Production Function, Average and Marginal Products, Returns to Scale, Change of Variables

Production Function:

links inputs to amont of output. Assume we have 2 inputs: Labor (L) and Capital (K), and we use Y for output . Then we write:
Y = F (L , K) , where F ( ) is the production Function**.** We make a number of assumptions about this function.

Examples:

(1) $Y = L . K$
(2) $Y = L + K$
(3) $Y = L^{1/3} . K^{1/3}$
(4) $Y = L^{1/2} . K^{1/2}$

Average Product and Marginal Product of a Particular Input

Labor:
Average Product of Labor (APL): Y/ L
Marginal Product of Labor (MPL): changes in Y / Changes in L (for small changes) = partial derivative of F(L, K) with respect to L.
Capital:
Average Product of Capital (APK): Y/ K
Marginal Product of Capital (MPK): changes in Y/ Changes in K (for small changes) = partial derivative of F(L, K) with respect to K

Returns to Scale:

Percentage of change in Y when we change all inputs in the same proportion.

Increasing Returns to Scale (IRS) :
% change in Y > % change in L = % change in K
Constant Returns to Scale (CRS) :
% change in Y = % change in L = % change in K
Decreasing Returns to Scale (DRS):
% change in Y < % change in L = % change in K

Formal Proof for Cobb Douglas

Let $Y = F(K,L)= K^{\alpha} L^{\beta}$, where α, β are positive constants.
We want to see what happens to Y when K and L increase in the same proportion. Let λ be a constant bigger than 1 (i.e. if $\lambda = 1.5$ this means that K and L increase by 50%).

$$F(\lambda K, \lambda L) = (\lambda K)^{\alpha} (\lambda L)^{\beta} = \lambda^{\alpha} K^{\alpha} \lambda^{\beta} L^{\beta} = \lambda^{\alpha+\beta} K^{\alpha} L^{\beta} = \lambda^{\alpha+\beta} F(K,L) = \lambda^{\alpha+\beta} Y$$

If $\alpha+\beta > 1$, $\lambda^{\alpha+\beta} > \lambda$ and % change in Y > % change in L = % change in K , then the function has IRS.
If $\alpha+\beta = 1$, $\lambda^{\alpha+\beta} = \lambda$ and % change in Y = % change in L = % change in K , then the function has CRS.
If $\alpha+\beta < 1$, $\lambda^{\alpha+\beta} < \lambda$ and % change in Y < % change in L = % change in K , then the function has DRS.

Some production functions exhibit the same type of returns to scale everywhere (like the 4 examples presented here), while others don't.
In our examples it is easy to find the type of returns to scale by looking at a couple of points.

Ex 1: $Y = L \cdot K$

L	K	Y= L . K	
1	1	1	
2	2	4	⇒ IRS
3	3	9	

Ex 2: $Y = L + K$

L	K	Y= L+K	
1	1	2	
2	2	4	⇒ CRS
3	3	6	

Ex. 3 : $Y = L^{1/3} . K^{1/3}$

L	K	$Y = L^{1/3} . K^{1/3}$	
1	1	1	
8	8	**4**	⇒ DRS
27	27	9	

Ex. 4: $Y = L^{1/2} . K^{1/2}$

L	K	$Y = L^{1/2} . K^{1/2}$	
1	1	1	
4	4	4	⇒ CRS
9	9	9	

Change of Variable

Sometimes it is convenient to make a change of variable in order to reduce the number of variables in our problem by one. When the production function exhibits CRS we can do this very easily.

Ex. 2: $Y = L + K$

Lets divide both sides by L,

$$Y/L = 1 + K/L$$

define two new variables : $y = Y/L$ and $k = K/L$, then

$$y = 1 + k$$

and we have only 2 variables in our problem.

Ex. 4: $Y = L^{1/2} . K^{1/2}$

Lets divide both sides by L (or $L^{1/2} . L^{1/2}$),

$$Y/L = (L^{1/2} . K^{1/2}) / L = (L^{1/2} / L^{1/2}). (K^{1/2} / L^{1/2}) = 1 . (K/L)^{1/2}$$

define two new variables : $y = Y/L$ and $k = K/L$, then

$$y = k^{1/2}$$

and we have only 2 variables in our problem.

Notice that our new function looks like this:

Since,

L	K	k= K/L	$y = k^{1/2}$
2	2	1	1
2	8	4	2
2	18	9	3