for all $t \geq T$, $N_t \geq m$ almost surely.

For any $t \geq T$,

$$P(z_{t+1} = 2 \mid z_t = 1, \overline{N}_t) = P(M_t \leq -\overline{N}_t(1 - x^*)) = P\left(\sum_{i=1}^{\overline{N}_t} X_{t,i} \geq \overline{N}_t(1-x^*)\right).$$

Since $N_t \leq U(t)$ a.s., we see that for almost every $\overline{N}_t$,

$$(2) \qquad P(z_{t+1} = 2 \mid z_t = 1, \overline{N}_t) > (\rho e^{-\gamma})^{U(t)} \geq (\rho e^{-\gamma})^{(u-\alpha)\ln t} = t^{-1}.$$

Therefore,

$$(3) \qquad \sum_{t=T}^{\infty} (\rho e^{-\gamma})^{U(t)} > \sum_{t=T}^{\infty} t^{-1} = \infty.$$

Now let $V = \max\{T, R\}$. Then since $\{(N_t, z_t)\}_{t=0}^{\infty}$ is a Markov chain, inequalities (2) and (3) and Lemma 3 imply that

$$\begin{aligned}P(z_t = 1 \text{ for all } t \geq R) &\leq P(z_t = 1 \text{ for all } t \geq V)\\ &= P(z_V = 1) \prod_{t=V}^{\infty} P(z_{t+1} = 1 \mid z_s = 1, V \leq s \leq t)\\ &\leq \prod_{t=V}^{\infty} \left(1 - P(z_{t+1} = 2 \mid z_s = 1, V \leq s \leq t)\right)\\ &\leq \prod_{t=V}^{\infty} (1 - (\rho e^{-\gamma})^{U(t)}) = 0.\end{aligned}$$

Hence, $P(z_t = 2 \text{ i.o.}) = 1$.

Now, we want to show that $P((z_{t+1} = 1 \text{ and } z_t = 2) \text{ i.o.}) = 0$. Let $\rho = \rho(\varepsilon, x^*)$ and $l = l(\varepsilon, x^*)$. Since $N_t >^u l \ln t$, there exist a function $L(t)$, an integer T, and an $\alpha > 0$ such that $N_t \geq L(t)$ a.s. and $L(t) \geq (l + \alpha) \ln t$ for all $t \geq T$. Observe that

$$(4) \qquad P(z_{t+1} = 1, z_t = 2 \mid \bar{z}_t, \bar{N}_t) = \begin{cases} 0 & \text{if } \bar{z}_t = 1 \\ P(z_{t+1} = 1 \mid z_t = 2, \bar{N}_t) & \text{if } \bar{z}_t = 2. \end{cases}$$

Also, observe that

$$P(z_{t+1} = 1 \mid z_t = 2, \bar{N}_t) = P(M_t > \bar{N}_t x^*) \leq P\left(\sum_{i=1}^{\bar{N}_t} X_{t,i} \geq \bar{N}_t x^*\right) \leq \rho^{\bar{N}_t}.$$

Fix $t \geq T$. Since $\bar{N}_t \geq L(t)$ a.s., for almost every $\bar{N}_t$, we have that

$$P(z_{t+1} = 1 \mid z_t = 2) \leq \rho^{L(t)} \leq \rho^{(l+\alpha) \ln t} = t^{(l+\alpha) \ln \rho}.$$

Hence, for such t, $P(z_{t+1} = 1 \text{ and } z_t = 2) \leq t^{(l+\alpha) \ln \rho}$. Therefore, since $l \ln \rho = -1$, $(l + \alpha) \ln \rho < -1$, so for almost every $\omega \in \Omega$, we have that

$$\sum_{t=0}^{\infty} P(z_t = 2, z_{t+1} = 1) \leq T + \sum_{t=T}^{\infty} t^{(l+\alpha)\ln \rho} < \infty.$$

So, Lemma 2 (*i*) implies that $P((z_t = 2 \text{ and } z_{t+1} = 1) \text{ i.o.}) = 0$. This completes the proof of part (*ii*). ❑

Proof of part (*iii*): We begin with the proof of (*a*). Observe that

$$P(z_t \neq z_{t+1} \mid \bar{z}_t,\ \overline{N}_t) = \begin{cases} P(\sum_{t=1}^{\overline{N}_t} X_{t,i} \geq \overline{N}_t(1 - x^*)) & \text{if } \bar{z}_t = 1 \\ P(\sum_{t=1}^{\overline{N}_t} X_{t,i} > \overline{N}_t x^*) & \text{if } \bar{z}_t = 2. \end{cases}$$

For any fixed $\overline{N}_t$, the former expression is larger than the latter. Therefore, letting $\rho = \rho(\varepsilon, 1 - x^*)$ and $u = u(\varepsilon, x^*)$, the proof is completed by repeating the argument following equation (4), making the appropriate substitutions.

We conclude with the proof of part (*b*). We prove that $P(z_t = 2 \text{ for all } t \geq 1) > 0$; that $P(z_t = 1 \text{ for all } t \geq 1) > 0$ can be proved in like fashion. Let $\rho = \rho(\varepsilon, x^*)$ and $l = l(\varepsilon, x^*)$. Since $N_t >^u l \ln t$, there exist a function $L(t)$, an integer T, and an $\alpha > 0$ such that $N_t \geq L(t)$ a.s. and $L(t) \geq (l + \alpha) \ln t$ for all $t \geq T$. Then, for all $t \geq T$,

$$(5) \qquad P(z_{t+1} = 1 \mid z_t = 2,\ \overline{N}_t) = P\left(\sum_{i=1}^{\overline{N}_t} X_{t,i} \geq \overline{N}_t x^*\right) \leq \rho^{\overline{N}_t} \leq \rho^{(l+\alpha) \ln t}.$$

Since $l \ln \rho = -1$, $(l + \alpha) \ln \rho < -1$, which implies that

$$(6) \qquad \sum_{t=1}^{\infty} \rho^{(l+\alpha)\ln t} < \infty.$$

Consequently, as $\{(N_t, z_t)\}_{t=0}^{\infty}$ is a Markov chain, inequalities (5) and (6) and Lemma 3 imply that

$$\begin{aligned} (7) \qquad P(z_t = 2 \text{ for all } t \geq 1) &= P(z_1 = 2) \prod_{t=1}^{\infty} P(z_{t+1} = 2 | z_s = 2\ \forall s = 1, \ldots, t) \\ &= P(z_1 = 2) \prod_{t=1}^{\infty} \left(1 - P(z_{t+1} = 1 | z_s = 2\ \forall s = 1, \ldots, t)\right) \\ &\geq P(z_1 = 2) \prod_{t=1}^{\infty} (1 - \rho^{(l+\alpha)\ln t}) > 0. \end{aligned}$$

This concludes the proof of Theorem 2. ■

Theorem 3 (*History Dependence*): *Suppose that* $N_t \in \Omega^u(\ln t)$. *Then under the best response dynamics* D^{BR}, $\lim_{\varepsilon \to 0} P(z_t = z_0 \text{ for all } t) = 1$.

Proof: We give the proof for the case in which $z_0 = 2$; the proof when $z_0 = 1$ is nearly identical, replacing $\rho(\cdot, x^*)$ and $l(\cdot, x^*)$ with $\rho(\cdot, 1 - x^* - \beta)$ and $u(\cdot, x^* + \beta)$ for some $\beta \in (0, \frac{1 - x^*}{2})$.

Since $N_t \in \Omega^u(\ln t)$, there exist a function $L(t)$, an integer T, and a constant $k > 0$ such that $N_t \geq L(t)$ a.s. for all $t \geq T$ and $L(t) \geq k \ln t$ for all $t \geq T$. Choose $\tilde{\varepsilon}$ small enough that $l(\tilde{\varepsilon}, x^*) = \frac{k}{2}$. Then for any $\varepsilon < \tilde{\varepsilon}$, $N_t >^u l(\varepsilon, x^*) \ln t$, so mimicking expressions (5) and (6), we see that for all $t \geq T$, all $\varepsilon < \tilde{\varepsilon}$, and almost every $\overline{N}_t$,

$$(8) \qquad P(z_{t+1} = 1 \mid z_t = 2, \overline{N}_t) \leq \rho(\varepsilon, x^*)^{k \ln t} \leq \rho(\tilde{\varepsilon}, x^*)^{k \ln t},$$

and that

$$(9) \qquad \sum_{t=T}^{\infty} \rho(\tilde{\varepsilon}, x^*)^{k \ln t} = \sum_{t=T}^{\infty} t^{-2} < \infty.$$

Since $z_0 = 2$ by assumption,

$$P(z_t = 2 \ \forall \ t \geq 0) = \prod_{t=0}^{\infty} P(z_{t+1} = 2 \mid z_s = 2 \forall s \leq t).$$

Now for all $t \geq 0$,

$$P(z_{t+1} = 2 \mid z_s = 2 \forall s \leq t) \geq P(X_{t,i} = 0 \ \forall i \leq N_t \mid z_s = 2 \forall s \leq t) \geq P(X_{t,i} = 0 \ \forall i \leq N_t).$$

Lemma 6 implies that this last expression approaches one as ε vanishes. Consequently, letting $f(t, \varepsilon) = P(z_{t+1} = 1 \mid z_s = 2 \forall s \leq t)$, we see that for all $t \geq 0$, $\lim_{\varepsilon \to 0} f(t, \varepsilon) = 0$.

Finally, define

$$g(t) = \begin{cases} 1 & \text{if } t < T, \\ \rho(\tilde{\varepsilon}, x^*)^{k \ln t} & \text{otherwise.} \end{cases}$$

By inequality (9), $g(t)$ is summable; furthermore, since $\left\{(N_t, z_t)\right\}_{t=0}^{\infty}$ is a Markov chain, inequality (8) implies that $f(t, \varepsilon) \leq g(t)$ for all $\varepsilon < \tilde{\varepsilon}$. Therefore, by Lemma 7,

$$\lim_{\varepsilon \to 0} P(z_t = 2 \ \forall t \geq 0) = \lim_{\varepsilon \to 0} \prod_{t=0}^{\infty} P(z_{t+1} = 2 \mid z_s = 2 \forall s \leq t) = \lim_{\varepsilon \to 0} \prod_{t=0}^{\infty} (1 - f(t, \varepsilon)) = 1. \ \blacksquare$$

Theorem 4: *Fix the Bernoulli dynamics* D^θ *for some* $\theta > 0$, *and suppose that* $z_0 \neq 0$. *If* $N_t >^u k \ln t$ *for some sufficiently large k, then* $\lim_{\varepsilon \to 0} P(z_t = z_0 \text{ for all } t) = 1$.

Proof: Assume that $z_0 = 2$; the other case is proved in a similar fashion. Let $l = \min\{j \in$

$\mathbf{N}$: $j \geq \frac{8-2x^*}{x^*}\}$. Let $\tilde{\varepsilon} = \min\{\frac{\theta}{2l}, \frac{x^*}{8}\}$. Fixing $\varepsilon = \tilde{\varepsilon}$, observe that for every t,

$$Y_i = \sum_{j=1}^{l} X_{t,l(i-1)+j} - U_{t,i}$$

has finite support and satisfies $EY_i = l\varepsilon - \theta \leq \frac{\theta}{2} - \theta < 0$. Therefore, by Lemma 1 (*i*), there exists an $r < 1$ such that

$$P\left(\sum\nolimits_{i=1}^{n} Y_i \geq 0\right) \leq r^n .$$

This inequality continues hold with the value of r fixed if ε takes any value less than $\tilde{\varepsilon}$.

Let $s = r^{(x^*/8)}$. Choose $k = 2 \max\{-(\ln \rho(\tilde{\varepsilon}, \frac{x^*}{4}))^{-1}, -(\ln s)^{-1}\}$. Since by assumption $N_t \geq^u k \ln t$, there exist a function $L(t)$ and an integer T such that $N_t \geq L(t)$ a.s. and $L(t) \geq k \ln t > \max\{2N_0, \frac{8}{x^*}\}$ for all $t \geq T$. Since entrants play best responses, this implies that if no mutations occur through time T, $\zeta_T \leq \frac{N_T x^*}{2}$ almost surely.

Suppose $t \geq T$, and fix $\varepsilon < \tilde{\varepsilon}$. If $(\overline{\zeta}_t, \overline{N}_t)$ satisfies $\overline{\zeta}_t \leq \frac{\overline{N}_t x^*}{4}$, we can bound the probability that ζ_{t+1} jumps beyond $\frac{N_{t+1} x^*}{2}$ as follows:

$$\begin{aligned} P\left(\zeta_{t+1} \geq \tfrac{N_t x^*}{2} \middle| \overline{\zeta}_t, \overline{N}_t\right) &\leq P\left(M_t \geq \tfrac{\overline{N}_t x^*}{4}\right) \\ &\leq P\left(\sum\nolimits_{t=1}^{\overline{N}_t} X_{t,i} \geq \tfrac{\overline{N}_t x^*}{4}\right) \\ &\leq \rho(\varepsilon, \tfrac{x^*}{4})^{\overline{N}_t} . \end{aligned}$$

On the other hand, if $(\overline{\overline{\zeta}}_t, \overline{\overline{N}}_t)$ satisfies $\frac{\overline{\overline{N}}_t x^*}{4} \leq \overline{\overline{\zeta}}_t \leq \frac{\overline{\overline{N}}_t x^*}{2}$, we can bound the probability that $\zeta_{t+1} > \zeta_t$.

$$\begin{aligned} P\left(\zeta_{t+1} > \zeta_t \middle| \overline{\overline{\zeta}}_t, \overline{\overline{N}}_t\right) &\leq P\left(M_t + D_t > 0 \middle| \overline{\overline{\zeta}}_t, \overline{\overline{N}}_t\right) \\ &= P\left(M_t > \tfrac{N_t x^*}{4}, M_t + D_t > 0 \middle| \overline{\overline{\zeta}}_t, \overline{\overline{N}}_t\right) + P\left(0 < M_t \leq \tfrac{N_t x^*}{4}, M_t + D_t > 0 \middle| \overline{\overline{\zeta}}_t, \overline{\overline{N}}_t\right) \\ &\qquad + P\left(M_t \leq 0, M_t + D_t > 0 \middle| \overline{\overline{\zeta}}_t, \overline{\overline{N}}_t\right) \\ &= P\left(M_t > \tfrac{N_t x^*}{4}, M_t + D_t > 0 \middle| \overline{\overline{\zeta}}_t, \overline{\overline{N}}_t\right) + P\left(0 < M_t \leq \tfrac{N_t x^*}{4}, M_t + D_t > 0 \middle| \overline{\overline{\zeta}}_t, \overline{\overline{N}}_t\right). \end{aligned} \tag{10}$$

The last equality holds because if mutations reduce the number of s^1 players, so too will updates: if $z_t = 2$ and $M_t \leq 0$, then $D_t \leq 0$.

The first term in expression (10) can be bounded as follows:

$$P\left(M_t > \tfrac{N_t x^*}{4}, M_t + D_t > 0 \middle| \overline{\overline{\zeta}}_t, \overline{\overline{N}}_t\right) \le P\left(M_t \ge \tfrac{\overline{\overline{N}}_t x^*}{4}\right) \le \rho(\varepsilon, \tfrac{x^*}{4})^{\overline{\overline{N}}_t}.$$

On the other hand, if $M_t \le \frac{\overline{\overline{N}}_t x^*}{4}$, then $\zeta_t + M_t \le \frac{\overline{\overline{N}}_t x^*}{2} < \overline{\overline{N}}_t x^*$, so s^1 players who receive the learning draw update to s^2. Thus, to bound the second term in expression (10), we use the law of large deviations to show that if the proportion of s^1 players is not too small, mutations from s^2 to s^1 are very likely to be canceled by s^1 players updating their strategies to s^2. Let $[\cdot]$ denote the greatest integer function. Since $l \ge \frac{8-2x^*}{x^*}$, for $n \ge \frac{8}{x^*}$ we see that

$$\begin{aligned} l\left[\tfrac{nx^*}{4}\right] &\ge \left(\tfrac{8-2x^*}{x^*}\right)\left(\tfrac{nx^*}{4} - 1\right) \\ &= \left(1 - \tfrac{x^*}{4}\right)\left(\tfrac{x^*}{4} - \tfrac{x^*}{8}\right)^{-1}\left(\tfrac{nx^*}{4} - 1\right) \\ &\ge \left(1 - \tfrac{x^*}{4}\right)\left(\tfrac{x^*}{4} - \tfrac{1}{n}\right)^{-1} n\left(\tfrac{x^*}{4} - \tfrac{1}{n}\right) \\ &= n\left(1 - \tfrac{x^*}{4}\right) \ge n\left[1 - \tfrac{x^*}{4}\right]. \end{aligned}$$

So, since $\overline{\overline{N}}_t \ge \frac{8}{x^*}$ a.s., we see that for almost every $\overline{\overline{N}}_t$,

$$\begin{aligned} &P\left(0 < M_t \le \tfrac{N_t x^*}{4}, M_t + D_t > 0 \middle| \overline{\overline{\zeta}}_t, \overline{\overline{N}}_t\right) \\ &\quad = P\left(0 < M_t \le \tfrac{N_t x^*}{4}, M_t - \textstyle\sum_{i=1}^{\zeta_t + N_t} U_{t,i} > 0 \middle| \overline{\overline{\zeta}}_t, \overline{\overline{N}}_t\right) \\ &\quad \le P\left(-\textstyle\sum_{i=1}^{\zeta_t} X_{t,i} + \sum_{j=\zeta_t+1}^{N_t} X_{t,i} - \sum_{i=1}^{\zeta_t + N_t} U_{t,i} > 0 \middle| \overline{\overline{\zeta}}_t, \overline{\overline{N}}_t\right) \\ &\quad \le P\left(\textstyle\sum_{j=1}^{[N_t(1-(x^*/4))]} X_{t,i} - \sum_{i=1}^{[N_t(x^*/4)]} U_{t,i} > 0 \middle| \overline{\overline{\zeta}}_t, \overline{\overline{N}}_t\right) \\ &\quad \le P\left(\textstyle\sum_{j=1}^{l[\overline{\overline{N}}_t(x^*/4)]} X_{t,i} - \sum_{i=1}^{[\overline{\overline{N}}_t(x^*/4)]} U_{t,i} > 0\right) \\ &\quad \le P\left(\textstyle\sum_{i=1}^{[\overline{\overline{N}}_t(x^*/4)]}\left(\sum_{j=1}^{l} X_{t,l(i-1)+j} - U_{t,i}\right) > 0\right) \\ &\quad \le r^{[\overline{\overline{N}}_t(x^*/4)]}. \end{aligned}$$

If $n \ge \frac{8}{x^*}$, then $\left[\frac{nx^*}{4}\right] \ge \frac{nx^*}{4} - 1 = n\left(\frac{x^*}{4} - \frac{1}{n}\right) \ge \frac{nx^*}{8}$. Therefore, since $N_t \ge \frac{8}{x^*}$ a.s., and since $s = r^{(x^*/8)}$, we see that for almost every $\overline{\overline{N}}_t$,

$$r^{[\overline{\overline{N}}_t(x^*/4)]} \le r^{\overline{\overline{N}}_t(x^*/8)} = s^{\overline{\overline{N}}_t}.$$

Since $N_t \ge L(t)$ a.s. for all $t \ge T$, we conclude that for these t and for any $(\overline{\zeta}_t, \overline{N}_t)$ satisfying $\overline{\zeta}_t \le \frac{\overline{N}_t x^*}{2}$,

$$(11) \qquad P\left(\zeta_{t+1} > \tfrac{N_t x^*}{2} \,\middle|\, \overline{\zeta}_t, \overline{N}_t\right) \le \rho(\varepsilon, \tfrac{x^*}{4})^{\overline{N}_t} + s^{\overline{N}_t} \le \rho(\varepsilon, \tfrac{x^*}{4})^{L(t)} + s^{L(t)}.$$

Moreover, by our choice of k we have that

$$(12) \qquad \sum_{t=T}^{\infty}\left(\rho(\varepsilon, \tfrac{x^*}{4})^{L(t)} + s^{L(t)}\right) \le \sum_{t=T}^{\infty}\left(\rho(\tilde{\varepsilon}, \tfrac{x^*}{4})^{k \ln t} + s^{k \ln t}\right) \le \sum_{t=T}^{\infty} 2t^{-2} < \infty.$$

We want to show that $\lim_{\varepsilon \to 0} P(z_t = 2 \ \forall t \ge 0) = 1$. For $t \ge T$, let $S_t \subseteq \Omega$ be given by $S_t = \{z_s = 2 \ \forall\ s \le T,\ \zeta_u \le \tfrac{N_u x^*}{2} \ \forall\ u = T, \dots, t\}$. Then for all $\varepsilon < \tilde{\varepsilon}$,

$$\begin{aligned} P(z_t = 2\ \forall\ t \ge 0) &\ge P(z_t = 2\ \forall\ t \le T,\ \zeta_T \le \tfrac{N_T x^*}{2}) \prod_{t=T}^{\infty} P(\zeta_{t+1} \le \tfrac{N_{t+1} x^*}{2} \mid S_t) \\ &\ge P(z_t = 2\ \forall\ t \le T,\ \zeta_T \le \tfrac{N_T x^*}{2}) \prod_{t=T}^{\infty} P(\zeta_{t+1} \le \tfrac{N_t x^*}{2} \mid S_t). \end{aligned}$$

Applying Lemma 6 in periods zero through $T - 1$, and recalling the observation that if no mutations occur through period T, $\zeta_T \le \tfrac{N_T x^*}{2}$, we see that $\lim_{\varepsilon \to 0} P(z_t = 2\ \forall\ t \le T - 1,\ \zeta_T \le \tfrac{N_T x^*}{2}) = 1$. To bound the second term, we apply Lemmas 6 and 7. Since $\left\{(N_t, z_t)\right\}_{t=0}^{\infty}$ is a Markov chain, inequality (11) implies that for each $t \ge T$, $f(t, \varepsilon) = P(\zeta_{t+1} > \tfrac{N_t x^*}{2} \mid S_t)$ is bounded above by $g(t) \equiv \rho(\tilde{\varepsilon}, \tfrac{x^*}{4})^{L(t)} + s^{L(t)}$. Inequality (12) tells us that $\sum_{t=T}^{\infty} g(t) < \infty$. Furthermore, by Lemma 6,

$$\lim_{\varepsilon \to 0} P(\zeta_{t+1} \le \tfrac{N_t x^*}{2} \mid S_t) \ge \lim_{\varepsilon \to 0} P(X_{t,i} = 0\ \forall i \le N_t \mid S_t) \ge \lim_{\varepsilon \to 0} P(X_{t,i} = 0\ \forall i \le N_t) = 1$$

for all $t \ge T$. Therefore, for such t, $\lim_{\varepsilon \to 0} f(t, \varepsilon) = \lim_{\varepsilon \to 0} P(\zeta_{t+1} > \tfrac{N_t x^*}{2} \mid S_t) = 0$. Thus, by Lemma 7, $\lim_{\varepsilon \to 0} \prod_{t=T}^{\infty} P(\zeta_{t+1} \le \tfrac{N_t x^*}{2} \mid S_t) = 1$, and so we conclude that $\lim_{\varepsilon \to 0} P(z_t = 2\ \forall\ t \ge 0) = 1$. ■

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