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Economics 714
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# Problem Set 3

Due in Class on 3/7

1. Suppose there is a population of risk-neutral borrowers who each own a project that requires an initial outlay of 1 unit of goods and yields a random return $X \in \{0, R\}$, and denote $p = P(X = R)$. Borrowers have no wealth and must obtain funds from an outside source. A borrower can be of two types $i = s, r$ where $s$ denotes "safe" and $r$ denotes "risky". A borrower of type $i$ has a project with return characteristics $(p_i, R_i)$. Assume $p_i R_i = m > 1$ and $p_s > p_r$ but $R_s < R_r$. There is a mass of $\beta$ safe borrowers and $1 - \beta$ risky, and a single bank with total funds $\alpha < 1$ which distributes funds. The bank offers contracts $(x_i, D_i)$ where $x_i$ is the probability of financing type $i$ and $D_i$ the repayment the bank obtains from type $i$. Note that borrowers will only apply for funds if $D_i \leq R_i$. The bank cannot observe a borrower's type.

   (a) Suppose that the bank is constrained to offering the same terms to each type, with $x_i = x = 1$ and $D_s = D_r = D$. Show that when $1 - \beta$ is small and $p_s$ is close to $p_r$ the bank sets a repayment rate so both types apply, but some risky types are willing to pay a higher $D$. (Thus there is "credit rationing.")

   (b) Now consider the full incentive-constrained problem, and find the optimal contract. Show that credit rationing disappears and that risky borrowers earn rents.

2. Consider the setup of the previous problem set: a two period endowment economy with a continuum of individuals of two types. The total population size is normalized to 2 and each type has measure 1. Endowments are i.i.d. over time, common across agent types, and take on the values 0 or 1 in each period with equal probability. The aggregate endowment is constant at 1 in each period, and the relevant state of nature $s_t \in \{0, 1\}$, $t = 1, 2$ is an i.i.d. random variable indicating which type receives the high endowment in period $t$. All agents have common preferences, defined over their own consumption $c_t^i$ $i = 1, 2$ at the two dates:

$$U(c_1^i, c_2^i) = -E\left[(c_1^i - 1)^2 + \beta(c_2^i - 1)^2\right].$$

   Now suppose that the endowment realizations are private information for the agents. We will consider efficient allocations in which agents receive transfers from a benevolent social planner who does not observe the agents' endowments. In the first period, after observing his endowment realization, an agent reports either 0 or 1 to the planner, and in return receives a transfer $b_1^0$ or $b_1^1$. In the second period, the agent also makes a report, and the transfer may depend on the current report as well as the previous report. Thus we denote $b_2^{ij}$ the transfer an agent gets in period 2 if he reported $i$ in period 1 and $j$ in period 2, with $i, j \in \{0, 1\}$. Goods are nonstorable, so each agent's consumption is his endowment plus his transfer. A feasible allocation is one in which total consumption is less than total endowment, and an incentive compatible allocation is one in which each agent chooses to report truthfully in each date and state. An incentive feasible allocation is feasible and incentive compatible.

(a) Show that in any incentive compatible allocation transfers in the second period must be independent of the second period report (but may depend on the first period report): $b_2^{i0} = b_2^{i1} \equiv b_2^i$.

(b) The social planner chooses the transfers $\{b_1^i, b_2^{ij}\}$ to maximize the ex-ante utility of the agents (recall that given all the symmetry in the problem they are identical ex-ante) subject to the consumption allocation being incentive feasible. Find the optimal allocation.

3. Consider a representative agent exchange economy with money, where the aggregate endowment $Y_t$ is governed by an exogenous process:

$$\log \frac{Y_t}{Y_{t-1}} = \mu + \sigma W_t \tag{1}$$

where $\mu \geq 0$ is the mean growth rate, and $W_t$ is an i.i.d. standard normal endowment shock. Preferences over consumption $c_t$ and real money balances $m_t = M_t/P_t$ are:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[ \frac{c_t^{1-\gamma}}{1-\gamma} + v(m_t) \right],$$

where $v$ is strictly increasing, strictly concave, and differentiable. The agent can trade in a stock (claim to the endowment stream) with price $S_t$, a risk-free real bond (paying one unit of real goods) with price $1/R_t$, and a risk-free nominal bond (paying one unit of nominal goods with real value $P_t/P_{t+1}$) with price $1/I_t$.

Denote household wealth $x_t$ and suppose the agent is endowed with the stock and the initial money: $x_0 = S_0 + M_0/P_0$. The agent then chooses his consumption $c_t$, real money holdings $m_t$, holdings of the real bond, the nominal bond, and the stock.

(a) Find the agent's optimality conditions, then impose the equilibrium conditions (with nominal and real bonds in zero net supply) to characterize equilibrium prices and interest rates.

(b) Given the specification for the endowment process, solve explicitly for the net real interest rate $r_t = \log(R_t)$ and describe how it depends on the growth and volatility of output and the agent's preferences.

(c) We will solve for equilibria of the form $P_t = Y_t^a$ for some $a$. Define $\hat{\pi}_t = \log E_t(P_{t+1}/P_t)$ as the net expected inflation rate. Show that a given $\hat{\pi}_t$ is (typically) consistent with two values of $a$.

(d) Solve for equilibrium nominal interest rate $i_t = \log(I_t)$ in this class of equilibria.

(e) Suppose that monetary policy pegs a constant interest rate $i_t = \bar{\imath}$. Show that if $\sigma = 0$ there is a unique equilibrium, but if $\sigma > 0$ there are two equilibria. Interpret your answer in terms of the Fisher equation and inflation risk.

4. Consider a variation on the basic search model with exogenous money loss and gain. In particular, there is a unit mass of agents agents who can each hold a single unit of money which is in total supply $M \in [0,1]$, and agents cannot produce while they hold money. They get utility $u$ from consumption and pay cost $c < u$ to produce. Suppose there is no barter ($y = 0$ in the notation from class), and we normalize the meeting rate $\alpha$ and the probability of gains from trade $x$ so that $\alpha x = 1$. Now suppose that there is an exogenous chance that an agent with money loses it, which is governed by a Poisson process with arrival rate $d$. In addition, there is an exogenous chance that an agent without money finds some money, which is governed by a Poisson process with arrival rate $f$.

   (a) Write down the Hamilton-Jacobi-Bellman equations determining the value functions $V_0$ and $V_1$ for agents without and with money, and characterize the optimal strategies.

   (b) Characterize the set of equilibria, and interpret your answer.